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THEOLOGICAL UNDERPINNINGS OF THE MODERN PHILOSOPHY OF MATHEMATICS. PART I: MATHEMATICS ABSOLUTIZED

Abstract. The study is focused on the relation between theology and mathematics in the situation of increasing secularization. My main concern is nineteenth-century mathematics. Theology was present in modern mathematics not through its objects or methods, but mainly through popular philosophy, which absolutized mathematics. Moreover, modern pure mathematics was treated as a sort of quasi-theology; a long-standing alliance between theology and mathematics made it habitual to view mathematics as a divine knowledge, so when theology was discarded, mathematics naturally took its place at the top of the system of knowledge. It was that cultural expectation aimed at mathematics that was substantially responsible for a great resonance made by set-theoretic paradoxes, and, finally, the whole picture of modern mathematics.

DON JUAN. What do I believe?
SGANARELLE. Yes.
DON JUAN. I believe that two and two are four, Sganarelle, and that twice four are eight.
SGANARELLE. A fine belief, and nice articles of faith! As far as I can see, your religion is arithmetic.

Molière, *Don Juan* (1665). Act III. Scene I.[1]

1. Introduction

When we speak of "theology in mathematics", what do we mean? First of all, what does the preposition "in" mean? It originally had a spatial meaning; hence, we metaphorically imagine mathematics to be a sort of receptacle or "container". What should count as lying *inside* the container labeled "mathematics" and what is to be found *outside* of it?

If we restrict the content of the word *mathematics* to mathematical *theories*, then there is apparently no theology in modern mathematics. God,

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gods, or divine attributes can hardly be found among mathematical objects (or at least nowadays we tend to think so). Dominic O'Meara (1989, pp. 199–209) called the opposite view "theologizing mathematics" and attributed it to Iamblichus (3^{rd}-4^{th} centuries). According to Svetla Slaveva-Griffin, Iamblichus established an "ontological affiliation between numbers and gods" (2014, p. 206). Despite Iamblichus's wishes, mathematical discourse failed to be a discourse about divine reality in any straightforward way. Even Georg Cantor, "a mathematician with deep theological inclinations" (Tapp, 2011, p. 96), insisted that Absolute Infinite (= God) cannot be a subject for mathematical demonstrations. Already, in Antiquity, there was a tendency to make a clear distinction between *mathematics of number* and *theology of number*, as well as *mathematical* and *divine* numbers, despite their strong correlations. It dates back to debates in the Old Academy documented by Aristotle (mathematical *vs.* ideal numbers, *Met.* 1083a), and was transformed into a complex hierarchy of different types of numbers by the Neopythagoreans and Neoplatonists (Slaveva-Griffin, 2014, pp. 200–215; O'Meara, 1989, pp. 79–81, 198–207).

Nevertheless, sometimes, even in modern times, theological entities can count as mathematical objects (or *vice versa*) after all. If we believe that mathematics describes some aspects of the divine mind, God's plan of creation or "the products of God's intellective activity" (Menzel, 2001, p. 75), or, perhaps, some parts of the supernatural angelic hierarchy, we thus introduce theology into mathematics. Let me make here some remarks on "mathematical angelology". Some version of it is present in Eriugena (9^{th} century) (Gersh, 1978, pp. 308–312) or Otloh of St. Emmeram (11^{th} century) (Gersh, 1996, pp. 233–236). Those medieval ideas were revived and transformed in the 20^{th} century, namely by Pavel Florensky (Shaposhnikov, 2009, pp. 355–357). In the early medieval texts, angelology is no more than a sphere of application for the universal principles of Pythagorean harmonics and number symbolism. On the contrary, in Florensky, numbers or numerical ratios are not the way of putting in order the realm of angels, but rather numbers in a sense *are* angels. In fact he *identifies* angels with Platonic Forms, which have two *inseparable* "dimensions": spiritual and physical, that is Names and Numbers. A hint on possible convergence between angelology and mathematics can also be found in Vladimir Lossky's mid-1950s lectures on dogmatic theology (2012, pp. 11–16).

Is there some specific theological *method* of inquiry that can be used in mathematics? Perhaps it is possible to introduce some form of *divine revelation* to mathematics. On this basis, we could then even distinguish between natural and revealed mathematics quite similar to the distinction in theology. For instance, the famous South Indian mathematician Srinivasa Ramanujan (1887–1920) can be seen as one who practiced revealed mathematics. Ramanujan seemed to be a true adherent of Hinduism under special inspiration from the goddess Namagiri and her consort Narasimha (the Lion incarnation of Vishnu). The goddess "inspired him with the [mathematical] formulae in dreams" (Seshu Aiyar & Ramachandra Rao, 1927, pp. xi–xii).

> It was goddess Namagiri, he would tell friends, to whom he owed his mathematical gifts. Namagiri would write the equations on his tongue. Namagiri would bestow mathematical insights in his dreams. [...] T.K. Rajagopolan [...] would tell of Ramanujan's insistence that after seeing in dreams the drops of blood that, according to tradition, heralded the presence of the god Narasimha, 'scrolls containing the most complicated mathematics used to unfold before his eyes'. [...] Ramanujan, showing his work to mathematics professor R. Srinivasan, made the statement in which he pictured equations as products of the mind of God. (Kanigel, 1991, pp. 36, 281–282).

