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THOUGHT AND BEING ARE THE SAME: CATEGORIAL RENDITION OF THE PARMENIDIAN THESIS

Abstract. Classic understanding of logic as an instrument of cognition, which, in effect, pertain rather to human's mind than to reality itself, gives rise to the fundamental mapping problem of reconciliation of this reality with any possible practices of its representations in thought. In other words, it is essentially not the same thing that can be thought and that can be. However, after unusual and highly abstract (essentially geometric) Grothendieck constructions gave rise to so called categorial analysis of logic, it became possible to show, that (up to categorial equivalence) Parmenides after all was right.

Keywords: appearing, categorial equivalence, Grothendieck topos, ontology, sheaves, theory of Ω-sets.

Introduction

...τὸ γὰρ αὐτὸ νοεῖν ἐστίν τε καί εἶναι[1] – this notoriously famous thesis of ancient Greek philosopher Parmenides still seems in a highest extent counterintuitive for our contemporary mind. Yet we know a number of so called "hunches" of quite the same sort Greek natural philosophers were renowned for, which later turned out to be scientifically proven truths. Thus Democritus insisted that everything consists of *atoms* – some tiny, physically indivisible parts – and in some sense it was perfectly right – if bold – guess. Anaximander speculated that, considering humans' extended infancy, we could not have survived in the primeval world in the same manner we do presently – so humans necessarily must have evolved. As we all know by now evolutionary theory confirms this intuition of the ancient thinker in precision and detail.

Nevertheless, both traditional and contemporary classic understanding of logic as an instrument of cognition, which, in effect, pertain rather to

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human's mind than to reality itself, gives rise to the fundamental mapping problem of reconciliation of this reality with any possible practices of its representations in thought. In other words, *it is essentially not the same thing that can be thought and that can be.* And only thanks to relatively recent formal theoretical results it became possible to study logic not as an arbitrary instrument of knowledge, but rather as a special case of some abstract topological construction, and thereby – in more or less strict sense – as an aspect of reality itself. These aforementioned results were obtained in the sixties of the last century mostly by great French mathematician Alexander Grothendieck. Radically new foundations for algebraic geometry, which he introduced and developed at that time, were formulated by him in a very special language – language of category theory. This theory itself emerged in the field of mathematics a bit earlier – in the forties of the last century – both as very abstract, though very productive algebraic tool and as an conceptual alternative to traditional set-theoretic foundations of mathematics.

Thus, at first unusual and highly abstract (but essentially geometric) Grothendieck constructions quite surprisingly gave rise to so called categorial analysis of logic, while particularly such constructions as *Grothendieck topos* and *categorial equivalence* will be shown as crucial in the rigorous proof of Parmenidian thesis.

1. What exactly identity means

Great German logician Gottlob Frege once famously noticed that propositions of the logical form $a = a$ are considerably less informative than propositions of the form $a = b$, – while the former are obvious and trivial tautologies like "to be is to be" or "the morning star is the morning star", which, being formally admissible as logic sentences, are barely make sense, the latter contain substantial amount of new information and thereby, generally speaking, must be justified somehow. For instance, proposition "the morning star is the evening star" states that the Morning Star, known to the ancients as Phosphorous, and the Evening Star, known to the ancients as Hesperus are one and the same heavenly body – Venus. Today this is a common fact, but the discovery of this *identity* was a prominent early advance in astronomy. Even now, although we readily understand and accept the hypothesis, only a few of us could formulate the argument and collect the crucial evidence without looking for help in the textbook.

So, first of all, as we just seen there are different kinds of identities – some more productive than the others, and second of all – unlike poetic metaphors, they admit rigorous proof. Let us see what kinds of identities are used in mathematics. First kind of identity is almost trivial – it is equality of sets. Nevertheless in ZFC axiomatics[2] this equality is guarantied by special axiom – axiom of extensionality, which posits that two sets are equal (identical) if the multiples of which they are the multiple, are "the same". Therefore, the identity of sets is founded on the indifference of their belonging. This is written:

$$\forall z \quad z \in x \Leftrightarrow z \in y \Rightarrow x = y$$

Despite its "almost-triviality" this identity also can be quite productive and unexpected. Let's consider the set of first four natural numbers $F = \{1, 2, 3, 4\}$ and the set D of all degrees of the equations solvable by radicals[3]. There was a time in 18th century when people thought that these two sets are completely different – for the latter thought to be infinite due to the common faith among the mathematicians in the early years of 18th century that sooner or later they will find formulas for calculating the values of the solutions of equations of arbitrary degree N. And only in 1832 young French genius Evariste Galois basically showed that two sets F and D are the same.

Sometimes equality of sets points us to the equality of some routes from one place (set) to another. For instance, for *additive* function f there is always the case that $f(ab) = f(a) + f(b)$. Or, in other words, there is no difference at all where we "go" first: from the elements of set A to the result of multiplication (m) and then – along f – to the set of destination C, or along f – to the elements of C which we will add (a) to each other later. In both cases we will get to the same element of C. We could write that two sets which correspond to the result of the consecutive applications of two operations to A are the same: $fm(A) = af(A) = B \subset C$[4].