Despite the examples given, in general these ideas about theology in mathematics are quite alien and marginal to modernity. *Nulla regula sine exceptione, sed exception non impedit regulam.*

On the contrary, if we identify mathematics with mathematical *community*, than perhaps we can find another (more common) room for theology in modern mathematics. It is the community of mathematicians that brings mathematical theories into being, that accepts or rejects them, preserves them unaltered or modifies them, uses or forgets them. If we dare to put *mathematicians* inside our "container" we can hardly leave their views about mathematics outside of it. Mathematicians see their own activities (and have expectations of mathematics) in a way that is unique to their cultural environment. This very environment may and does include theology, making the impact of the latter on mathematics worth considering. In a nutshell, the main way theology is present in modern mathematics is not *immediate* (i.e. in mathematical theories); it is rather *mediated* by the philosophy of mathematicians.

2. A shift from theology to mathematics

Since the time of Plato, theoretical mathematics has been closely associated with theology through the philosophy of mathematics. It was Plato who stated unequivocally that mathematical entities belonged to the realm of true being, which they shared with the pure forms and Olympian gods. The main tradition of Greek Philosophy, which ran from Plato and his Academy to Aristotle to the Stoics to the Neoplatonists, was theology-oriented, and correlatively held mathematics in great esteem. The opposition from Sophists to Epicureans and Skeptics, which minimized the role of theology, criticized theoretical mathematics as well.

The alliance between theology and mathematics survived through the Middle Ages (Albertson, 2014), and got even stronger during the Scientific Revolution of the 17th century. In the early modern period, a sort of natural or philosophical theology played a pivotal role in the philosophy of mathematics. It was the sort of theology that Blaise Pascal decisively rejected in his *Memorial*: "God of Abraham, God of Isaac, God of Jacob, not of the philosophers and of the learned (Dieu d'Abraham, Dieu d'Isaac, Dieu de Jacob, non des philosophes et des savants)" (Pascal, 1654). No matter how famous this remark turned out to be later on, Pascal's position was unable to shatter the God of philosophers and scientists. Its proponents followed a strong, long-standing tradition: that God is the greatest mathematician, who "has arranged all things by measure and number and weight" (*Wisdom 11:20*). In these words, a Platonic theme from *Timaeus* is easily heard. Plutarch commemorated it in the famous phrase: "God always geometrizes" ("ἀει γεωμετρεῖν τὸν θεόν", Plutarch, *Questiones convivales*, 8.2, 718b–720c). This Platonic theme was inherited by the Christian thinkers of the Middle Ages and, in their turn, by the creators of the Scientific Revolution.

The situation changed dramatically in the 19th century. Secularized science was separated from theology, and even from metaphysics, as the latter was too closely associated with theology. Let us compare the classification of sciences in the famous *Encyclopédie* (mid-18th century), the so-called tree of Diderot and d'Alembert ("Système figure", 1751; "Map of the System", 1751; Diderot, 1751/1995), with the no-less-famous classification of Auguste Comte (Comte, 1830/1934, pp. 32–63; Martineau, 1896, vol. I, pp. 19–35). In both cases, we have a hierarchy of knowledge, but in the first one, general metaphysics (ontology) and the science of God (theology) still take the highest ranks in the sphere of reason (as it already was in Aristotle), while in the second one, both theology and metaphysics are eliminated from the list of sciences (by the law of the three stages) and mathematics takes their place on the top of the hierarchy.

Mathematics was considered in Comte's *Cours* as the most perfect science, "the science beyond all others – the science of sciences". Moreover, mathematics has "a naturally indefinite extent and even a rigorous logical universality". All limitations of the applicability of mathematics are of mere practical and not theoretical nature: they are due to the complexity of phenomena and "the feebleness of our intellect". Mathematics consisted, for Comte, of two parts: *abstract* (analysis, including arithmetic and algebra) and *concrete* (geometry and mechanics). "Mathematical analysis is, then, the true rational basis of the entire system of our actual knowledge. It constitutes the first and the most perfect of all the fundamental sciences." (Comte, 1830/1851, pp. 17–41). It is worth noting that Comte's classification of sciences turned out to be the most famous one throughout the 19$^{\text{th}}$ century.

That change is symbolic: the holy place is never empty, so mathematics (though rather unconsciously) began to play the role of the departed theology and metaphysics; the ally turned into a rival. While the queen of sciences (*regina scientiarum*) of the Middle Ages was theology (Zakai, 2007; Zakai, 2010; Huttinga, 2014, pp. 143–144), the newly brought to light queen of 19$^{\text{th}}$-century science was mathematics, as C.F. Gauss famously put it. According to his biographer,

> To use Gauss's own words, mathematics was for him the queen of sciences, and arithmetic the queen of mathematics. It then often condescends to render a service to astronomy and other natural sciences, but in all relations mathematics is entitled to the first rank. (Sortorius von Waltershausen, 1856, S. 79; Ferreirós, 2007b, p. 248)

The same biographer also testified to the effect that the divine perspective on mathematics was quite clear to Gauss:

> [H]e placed arithmetic at the top, and, in connection with questions that we cannot ascertain scientifically, he loved to employ the words: ὁ Θεὸς ἀριθμετίζει [God arithmetizes], with which he acknowledged the logic that goes through the whole cosmos, also for those domains in which our mind is not allowed to penetrate. (Sortorius von Waltershausen, 1856, S. 97; Ferreirós, 2007b, p. 236)

Pure mathematics (under the names of arithmetic, analysis or some other) has taken the top position in the hierarchy of sciences instead of theology and metaphysics since the first half of the 19$^{\text{th}}$ century. By now, if it has a likely challenger, it is not a particular branch of knowledge but an alternative to the very principle of the linear hierarchy of knowledge such as a net-like dynamic structure.

3. A popular philosophy of mathematics

Mathematics stood now alone on the cultural stage without powerful support from theology (Shaposhnikov, 2014, pp. 187–189). Nevertheless, mathematics inherited certain cultural expectations aimed at Her Majesty from its long-lasting alliance with theology.

These expectations formed a sort of *popular* philosophy of mathematics. It was well explicated already in the end of the 18th century by Kant, whose influence on philosophers and scientists throughout the 19th and even 20th centuries can hardly be overestimated.[2] This philosophy reached maturity at the beginning of the 20th century (in an obvious correlation with the development of highly abstract mathematics) and was made explicit by Cantor, Poincaré, Hilbert and others.[3] Let me dare to sum it up in the five following items.

(1) *Mathematics is <u>certain and infallible</u>. That is, it is always true and never wrong; hence, it is absolutely reliable. Mathematical results are final. No revision is needed, only accumulation and expansion.*

Kant meant it when he said that mathematics "has [...] travelled the secure path of a science (den sicheren Weg einer Wissenschaft gegangen)" since the time of the ancient Greeks (*Kritik der reinen Vernunft*, B X; Kant, 1781/1787/1998, p. 107). David Hilbert called mathematics "[the] paragon of reliability and truth ([das] Muster von Sicherheit und Wahrheit)" (Hilbert, 1926, p. 170; van Heijenoort, 1967, p. 375).

(2) *Mathematics is <u>necessarily and universally</u> valid. There is only one science of mathematics.*

This was famously declared by Kant (*Kritik der reinen Vernunft*, B 3; Kant, 1781/1787/1998, p. 137).

(3) *Mathematical knowledge is <u>consistent and rigorous</u>, that is, free from contradictions and ambiguities.*

Henri Poincaré publicly claimed in his address *Du róle de l'intuition et de la logique en mathématiques* at the 1900 congress in Paris: "The question arises, is this evolution [of mathematical rigor] ended – have we at last attained to absolute rigor, or do we deceive ourselves as our fathers did? [...] [W]e may say that at last absolute rigor is attained" (Scott, 1900, p. 72).

(4) *Mathematics is free: Every theory that is free from contradiction is acceptable in mathematics, no matter whether it finds real-world applications or not. Moreover, there are no mathematical problems unsolvable in principle.*

Georg Cantor proposed to call pure mathematics "*free mathematics* (der *freien Mathematik*)" and declared that "the *essence* of *mathematics* lies precisely in its *freedom* (das *Wesen* der *Mathemtik* liegt gerade in ihrer *Freiheit*)".

> Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. (Cantor, 1883/1932, S. 181–183; Ewald, 1996, pp. 895–897).

It was most notably David Hilbert who *identified* existence in mathematics with lack of contradiction, though the idea dates back to the time of Leibniz (Hilbert, 1900/1902, p. 448).

Hilbert also said:

> This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no *ignorabimus*. (1900/1902, p. 445).

Hilbert's *optimism* about mathematics has its parallel in Kant's (Posy, 1998, pp. 317–318). "Nothing here can escape us, because what reason brings forth entirely out of itself cannot be hidden" (I. Kant, *Kritik der reinen Vernunft*, A XX; Kant, 1781/1787/1998, p. 104). In those words, Kant spoke of metaphysics that seemed for him to be on a par with logic (i.e. a sphere of knowledge all problems of which can be solved), while mathematics appeared to be close enough to physics. One of the leading motifs of the neo-Kantian philosophy was to separate mathematics from physics and to blend it with logic. To the time of Hilbert, pure mathematics, as an *a priori* activity, was apparently closer to Kant's idea of metaphysics than to that of physics.