Our final and most striking example would consist in equating elliptic functions with tori ("doughnuts" in laymen terms). Algebraic equation of form $y^2 = x^3 + ax + b$ defines an elliptic curve on a plane. But when considered over complex numbers the set of its solutions is equal to the set of points that define some torus embedded in the complex projective plane. That is, topologically speaking, a complex elliptic curve is torus.

Much more subtle type of identity is called *isomorphism.* It is identity of structures, and not just a quantitative equality of sets. They also say that there is one-to-one correspondence between sets which preserves structure. Set is said to be equipped with structure when it is closed under some algebraic operations (e.g. together with its every two elements the sum (product etc.) of these elements also belongs to this set), or its elements are not "equal" – one is "larger" or "smaller" then another, or there are some other kind of relations between the elements. Sets may seem completely different, but nevertheless structurally identical.

So $f : X \to Y$ is isomorphism when there is $g : Y \to X$, such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. In another words, two objects are isomorphic if we can define at least one reversible map between them. This definition does not say anything explicitly about preserving structures, but it follows from the above conditions that if sets X and Y are endowed with some structure (for instance, $(X, *)$ and $(Y, \bullet)$), then $\forall a, b \in X$: $f(a) \bullet f(b)$ must be equal to $f(a * b)$[5]. This property of f often call *functoriality* whereas map g is called *inverse* for f.

Magnificent example of structural identity is isomorphism between infinite-dimensional space and set of functions of a given kind. A "curve" in such function space is a subset of functions with some additional conditions – let's say it's a set of all possible ways from A to B. If we find a "point" on that curve where its derivative equals zero, we thereby find a function which defines shortest (or longest) way from A to B.

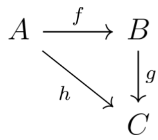
Another example is actually just the extension of aforementioned example with our additive function. It can be shown that there is one-to-one correspondence between the set of positive real numbers equipped with multiplication and the set of all real numbers with addition. We all know such f and g very well:

$$(\mathbb{R}^+, *) \underset{g=\ln x}{\overset{f=e^x}{\rightleftarrows}} (\mathbb{R}, +)$$

Isomorphism between these two structures is used in a slide rule, when we take two positive numbers whose product we would like to get, go along f to another set, where we add two corresponding elements (which is much easier operation), and go back along g to desired product. It means that e^x is an additive and, therefore, also a functorial map. And at last, most profound and most unusual kind of identity is *categorial equivalence*. Preeminently this kind of identity is necessary to justify Parmenidian thesis. In particular we will show that "everything that can be thought", or *thinkable world* is a category of some sort, and everything that can be, or *real world* is a category of another sort, and then we will prove the equivalence of these two categories. But in order to do this first of all we need to say some words about such special entities as categories and functors.

2. "Categorial" approach

From axiomatic point of view we have a category $\underline{C}$ whenever we define the class[6] of *objects* of the category and for every two objects $A, B \in \underline{C}$ we define a set of *arrows* $\mathbf{Hom}_{\underline{C}}(A, B)$ and for every two arrows $f : A \longrightarrow B$ and $g : B \longrightarrow C$ we always have an arrow $h : A \longrightarrow C$ such that $h = g \circ f$. Whenever this condition is met they say that the triangle below commutes:



and typical categorial proof has this form of commuting diagrams. It is easy to notice that the concept of an arrow (or a map) is a vast generalization of set-theoretic function, whereas condition $h = g \circ f$ is naturally emerging rule of composition. So, an arrow can be thought as function, but in a very informal sense – as some abstract sort of action, transformation or relation, depending on context. Later we'll see that functoriality is the minimal universal requirement for all structure preserving actions, e.g. for homeomorphic map: hence, the abbreviation "Hom" for a set of arrows and hence, another name for an arrow – morphism. Final and also natural request then for the composition of arrows is to be associative which means that $(h \circ g) \circ f = h \circ (g \circ f) = h \circ g \circ f$ and it is also required that for every object $A \in \underline{C}$ we have a special arrow $id_A \in \mathbf{Hom}_{\underline{C}}(A, A)$ (it's called identity arrow or identity map) such that for every two objects $A, B \in \underline{C}$ and for every arrow $f \in \mathbf{Hom}_{\underline{C}}(A, B) : id_A \circ f = f \circ id_B = f$. If we look at the composition as at some kind of product we will see that identity arrow plays role identical to multiplicative unit. So, we see that as an abstract algebraic object a category is very poor and almost meaningless construction. But, on the other hand, exactly the same axiomatic poverty sometimes allows to draw meaningful and very important consequences based on quite unexpected unification (categorial equivalence) of seemingly distant domains of formal discourse (categories). To see why it happens we must notice that highly abstract character of categorial language is relatively deceptive – from one hand, a huge variety of entities comply with the axioms of category: partially ordered set, set of all natural numbers, singleton together with identity arrow as well as such gargantuan constructions as all sets together with all functions or all topological spaces together with all continuous transformations are all categories. But from the other hand, the more additional structure a particular category has, the more intricately organized its arrows must be to make formally "primitive" composition rule work. It means that if an object of category has any additional internal structure, it is always preserved by an appropriate arrow in category in question. As a result, simple and elegant categorial statements are always supported by implicit rigorous machinery eventually defined on those "atomic elements" of objects which category theory "shouldn't take into account".