(5) *Mathematics is applicable to the world without restriction. There is a sort of pre-established harmony between pure mathematics and the real world. Any correctly built mathematical concept must eventually find some real application and vice versa: for any real occasion an appropriate mathematical tool can be found.*

Georg Cantor distinguished between two kinds of reality: *intrasubjective* (or *immanent*) and *transsubjective* (or *transient*). The first kind of reality of a mathematical concept means a proper immanent fit into the whole network of mathematics within the intellect (der Intellekt), while the second presupposes a sort of transcendence through correspondence with something in the external world (die Außenwelt). Cantor was convinced that transient reality could be inferred from immanent reality for mathematics:

> I have no doubt that these two sorts of reality always occur together in the sense that a concept designated in the first respect as existent always also possesses in certain, even infinitely many, ways a transient reality. To be sure, the determination of this transient reality is often one of the most troublesome and difficult problems of metaphysics, and must frequently be left to the future, when the natural development of one of the other sciences will uncover the transient meaning of the concept in question. This linking of both realities has its true foundation in the *unity* of the *all to which we ourselves belong.* (1883/1932, S. 181–182; Ewald, 1996, pp. 895–896).

According to David Hilbert,

> We are confronted with the peculiar fact that matter seems to comply well and truly with the formalism of mathematics. There arises an unforeseen unison of being and thinking, which for the present we have to accept like a miracle (Hilbert, 1992, S. 69; Wilholt, 2006, p. 69).

Later, the idea was popularized by Eugene Wigner (1960). The term "pre-established harmony (harmonie préétablie)" was coined by Leibniz, but in the context of the interrelations between pure mathematics and physics became especially popular around the year 1900. It was used by D. Hilbert, H. Minkowski, A. Einstein, F. Klein, H. Weyl and others (Pyenson, 1982; Vizgin, 1998).

These five items of the popular philosophy of mathematics, made explicit and brought together, reveal a picture too strange for a secular mind. Mathematics is unique, universal and unchangeable in its final rigor. This means that mathematics is an inhabitant of some eternal sphere that is a part of the supernatural rather than natural world. Mathematics is free from the natural world and rules this very world at the same time. It is exactly the way God is related to the natural world in Christian theology. A clue to this puzzle can be easily found in the dependence of the traditional philosophy of mathematics on theology. Nevertheless, in secular philosophy, it should count as a sort of contraband.

To sum up the five items, I can say that popular philosophy views mathematical knowledge as *absolute*. From a naturalistic point of view, if some knowledge is absolute, it cannot be solely human, and *vice versa*.

4. Autonomous pure reason and its paradox

A deep conviction of the majority of mathematicians on the brink of the 20th century was that mathematics is (or at least must be) certain and infallible, necessary and universal, consistent and rigorous, as well as free but nonetheless applicable to the world without restriction. In my view, it was this popular philosophy of mathematics that was basically responsible for a heated argument provoked by set-theoretic paradoxes (Giaquinto, 2002; Ferreirós, 1999/2007a), which were interpreted as an indication of the foundational crisis of mathematics (Grundlagenkrise der Mathematik). The term was coined by Hermann Weyl (1921; Mancosu, 1998, pp. 86–118). According to José Ferreirós:

> The foundational crisis is usually understood as a relatively localized event in the 1920s, a heated debate between the partisans of 'classical' (meaning late-nineteenth-century) mathematics, led by Hilbert, and their critics, led by Brouwer, who advocated strong revision of the received doctrines. There is, however, a second, and in my opinion very important, sense in which the 'crisis' was a long and global process, indistinguishable from the rise of modern mathematics and the philosophical and methodological issues it created. (2008, p. 142).

I completely agree with this characteristic of the foundational crisis as "indistinguishable from the rise of modern mathematics", for the paradox of theological pretense (to obtain absolute knowledge) without theology was there from the very start.

The quest for *the foundations of mathematics (die Grundlagen der Mathematik, les fondements des mathématiques)* seems to be an activity specific to the 19th century. I mean here a *very long* 19thcentury (even longer than Eric Hobsbawm's one): from the 1780s through the 1920s, from Kant's *Kritik der reinen Vernunft* (1781) to Gödel's famous theorems (1931).

There exists a striking similarity between the problems with the foundations of metaphysics in Kant and the foundations of mathematics up to Gödel (Kovač, 2008, pp. 150–155). The foundational crisis of metaphysics, which turned this traditional intellectual activity into the building of castles in the air, was documented by Kant. The same thing happened a century later with abstract mathematics (as the successor to metaphysics); being detached from the support of both empirical data and divine revelation, it turned into an enterprise (at its own risk) of *autonomous* pure reason. The status of this pure reason remained highly obscure, from Kant's "reine Vernunft" to Frege's "drittes Reich der Gedanken".

In Richard Dedekind's widely known words, "numbers are free creations of the human mind (die Zahlen sind freie Schöpfungen des menschlichen Geistes)" (1888/1893, S. VII; 1901, p. 31). While Leopold Kronecker at least shared the work between God and man[4], Richard Dedekind liberated God from this job completely. Dedekind (a former pupil of Gauss) made his own version of Gauss's words: "ὁ Θεὸς ἀριθμετίζει (God arithmetizes)"; he replaced them with "ἀεὶ ὁ ἄνθρωπος ἀριθμετίζει (man always arithmetizes)" (1888/1893, title and p. X; Ferreirós, 1999/2007a, pp. 215–217; Ferreirós, 2007b, p. 264).

The strange pure reason that mathematicians and philosophers appealed to, and which made abstract mathematics possible, was, on the one hand, solely *human*, but, on the other, was believed to grant *divine* potential. As R. Dedekind famously put it in a letter to H. Weber (Jan. 24, 1888):

> Wir sind göttlichen Geschlechtes und besitzen ohne jeden Zweifel schöpferische Kraft nicht bloß in materiellen Dingen (Eisenbahnen, Telegraphen), sondern ganz besonders in geistigen Dingen. [...] Wir haben das Recht, uns eine solche Schöpfungskraft zuzusprechen [...] (Dedekind, 1932, S. 489)
>
> We are a divine race and undoubtedly possess creative power, not merely in material things (railways, telegraphs) but especially in things of the mind. [...] We have the right to ascribe such a creative power to ourselves [...]. (Ewald, 1996, p. 835).

Dedekind's expression "das göttliche Geschlecht" is better translated as "the offspring of God" or "God's offspring" than "a divine race", for it is a direct quote from the Luther Bible (*Apostelgeschichte 17:29*). This expression is from St. Paul's Areopagus sermon (*Acts 17: 22–31*):

> [17:28] Denn in ihm leben, weben und sind wir; wie auch etliche Poeten bei euch gesagt haben: "Wir sind seines Geschlechts." [17:29] So wir denn göttlichen Geschlechts sind, sollen wir nicht meinen, die Gottheit sei gleich den goldenen, silbernen und steinernen Bildern, durch menschliche Kunst und Gedanken gemacht.
>
> [17:28] For in him we live and move about and exist, as even some of your own poets have said, "For we too are his offspring." [Aratus, *Phenomena*, 1–5; Cleanthes, *Hymn to Zeus*] [17:29] So since we are God's offspring, we should not think the deity is like gold or silver or stone, an image made by human skill and imagination.

I use here the NET Bible English translation, but already the King James Version had used "the offspring of God" to translate "γένος οὖν ὑπάρχοντες τοῦ θεοῦ".

The *absolute* character of the human mathematical reason inevitably means the *autonomy* of this reason. '*Absolutus*' in Latin means just this: 'freed', 'unrestricted', 'autonomous'. To be autonomous for mathematics means to be governed by intrinsic laws of pure reason, to be independent of both natural and supernatural worlds, and both the laws of physics and theology. The main paradox of autonomous pure reason (and the pure mathematics it produces) is due to its obscure ontological status. If it is merely human, how can it give absolute results? I take here "human" to mean related to the *Homo sapiens* species as something subjected to historical change.

5. A brief history of autonomous mathematics

Let us skim over the history. From the early Modern Era on and up to the last decades of the 18th century, an intimate coordination of the absolute character of mathematics with the Absolute was the philosophers' credo. Kepler, Galileo, Descartes and many other personalities of the Scientific Revolution shared a belief that can be visualized as the *Theo-Cosmo-Anthropological Triangle* (TCA-triangle) (Shaposhnikov, 2014, pp. 187–189).

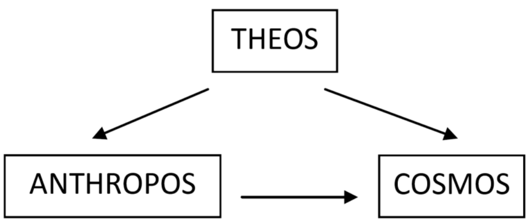

Figure 1. TCA-triangle

This world and human beings in it have been created by one and the same God. This God cannot be a deceiver (*Meditationes de prima philosophia*, 1641, Descartes, 1985, vol. II, p. 35), which is why the world (created according to divine mathematics) fits perfectly to the mathematics of human reason.

The points I would like to stress in Kepler's (*Harmonices Mundi*, 1619/1997, p. 304) and Descartes' (a letter to M. Mersenne, April 15, 1630, Descartes, 1991, p. 23) views are as follows:

1) Mathematics is eternal (both Kepler and Descartes).
2) Mathematics "depends on God entirely" (Descartes) or even "is God himself" (Kepler).
3) Mathematics gives "patterns for the creation of the world" (Kepler); "God has laid mathematical laws in nature" (Descartes).
4) Mathematics is "inborn in our minds" (Descartes); it "passed over to Man along with the image of God" (Kepler).

According to Galileo (Galilei, 1632/1967, p. 103), human mathematics differs from the divine one only in scope (extensively) but not in quality (intensively). Hence, even human mathematics is objective, absolutely certain and necessary, though perhaps limited in scope. Galileo also stresses the *unique* position of mathematics in human knowledge; no other part of our science or philosophy can boast such high status.

Mathematics is *universal*, for the whole universe has only one almighty creator (strict monotheism). Mathematics is *necessary*, for God never changes. No less than the truth and reliability of God is what makes mathematics *certain*. And, to be certain for mathematics means also to be *consistent*, *rigorous* and *infallible*.