If anything could be called an object of a category as soon as some fundamental conditions met (see above), then we can make categories themselves serve as objects of some other category. How then an arrow in such category should look? Let $\underline{C}$ and $\underline{D}$ be two categories which are in our case considered as objects. It can be noticed that in order to internal (categorial) structure of such special object be preserved by some arrow (and, accordingly, for any diagram in such category to commute) composition rule must be carried out not only on objects of the category but also on its arrows. In that way, we get definition of a *functor*:

Functor $F : \underline{C} \longrightarrow \underline{D}$ from category $\underline{C}$ to category $\underline{D}$ is an arrow which maps

- every object A of category $\underline{C}$ to some object $F(A)$ in category $\underline{D}$;
- every morphism $f \in \mathbf{Hom}_{\underline{C}}(A, B)$ to some morphism $F(f) \in \mathbf{Hom}_{\underline{D}}(F(A), F(B))$ so that

$$F(id_A) = id_{F(A)}$$

$$F(g \circ f) = F(g) \circ F(f).$$

Naturally, there can be more than one functor between two categories. When this is the case, some functors themselves can be related somehow. In particular, we can consider the following situation:

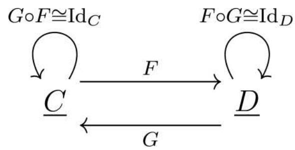
$$\begin{array}{cccc} X & F(X) & \xrightarrow{\alpha_X} & G(X) \\ \Big\downarrow f & \Big\downarrow F(f) & & \Big\downarrow G(f) \\ Y & F(Y) & \xrightarrow[\alpha_Y]{} & G(Y) \end{array}$$

If there exists an arrow $\alpha \in \underline{D}$ which every object $X \in \underline{C}$ puts in correspondence with family of morphisms $\alpha_X : F(X) \longrightarrow G(X)$ (called component α at X) so that for every $f \in \mathbf{Hom}_{\underline{C}}(X, Y)$ diagram above commutes, then such arrow α called *natural transformation* of functors F and

G. Natural transformation provides the way of transforming one functor into another while respecting the composition of morphisms of the categories involved. So, α can be construed as morphism of functors and it is quite naturally to expect that in some cases this morphism turns out to be isomorphism. And, indeed, if for every object X in $\underline{C}$ the morphism α is isomorphism in $\underline{D}$, then α is said to be *natural equivalence*, or natural isomorphism of functors:

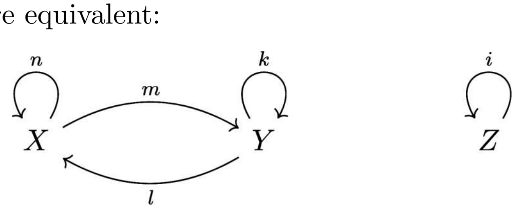
$$F \cong G.$$

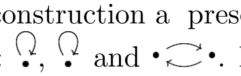
Remember, that isomorphism of two sets X and Y was completely determined by existence of two functions f and g such that $g \circ f = \mathrm{id}_X$ and $f \circ g = \mathrm{id}_Y$. But great mathematician Alexander Grothendieck noticed[7] that we get much deeper meaning from some special sort of identity between two categories $\underline{C}$ and $\underline{D}$ when from two functors F and G we do not demand equality of their composition to the identity functor Id – it is enough to be isomorphic to it!



As a result, two categories $\underline{C}$ and $\underline{D}$ are said to be *equivalent* when there are two functors F and G such that $G \circ F \cong \mathrm{Id}_C$ and $F \circ G \cong \mathrm{Id}_D$.

Again, as in case of isomorphism of two sets, "equality" of equivalent categories can be seen only at certain height of abstraction: *prima facie*, they are absolutely different constructions but nevertheless they have some fundamental structural identity. For example, it can be shown that following two categories are equivalent:



We can see that in this case categorial equivalence reveals inside quite complicated construction a presence of almost identical copies of much simpler structure: . In other cases categorial equivalence hepls to reveal striking similarities between seemingly distant domains of discourse. And, in particular, in case that will follow it will help us to rigorously unify ontology with logic.