For example, as far as Gottfried Leibniz's view is concerned (cf. Breger, 2005), mathematical knowledge is innate and necessary, for it is based on the principle of contradiction. So, mathematics is a block of absolutely necessary truths that have a place not only in our world, but in any possible world. The eternal truths of logic and mathematics *depend on God*, for "God's understanding is the realm of eternal truths or that of the ideas on which they depend; without him there would be nothing real in possibles, and not only would nothing exist, but also nothing would be possible". Nevertheless, this does not mean, according to Leibniz, that "they are arbitrary and depend on his will" (cf. Descartes, 1991, p. 23); "necessary truths depend solely on his understanding, and are its internal object" (*Monadology*, §§ 43 and 46, 1714, Leibniz, 1989, pp. 218–219). God is unable to create something that violates the principle of contradiction. Nevertheless, this is no external restriction on God's absolute freedom, for to avoid contradiction is a part of the very divine reason that is his own nature (*On Freedom and Possibility*, 1680–82? Leibniz, 1989, pp. 19–23).

In the early Modern Period, the absolutist account of mathematics was still grounded in theology, as we have seen in Descartes and Leibniz. Attempts to separate mathematics from God and to treat it as a result of solely human endeavors were made mainly by the heirs of Skeptical and Epicurean traditions, who developed a philosophy of mathematics along nominalistic and fictionalist lines (Sepkoski, 2007). A well-known example is Pierre Gassendi's objections to *Meditationes* (1641), where he mocked Descartes' idea that our belief in certainty and the truth of geometrical proofs has something to do with our knowledge of the true God:

> Of course it is quite true – as true as anything can be – that God exists, is the author of all things, and is not a deceiver; but these truths seem less evident than the geometrical proofs, as is shown by the fact that many people dispute the existence of God, the creation of the world, and so on, whereas no one impugns the demonstrations of geometry. (Descartes, 1985, vol. II, pp. 227–228)

Since Kant, pure reason (human reason?) has had some sort of obscure autonomy; it is an autonomy of "the head" that has been chopped off from "its body".[5] For Bernard Bolzano, despite his anti-Kantianism, this work of separation has been completed and "the chopped head" obtained its phantom life. It is he who clearly stated that, though the Third Realm (the realm of propositions in themselves, Sätze an sich) has a divine owner, *it does not actually need one* (Gottlob Frege preserved the second part of this claim, but totally omitted the first one). Bolzano's *Wissenschaftslehre*, §25, gives us an important piece of evidence:

> From God's omniscience it follows, to be sure, that every truth, even if it should be known to no other being, will not only be thought, but to Him, the all-knowing, it will be known, and will be permanently represented in his intellect (Verstand). Therefore there is really not a single truth that is not known to anyone at all. But this does not prevent us from speaking of truths in themselves, in the concept of which it is not at all presupposed that they have to be thought by anyone. [...]
>
> The word *proposition* [Satz] used, by way of its derivation from the verb, to propound [setzen], readily calls to mind an action, a something that has been propounded [gesetzt] by someone (and so produced or altered in some way). In fact there must be no such thought in the case of truths in themselves. For they are not propounded by anyone, not even by the divine intellect [Denn diese werden von Niemand, selbst von dem göttlichen Verstande nicht gesetzt]. A thing is not true because God knows it, but on the contrary God knows it because it is so [Es ist nicht etwas wahr, weil es Gott so erkennet; sondern im Gegentheile Gott erkennet es so, weil es so ist]. (Bolzano, 1837/1973, pp. 57–58)

In Frege, we can observe aftereffects. Frege argued *as if* God was still there to support his claims, only *incognito*. He was a staunch supporter of an absolutist account of logic and mathematics. Let me give two famous quotes as illustrations.

> "2 times 2 is 4" is true and will continue to be so even if, as a result of Darwinian evolution, human beings were to come to assert that 2 times 2 is 5. Every truth is eternal and independent of being thought by anyone and of the psychological make-up of anyone thinking it. (*17 Kernsätze zur Logik*, 1906 or earlier, Frege, 1979, p. 174)

> [T]houghts are neither things in the external world nor ideas [in the human consciousness]. – A third realm must be recognized. Anything belonging to this realm has it in common with ideas that it cannot be perceived by the senses, but has it in common with things that it does not need an owner so as to belong to the contents of his consciousness. Thus for example the thought we have expressed in the Pythagorean theorem is timelessly true, true independently of whether anyone takes it to be true. It needs no owner. It is not true only from the time when it is discovered; just as a planet, even before anyone saw it, was in interaction with other planets. (*Der Gedanke*, 1918–19, Frege, 1984, p. 363)

Frege believed in the timeless and eternal realm of absolute truths in addition to those of the physical world and the mental worlds of humans. True statements of mathematics and logic belong to it. His lifework was to discover the interconnections between some parts of that "ein drittes Reich der reinen Gedanken", but Frege, so far as I know, made no public attempt to connect his Third Realm to the divine reality, in contrast with a tradition that dates back to Plato and Aristotle.

Were there any religious underpinnings of Frege's belief? Unfortunately our knowledge about his inner life is next to nothing. We have his brief evidence that he "was brought up in the Lutheran faith" (*Curriculum vitae*, 1874, Frege, 1984, p. 92), and his notorious 1924 pre-death diary that exhibits not only right-wing political views but deep religious interests as well. In his diary, for instance, Frege was preoccupied with the idea "to separate clearly the religious from the civil and the legal" for both their sakes (Frege, 1996, p. 313). Here is the concluding excerpt (Frege died on July 26, 1925 at the age of 76):

> 8 May 1924
> We urgently need a revival of religion. The Lutheran church is to some extent hardened in orthodoxy. [...] But indeed, even where orthodoxy does not prevail, the effectiveness of the parson is impeded by dogma. [...] The old saying about new wine in old bottles still holds good. [...] It is hardly to be expected that the civil- or church-appointed clerics will change themselves. We must have prophets proclaim something new to come that really is something old, namely just the old religion of Jesus himself.

> 9 May 1924
> The life of Jesus must be told according to the results of the German scholarly research. To be sure, because of the nature of this project, one cannot rule out errors completely, but the intention of the narrator must be directed to the purest truth. As far as possible, he must not bring forward anything that does not appear certain to him. If he thinks none the less that he must relate something whose truth is not wholly certain for him, he must indicate the doubts, for example, by interpolating a 'perhaps'. [...] The way Gustav Frenssen describes the life of the savior does not really suit my purposes, because therein fiction and truth are mixed together. I want truth and nothing but the truth, at least in the intention of the narrator. A life of Jesus, as I have it in my mind, should, I think, give rise to the founding of a religion without that being obvious as the intention. ... (1996, pp. 340–342)

Was there any connection between "the purest truth" of "the old religion of Jesus himself" and the timeless truth of logic and mathematics for the Lutheran Frege? It seems that the realm of truth divided against itself in Frege.

6. Conclusion

Gottlob Frege is often thought to be the founder of modern logic, the modern philosophy of mathematics, and even analytic philosophy in general. This makes him one of the most representative figures for my study. Frege's absolutist account of logic and mathematics marks a mature form of *autonomous mathematics*, which is one of the three main perspectives on mathematics. What are they and how is autonomous mathematics connected with the other two?

In European thought, Aristotle's ideas predominated for centuries; it makes them an appropriate starting point for classification of the perspectives in question. Let us recall Aristotle's well-known classification of theoretical disciplines (*Met.* 1026a):

1) First philosophy = theology;
2) Mathematics;
3) Second philosophy = physics.

Mathematics (in a sense) takes the *intermediate* position in this classification. It shares the immutability of its objects with theology. Nevertheless, mathematical objects exist as embodied in matter, i.e. unlike theological entities but like physical ones.

According to this scheme – that places mathematics between theology (on one side) and physics in the broad Aristotelian sense (on the other) – one can classify accounts of mathematics. We can subordinate mathematics to the supernatural world or to the natural world, or to none of them, trying to keep mathematics autonomous. It gives us three basic accounts: *theology-based mathematics*, *physics-based mathematics*, and *self-based or autonomous mathematics*.

The straightforward opposition to the theology-based approach to mathematics is not physicalism or empiricism but *naturalism* about mathematics. *Physicalism* is a rather modern term (the 1930s); it gives a privileged status to physics in the 20th century sense of the word, as opposed to other sciences. On the contrary, according to Aristotle, *physics* means the study of the natural world in general. Empiricism, on the other hand, stresses the sensual experience and empirical evidence as opposed to reason, while the latter can also be treated as a natural capacity of human beings.

According to Michael Eldridge, "God has ceased to be an explanatory principle for philosophers. [...] This secular outcome is what philosophic naturalism attempts to understand and advocate" (2004, p. 52). Provided God's death (Heidegger, 1950/2002), naturalism tries to avoid any explicit or even implicit use of *supernatural* in our understanding of mathematics (or any other subject matter). In my opinion, this is the most adequate definition of the notoriously polysemic term *naturalism*.

Naturalism substitutes God, as the ultimate means of explanation, with Nature. The naturalistic idea of Nature can be explicated as the *Fundamental Naturalistic Pyramid*, which has three main levels – biological, social, and cultural. Every next level presupposes the previous levels but is in no way reducible to them. My interpretation of naturalism here owes a lot to Jean-Marie Schaeffer (2007); I refer to his book for a detailed account.

Naturalistic beliefs about mathematics constitute the straightforward opposition to the popular (absolutist) philosophy of mathematics, for naturalism questions every issue of the five listed in Section 3. According to mathematical absolutism, mathematics (*de facto* if not *de jure*) is *divine and eternal*, while, according to naturalism, it is *human and historically changing*. Here are naturalistic antitheses.

(1) *Mathematics is dubiously certain and fallible.*

Our mathematics is far from being perfect not only extensively (as Galileo believed), but intensively as well (cf. Kline, 1980). The term *fallibilism* was proposed by Charles Peirce and widely used by Karl Popper. Fallibilist methodology was applied to fight absolutism in the philosophy of mathematics by Imre Lakatos (1976). The opposition of absolutist and fallibilist philosophies of mathematics was popularized by Paul Ernest (1991; 1998).