3. Is a world really a Grothendieck topos?

The central statement of Alain Badiou's "Logics of worlds" is following: "A world is a Grothendieck topos" [1, p. 295] and first of all I'd like to show that this statement is equal to much more comprehensible proposition – "a world is thinkable" whereas the latter itself is a consequent of another hypothetical proposition "if a world is real, then it is thinkable". This hypothetical statement is true, because it is just an ontological interpretation of the theorem which states that in category of complete Heyting-valued sets the axiom of gluing holds, or, equivalently, that for every Heyting-valued set it is possible to define functor from suitable poset category into **Set**[8] which is a sheaf.

Now let's see if we can make sense from all aforesaid, because if we succeed, then we will have at our disposal at least the half of so called "Parmenidian equality": *the world is thinkable if and only if the world is real.* In order to do so we will need to examine more closely a special kind of category called topos and, in particular, quite profound ontological intuitions made by Alain Badiou regarding this fascinating categorial structure.

In mathematics concept of Heyting-valued set realizes an idea of *potentially* existing elements. For example, when elements of set A are functions defined on open subsets of topological space X. Then it makes perfect sense to speak about *actually* existing functions (defined on whole X) as well as about functions existing *in some extent*, depending on the size of subset where given function is defined. Moreover, if f is defined on $U \subseteq X$ and g is defined on $V \subseteq X$ it is also makes sense to compare the *extent of identity* of such functions, measuring the largest open subset of $U \cap V$ on which $f = g$. Emerging from this discussion is a generalized concept of a "set" as consisting of a collection of partially existing elements with the degree of identity of these elements measured on some Heyting algebra Ω[9]. An Ω-valued set **A** is then defined as a pair (A, **Id**) where **Id** is a function assigning to every pair of elements $x, y \in A$ an element $\mathbf{Id}(x, y) \in \Omega$, satisfying two conditions:

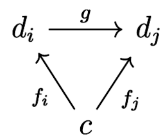
$$\mathbf{Id}(x, y) = \mathbf{Id}(y, x);$$

$$\mathbf{Id}(x, y) \cap \mathbf{Id}(y, z) \leq \mathbf{Id}(x, z).$$

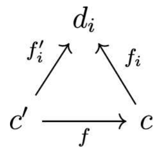
We see that function **Id** reminds some sort of "quasi-metric", however it gets its values in suitable poset. Partially ordered values of function **Id** emerge quite naturally in this formal setting because family of open subsets of topological space are partially ordered. But Badiou goes far beyond any formal setting and fairly points out on universal (transcendental) character of any appearing whatsoever – to appear is to be evaluated on partially ordered transcendental scale (locale). Surprisingly, his "phenomenological" intuition totally coincides with basically the same ideas of computer scientist Joseph Gougen who wrote: "A housewife faces a fairly typical optimization problem in her grocery shopping: she must select among all possible grocery bundles one that meets as well as possible several conflicting criteria of optimality, such as cost, nutritional value, quality, and variety. The partial ordering of the bundles is an intrinsic quality of this problem." [4, p. 145] So, evaluation on partially ordered scale really seems to be the basis of any differentiation and Badiou by no means accidentally borrows aforementioned categorial apparatus together with function **Id** which he calls now *function of appearing.* Special case of $\mathbf{Id}(x, y)$ when $x = y$, just like in the case of partially defined functions,[10] Badiou calls *existence* and denotes as $\mathbf{E}x$. Thus, he ascribes phenomenological and even existential meaning to these formal evaluations and shows that the values of $\mathbf{Id}(x, y)$ are inherently belong to every "world" in a sense that every single element of the world necessary appears in it as some quality/qualities which necessary manifest themselves with some intensity. In different "world" the same element x can be appeared with different intensity $p \in \Omega$, but its appearance nevertheless will have some inherent order completely determined by Ω and **Id**. "World" here could mean any situation whatsoever – it can be our world itself, or any part of it: still life, battle, painting of battle, somebody's perceptual picture of some event etc. The only important thing is that for every multiple $A \in \mathbf{m}$ and for every pair of elements $x, y \in A$ we are always able to evaluate the degree of their identity $\mathbf{Id}(x, y)$ as well as for every $x \in A$ – the degree of its existence in $\mathbf{m}$[11]. As a result, depending on aspect of its appearing, we always get a collection of some Ω-valued sets $\mathbf{m}$ and that is what we will call a world.

Now, it can be shown that Ω-valued sets form objects of a category denoted $\Omega - \mathbf{Set}$. Moreover, it proves to be *topos.* And this is what every world is, from categorial point of view. Topos is a category endowed with quite rich additional categorial structure, and in his most recent book "Mathematics of transcendental" [2] Alain Badiou defines topos almost poetically, saying that topos is a possible universe which is both big and centered, and which presents its own internal logic. Let's then examine more closely what does it mean for a category to be "big", "centered" and "to present logic".

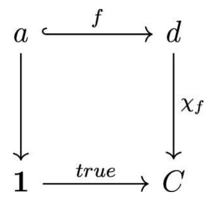
In contrast to set theory where there is straightforward answer to the question on size of universe – answer about its cardinality, it is not so clear in category theory. What rather matters here is which actions are possible in this universe and which compositions of actions, in particular. Another, and even more appropriate way to ask about the size of a category construed as a place of appearing being (world) is to ask whether we can "see" from a point within the category the vast configurations located elsewhere in the category. Since seeing, as with any action, must be thought in terms of arrows, for given configuration (it's called *diagram*[12]) it is natural to demand the existence of an object c of the category from which there exist arrows which go from c to every object of the diagram, whereas every arrow of the diagram enters to the composition of an arrow coming from c:



Such system is called a cone for the diagram D, or D*-cone.* The category will be big enough if many diagrams admit cones. Of course, diagram can admit several cones for we can "see" the same fragments of the category from different objects, but there can be *universal* object which is itself visible from every other object from which we see the diagram – a point where we see the diagram as closely as possible. Technically it means that for any other D-cone there exists unique arrow $f : c' \longrightarrow c$ such that following triangle commutes for every $d_i \in D$:

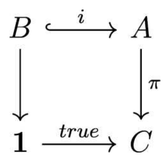


If this is the case they say that diagram admits the *limit* and that the limit cone of the diagram D has the *universal property* with regard to this diagram. The concept dual to the concept of limit is that of *co-limit* – a universal object c which is seen by the diagram from as far away as possible. And a world as a category is precisely big enough for all its finite diagrams to admit both limits and co-limits. But there is "little bit" more[13]. Every topos (and thereby – every world) has some sort of "central object" that is called *subobject classifier* – an object C of the category with marked element[14] $true : \mathbf{1} \longrightarrow C$ such that for every *monic* arrow $f : a \hookrightarrow d$ there exist unique characteristic (*centralizing*) arrow $\chi_f : d \longrightarrow C$ such that the following square is a *pullback*:



The problem is that there considerable ambiguity left whenever try to determine a part of an object (subobject) in the category – a monic arrow f (an categorial analog of set-theoretic injective inclusion), generally speaking, localizes part a of an object d only up to the class of equivalence (in ordinary set-theoretic sense), but in topos the situation is much more accurate: there exist an object C which is "seen" from any other object d included in monic configuration $a \hookrightarrow d$ (that's why Badiou calls topos centered) and reciprocally any part of any object d can be "seen" (located) up to isomorphism!

And this is where a world gains its inner logical resource for any predicate (partition) function $\pi : A \longrightarrow C$ "centers" monic arrow i assigning to every $x \in A$ the degree of intensity $p \in C$ with which an element $x \in \pi$.



Whereas in case of **Set**, where $C = 0, 1$ $\pi : A \longrightarrow C$ just separates in A its subset B of all such $x \in A$ that $\pi(x)$[15] (square on the diagram pulls *true* along π back into subset B of A specified by pullback condition – in this case $\pi(x) =$ "true")[16], in Ω-**Set** surprisingly enough the role of central object C plays Heyting lattice Ω itself, so subobject here looks like bizarre entity capriciously "glowing" with different intensities $p \in \Omega$. Formally, subset $\pi : A \longrightarrow \Omega$ of Ω-valued $\mathbf{A}$ also determines a monic arrow $i_\pi : B \longrightarrow A$, but $\mathbf{B}$ has the same collection A of elements as $\mathbf{A}$ with equality given by: $\mathbf{Id_B}(x, y) = \pi(x) \cap \pi(y) \cap \mathbf{Id_A}(x, y)$, i.e. x and y are identical in $\mathbf{B}$ to the extent they are identical in $\mathbf{A}$ and belong to π. From logical point of view these intensities of "belonging" correspond to certain "truth-values" in Ω with which elements of the world manifest their property π. Moreover, it can be shown that not only predicates of the form $\pi(x)$, but all propositional constructions, employing relations with n terms, as well as all other connectives and quantifiers of logical calculus also can be expressed by the arrows of topos[17] while nature of this logic every time is completely determined by *internal* structure of topos in question. It must be added that not only in case of Ω-**Set**, but in general case of topos as well its object classifier has the structure of Heyting algebra, whereupon internal logic of topos is essentially intuitionistic.