(2) *Mathematics is necessarily and universally valid only within cultural and biological limitations.*

On biological limitations see comments on point 4 below. Perhaps the easiest way to question the universality of mathematics is to point out an alternative. That is why the *problem of alternative mathematics* is crucial for naturalism in the philosophy of mathematics (Bloor, 1976/1991, pp. 107–130, 179–183).

(3) *Mathematics is never absolutely rigorous, normally ambiguous, and sometimes even inconsistent.*

On the ambiguity of mathematics, see for instance (Lakatos, 1976; Byers, 2007). Recently, the theme of inconsistent mathematics has been a popular one (Colyvan, 2008; 2009). There are articles on the subject in the leading Internet philosophy encyclopedias (Mortensen, 1996/2013; Weber, 2009). There was a special issue of "Synthese" on the inconsistency of science (Bueno & Vickers, 2014).

(4) *Mathematics is never free from cultural stresses and biological framework.*

On cultural pressure in the development of mathematics see (White, 1947; Wilder, 1981). On biological framework (mathematics is not a creation of pure reason or a free spirit but a product of the embodied mind) see (Lorenz, 1941; Lorenz, 2009; Lakoff & Núñez, 2000).

(5) *Mathematics is applicable to the world with a lot of restrictions and preconditions.*

What are the scopes and limits of mathematical knowledge and the mathematization of other disciplines? For any given time and place there are always some confines, though they are not absolute but historically changing. When we think that true mathematics is pure mathematics and then ask about applications, the whole situation, from a naturalistic point of view, is misrepresented. This very misrepresentation is responsible for delusions of "pre-established harmony" and the "unreasonable effectiveness of mathematics". "It is the tail that has come to wag the dog", as Willard Quine wrote on the matter (1981, p. 151). Here, applied mathematics is "the dog" while pure mathematics is no more than its "tail". Restrictions and preconditions in the applicability of mathematics are not universal limitations inevitable for mathematics but rather an aspect of a particular situation in the evolution of the whole organism of mathematics within its natural environment.

The self-based mathematics seems to be an *intermediate* position between theology-based and physics-based (naturalistic) mathematics. The 19th century witnessed the birth of pure mathematics in a radically new sense. This new mathematics was considered to be a child of the human spirit alone and a gap separated it from the natural world and another gap – from the supernatural world of the divine wisdom. The world of pure mathematics was found to be rather autonomous. The divine foundation of mathematics, provided by theology, had been rejected while any naturalistic foundation was still too immature and naïve to be convincing (John Stuart Mill's radical empiricism about mathematics can serve as an example); hence the foundational problem turned out to be a crucial one.

What is self-based (autonomous) mathematics after all? In my opinion, the very long 19th century's attempts to advocate autonomous mathematics seem to have been an ingenious device for avoiding a clear choice between naturalism and supernaturalism. The Third Realm preserved characteristics of the Divine Mind; but Frege sought to ground it without any appeal to God. Bolzano even subordinated God along with humans to the Third Realm, thus liberating God from His absoluteness. What arguments can be given in support of such an ontological monster?

Mill was right, when, criticizing "*a priori* fallacies", he wrote that human beings had "no right to mistake the limitation [...] of their own faculties for an inherent limitation of the possible modes of existence in the universe" (1843/1974, p. 753). If something looks necessary and universal to us, for we cannot even imagine a situation violating the rule in question, it means nothing more than our particular inability then and there. So, from a naturalistic point of view, our 'universality' can hardly be even species-wide. As I see it, self-based mathematics is an illusion, though a very beautiful and fruitful illusion.

It is hardly surprising that some key mathematicians of the period made an attempt (perhaps partially unconscious) *to substitute pure mathematics for departed theology.* Part II of this paper gives additional materials illustrating this thesis through three case studies: of Bertrand Russell, David

Hilbert, and L.E.J. Brouwer's views on mathematics. If theology (or quasi-theology at least) can be found in the philosophy of leading mathematicians, then the answer to the question 'is there theology in mathematics?' has to be positive.

NOTES

[1] An apparent conflict between mathematics and Christian belief can be easily noted in this passage from Molière (18??, p. 106). Moreover, mathematics in this joke pretends to take the place of religion. There is many a true word spoken in jest. As the motto hints, the case in question is mathematics as a rival to religion (and theology), and this rivalry is taken in this paper quite seriously.

[2] "A hundred years ago there was a saying in German academic circles: 'You can philosophize with Kant, or against Kant; you cannot philosophize without him'" (Beck, 1991, p. ix).

[3] On a Kantian background for the Brouwer-Hilbert controversy, the acme of the foundational crisis of mathematics, see: (Posy, 1998, pp. 291–325; Detlefsen, 1998).

[4] It is surely a hint at the famous dictum ascribed to Kronecker and dated to 1886 by Heinrich Weber: "Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk (good God made the integers, and all the rest is the work of man)" (Weber, 1893, p. 15; Ferreirós, 1999/2007a, p. 217; Ewald, 1996, p. 942). Kronecker was an opponent to the whole enterprise of autonomous pure reason, not only Cantor's transfinite mathematics. For an attempt to reconstruct his unique position, see (Boniface, 2005).

[5] I must apologize for these allusions to the guillotine and the French Revolution.

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