4. Thought and being are the same

Formally, a subset of Ω-valued set $\mathbf{A}$ is a function $\pi(x) : A \longrightarrow \Omega$ that has:

$$\pi(x) \cap \mathbf{Id}(x, y) \leq \pi(y);$$

$$\pi(x) \leq \mathbf{E}x.$$

These two axioms are natural conditions imposed on any action coherently marking out a part of something in situation with more then binary choice. We want to be sure that if x strongly belongs to the part $\mathbf{A}_\pi$ and x is very identical to y, then y itself must belong to $\mathbf{A}_\pi$ and belong strongly. Second condition is just an observation that degree of element's belonging to the part can't be superior to that of its own degree of presence (existence) in the whole. It happens sometimes that one more condition is met:

$$\pi(x) \cap \pi(y) \leq \mathbf{Id}(x, y)$$

The latter means that predicate π separates in $\mathbf{A}$ its part to which no more than one element belongs "absolutely". Or, two elements belong to the part only to the extent that they are identical. Such subobject is called *singleton*, but Badiou calls it an "atom". Now, for $a \in A$ (that is for an element of multiple A in strict ontological sense) we can define a function $a(x) = \mathbf{Id}(a, x)$ which associates to every $x \in A$ its degree of identity with some fixed element a. In terms of predicates it's the one which says something like: "to be like this thing a in world $\mathbf{m}$". This function is not only *atomic*, but it also is *real* for it has been *ostensively* defined by pointing on ontological being $a \in A$.

We saw that each element $a \in A$ yields singleton $a(x)$ and Ω-valued set $\mathbf{A}$ is called *complete* if each of its singletons is of the form of $a(x)$ for a unique $a \in A$.

Badiou does exactly the same thing defining his "object" as couple (A, **Id**) under the condition that every atomic predicate $\pi(x) : A \longrightarrow \Omega$ be equal to real atom $a(x) = \mathbf{Id}(a, x)$ for every $x \in A$ [1, p. 251]. So, basically he repeats the definition of complete Ω-set. But then we can ask ourselves:

is it true that every atomic predicate $\pi(x)$ of every $A \in \mathbf{m}$, however unrealistic or fancy, can be reduced to real ostensive form $\mathbf{Id}(a,x)$? Is our linguistic resource essentially sutured to reality itself? It seems that Badiou doesn't know how to rigorously justify positive answer to this question for he just "materialistically postulates": every atom is real. However, Ω-**Set** and subcategory of Ω-**Set** generated by the complete objects are known to be equivalent as categories (a result due originally to D. Higgs [7]). Which means, up to categorial equivalence, that *the world is real.*

Now, as Robert Goldblatt puts it, the completeness property for a Ω-set allows a very elegant abstract treatment of the idea of the restriction of a function to an open set [5, p. 389]. Given $a \in A$ and $p \in \Omega$ the function $\mathbf{Id}(a,x) \cap p$ happens to be a singleton, and if **A** is complete then there is exactly one $b \in A$ with $\mathbf{Id}(b,x) = \mathbf{Id}(a,x) \cap p$. We'll call such b the *restriction* of a to p and denote it as $a \restriction p$.

Let's see what formal instruments it gives us in the view of finishing categorial justification of Parmenidian thesis by considering the following construction: if we have a Ω-set **A**=(A, **Id**) we could try to associate to an element p of Ω all the elements $x \in A$ which have the degree of existence $\mathbf{E}x = p$. It would be our schema for thinking that seizes hold of objects of a world *analytically*, according to the existential stratification of their appearing. Badiou himself takes as an example for his analysis a "world" of the battle of Gaugamela (1 October 331 BC) in which Alexander destroyed the Persian army and the power of Darius III and "center of the Persian army", in particular (**A**). The degree of existence in this particular case means the combat capacity of different *parts* of Darius's setup: royal guard, the elephants, the Hyrcanian and Indian cavalry, the scythed chariots, the Greek mercenaries etc. taking into account the whole dynamic of the battle-world for these parts are modified by becoming of this world, which is also the dynamic of its appearance[18].

Generally speaking, to a fixed degree of combat capacity there correspond several elements of an "object". The problem then seems to be the following: we would totally recover world's logic (a world would be *thinkable*) if, based on the analysis of appearing of its parts, we would be able to choose a single element in **A** which has a synthetic, envelope-value with regard to objective appearing of the multiple in the world. Or, as Badiou asks about the case in question: "Does there exist an element of the object "center of the Persian army" which subordinates all others to itself in terms of the destiny of the object as a whole within the battle-world?" [1, p. 286]

In order to answer this question systematically, let's examine the following diagram:

$$\begin{array}{ccc}
p \in \Omega & \xrightarrow{F_A} & \begin{array}{c} F_A(p) \subseteq A, \\ \mathbf{E}x = p \end{array} \\
{\scriptstyle q \leq p}\downarrow & & \downarrow{\scriptstyle \varphi_q(x) = x \restriction q} \\
q \in \Omega & \xrightarrow[F_A]{} & \begin{array}{c} F_A(q) \subseteq A, \\ \mathbf{E}x = q \end{array}
\end{array}$$

Here we formalize our idea of stratification of $\mathbf{A}$ with the operator F_A which guarantees the correlation that goes from $p \in \Omega$ towards a subset of A:

$$F_A(p) = \{x : x \in A \wedge \mathbf{E}x = p\} .$$

What is the correlation between $F_A(p)$ and $F_A(q)$? Remember, that Ω is partially ordered set (and, hence – a category), so in general there are incomparable elements, but it can be shown that for a complete $\mathbf{A}$ if $q \leq p$ then for every $y \in F_A(p)$ the restriction $y \restriction p \in F_A(q)$, and therefore, an arrow $\varphi_q(x) = x \restriction q$ makes our diagram commute, which means that the operator F_A is a functor.

Now, going back to our problem, its analytical part would consist in choosing from every part $F_A(q)$ a "typical" *representative* x_q corresponding to each degree q. The element will be typical if the global importance of its existence is greater than that of elements with the same existential degree. But there is one more crucial condition that should be met: for a subset $B \subseteq A$ of representatives to admit a unique *supremum*, or synthetic term that corresponds, through analytical procedure of typical selection, to the degree that acts as an envelope in Ω, these representatives must be, loosely speaking, "of the same kind". Badiou insists that military genius of Alexander was able *to calculate* the synthetic position of the element "scythed chariots" for the object "center of the Persian army" only because "all other elements of thew object, in their spatial disposition and differential evaluations, *were compatible with each other*. (My emphasis.) They were under the sway of a battle plan which articulated them all with the supposedly desisive action of the 200 scythed chariots" [1, p. 287].

In formal setting to this "poetic" criterion correspond precise mathematical definition: in complete Ω-set $\mathbf{A}$ $a, b \in A$ are *compatible* if $a \restriction \mathbf{E}b = b \restriction \mathbf{E}a$ (we will denote this relation $a \frown b$). Probably, this property of compatibility will become more graphic when we notice that it follows almost immediately from the definition that if $a \backsim b$ then $\mathbf{E}a \cap \mathbf{E}b = \mathbf{Id}(a,b)$. So, actually, we could reverse the implication and say that a and b are compatible when their identity with regard to their mutual intensity of existence is maximal[19]. It is also important to keep in mind that at first singleton was defined as a function $\pi(x) : A \longrightarrow \Omega$, so is restriction $b(x) = \mathbf{Id}(a,x) \cap p$. Now, if we recall the motivation for Ω-set construction from the beginning of Part 3, then we'll see that compatibility of partial functions f and g, defined on open subsets of topological space X, is just a coincidence of their values on the full intersection of their domains – an absolutely natural condition, under which we are able to "glue together" these two functions. Essentially the same intuition can be applied for understanding onto-logical (or existential) compatibility of two singletons $a \backsim b$. Ω also happened to be topological space, so the restriction of singleton $a(x)$ on the open subset $Ex \subseteq \Omega$ is sort of re-evaluation of atomic part $\mathbf{A}_a$ on the scale, more adjusted to distinguish properties of b, and vice versa. And the equality $a \backsim b$ then $\mathbf{E}a \cap \mathbf{E}b = \mathbf{Id}(a,b)$ means that, being measured on the scale Ω properly calibrated relative to their coexistence $\mathbf{E}a \cap \mathbf{E}b$, singletons $a(x)$ and $b(x)$ behave identically as predicate functions, i.e. $a(x)$ and $b(x)$ "say the same about the others", or they are identically different from every other $x \in A$.[20]

Finally, compatibility induces "ontological" partial order $<$ on elements of A based on their degree of existence:

$$\forall a, b \in A : a < b \Leftrightarrow a \backsim b \wedge Ea \leq Eb.$$

Intuitively seems clear that inequality in being must entail inequality in existence but reciprocal is already not that obvious.

And this is exactly what needed to complete our analysis for it can be sown that in a complete Ω-set $\mathbf{A}$ every subset $B \subseteq A$ whose elements are pairwise compatible has a unique join. It means that function $\varepsilon(x) = \bigcup\limits_{b \in B} \mathbf{Id}(b,x)$ defines a singleton when B has pairwise compatible elements and this singleton prescribed by unique element $\varepsilon \in A$ such that $\varepsilon = \sup B$.

Now, if we define $\mathbf{Cov}_\Omega(p)$ to be the collection of all subsets C of Ω that have $\bigcup C = \Omega$, take $C \in \mathbf{Cov}_\Omega(p)$ and consider a selection B of elements $x_q \in F_A(q)$ corresponding to all $q \in C$, it can be shown that $\varepsilon \restriction q = x_q$!

In geometry this construction is called *gluing* (whereas functor F_A is a *sheaf*), but we saw that purely onto-logical sense also can be ascribed to it. For ε guarantees a comprehensive grasp of the unity of a multiple $B \subseteq A$ in terms its logic of appearing, (since $\varepsilon \restriction q = x_q$), in terms of compatibility (since $x_{q_1} \backsim x_{q_2}$) and in terms of its order in being (since $x_q < \varepsilon$).

Conclusion

An action in converse direction is also possible: given a sheaf F over Ω we can construct a corresponding Ω-set $\mathbf{A}_F$ which is complete. The constructions $\mathbf{A} \longrightarrow F_A$ and $F \longrightarrow \mathbf{A}_F$ can be extended to arrows[21] to give an equivalence between categories of sheaves over Ω ($\mathbf{Sh}(\Omega)$) and a category of complete Ω-valued sets which already has been shown to be equivalent to category Ω-**Set** itself.

The construction of sheaf was possible, in particular, because we could define $\mathbf{Cov}_\Omega(p)$ on Ω. It was relatively easy in case of poset category but can be quite tricky in general for what we do is define topology J on a category. It is called *Grothendieck topology*, a category $\underline{C}$ equipped with J is called *site* and a category of sheaves on a site $\mathbf{Sh}(\underline{C}, J)$ is called *Grothendieck topos.*

Finally, we can see more clearly why Badiou says that the world is Grothendieck topos if the world *is* real in a sense that it can be identified with a category of complete Ω-valued sets (but we showed above that it realy can be done) then there both analytic and synthetic procedures are *really* possible which totally recovers its logic as appearing multiplicity. Therefore, *a world is thinkable.* Reciprocal is also true: if transcendental thinking F of the object is achievable, i.e. we are able to pass analytically from the collections of parts of the object $\mathbf{A}$ to a collection of elements of $\mathbf{A}$ in such manner that comprehensive synthesis of that part is also available, then we always can define within a world a corresponding part which is real ($\mathbf{A}_F$ is complete). But it means that our work is done for Parmenidian equivalence just has been completely categorically justified: *a world is thinkable if and only if a world is real.*

N O T E S

1 ...For it is the same thing that can be thought and that can be (*Greek*).

2 Zermelo-Fraenkel set theory

3 The problem of equations "solvable by radicals" is the following: is it possible for a given type of algebraic equation to establish a sequence of the algebraic operations (four basic arithmetic operations plus n-th degree rooting) which, when applied to its coefficients, determine the value of the solutions?

4 Sometimes they call such subset $B \subset C$ the image of A under given transformation, or, as in our case – under composition of two transformations.

5 Note, that both $f(a)$, $f(b)$ and $f(a) \bullet f(b)$ are actually the elements of the set Y.

6 The concept of class allows to examine huge collections of objects that are not, strictly speaking, sets – for example, objects of category of sets by definition are all sets and we know that there is no such thing as set of all sets.

[7] The notion of equivalence of categories was introduced for the first time by Grothendieck's in his legendary "Tôhoku" paper – revolutionary article on homological algebra which was published in 1957 in Tôhoku Mathematical Journal after almost 3 years in redaction. (See: [6])

[8] Category of all sets.

[9] Complete distributive lattice, or Heyting algebra was introduced by Dutch logician Arend Heyting as a model for intuitionistic logic, and we will come back to this important fact in our later discussion of so called "internal logic of topos".

[10] For more formal details see: [5, p. 274-276].

[11] I'd like to notice once again that $\mathbf{E}x = \mathbf{Id}(x, x)$ not always needs to get maximum value (as would be the case in regular metrics where proximity of element to itself is always maximal). This is fortunate because it allows us to express truly existential idea of authenticity of presence in m, ranging it, figuratively speaking, from das Man to Dasein.

[12] Diagram must be thought as any possible fragment of given category. So it can be any collection of objects and arrows whatsoever – it may be empty, or it may have infinite number of objects and arrows, it may have objects but no arrows, it may have some objects connected with several arrows and other objects not connected at all etc.

[13] Besides finite limits, co-limits and subobject classifier, every topos also has so called *map object* B^A – categorial analog of set of all functions from set A into set B. It can be thought as kind of a limit of the following quasi-diagram: $(\ ) \times A \quad B$. This limit optimizes the property "being produced with A and seeing B from this product".

[14] A limit for the empty diagram, denoted $\mathbf{1}$, or *terminal object* – an object uniquely "visible" from every other object of given category. Formally, there exist, for every object of the category one and only one arrow which goes from this object toward $\mathbf{1}$. Since everything in a category must be defined in terms of arrows, an element of an object is an arrow $\mathbf{1} \xrightarrow{x} d$ that effectively chooses exactly one element x in d.

[15] Axiom of separation in ZFC.

[16] Pullback is a limit for diagram of the form $A \xrightarrow{f} C \xleftarrow{g} B$, i.e. subset D of $A \times B$ specified by pullback condition $f(x) = g(y)$.

[17] For elaborate study of expressive power of toposes see [8].

[18] It was F. W. Lawvere who first thought of Grothendieck topos as a space (or rather space-time) of sets continuously varying over Ω.

[19] Generally, $\mathbf{E}a \cap \mathbf{E}b \geqslant \mathbf{Id}(a, b)$. is the case.

[20] Let's take as an example the "office-world" and consider two its atomic parts: "redhead" and "blonde". These redhead and blonde individuals will be *compatible* if their overall presentation in the office structure is also similar (they both are workaholics, both are heads of their departments etc.)

[21] See: [3, p. 160–162]

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