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ALGEBRAIZATION OF JAŚKOWSKI'S PARACONSISTENT LOGIC D2

Abstract. The aim of this paper is to present an algebraic approach to Jaśkowski's paraconsistent logic $D2$. We present: a $D2$-discursive algebra, Lindenbaum-Tarski algebra for $D2$ and $D2$-matrices. The analysis is mainly based on the results obtained by Jerzy Kotas in the 70s.

Keywords: discursive (discussive) logic, discursive algebra, Jaśkowski's logic, $D2$, paraconsistent logic.

1. Introduction

Discursive (or discussive) logic, $D2$, introduced by Jaśkowski is seen as one of the earliest examples of so-called paraconsistent logic (cf. Priest, Tanaka, & Weber 1996). There are several definitions of paraconsistent logic. One of them is that paraconsistent logic is a logic which is not closed under the rule *ex contradictione sequitur quodlibet*, i.e. $\alpha, \sim \alpha \;/\; \beta$. At first sight, this definition seems too general because it includes some logics which have little in common with paraconsistency, but it shows a tendency to regard paraconsistent logic as a logic of the negation connective.[1]

Discursive logic shows the importance of a philosophical explanation for a formal approach to natural language. Usually in a discussion, people exchange ideas. They may use some vague predicates either purposefully or unintentionally. In thinking about sentences like 'He is rich', 'He is bald' or 'He is famous', one might be confused as to how to assign truth or falsity. This can lead to a seeming contradiction. Jaśkowski suggested that each sentence asserted by participants in a discourse should be interpreted as 'for a certain admissible meaning of the terms used'. If a sentence α 'is recorded in a discussive system, its intuitive sense ought to be interpreted so as if it were preceded by the symbol $\Diamond$, that is, the sense: *it is possible that* α' (Jaśkowski, 1999a, p. 43), where $\Diamond$ is the connective of $S5$-possibility.[2] So as it is seen, Jaśkowski's logic is intended in part to deal with problems of vagueness.

On the other hand, 'it is known that the evolution of the empirical disciplines is marked by periods in which the theorists are unable to explain the results of experiments by a homogenous and consistent theory, but use different hypotheses, which are not always consistent with one another, to explain the various groups of phenomena. This applies, for instance, to physics in its present-day stage. Some hypotheses are even termed working hypotheses when they result in certain correct predictions, but have no chance to be accepted for good, since they fail in some other cases' (Jaśkowski, 1999a, p. 37).

According to Jaśkowski, all these considerations raise an issue which should be formulated in terms of formal logic. It was his intention to 'find a system of the sentential calculus which: (1) when applied to the inconsistent systems would not always entail their overfilling, (2) would be rich enough to enable practical inference, (3) would have an intuitive justification.' (*Ibid.*, p. 38)

Let *var* denote a non-empty denumerable set of all propositional variables $\{p_1, p_2, \ldots\}$ The set of (well-formed) formulas of the discursive logic is inductively defined in the following way:

(i) if $\alpha \in var$, then α is a formula of $D2$

(ii) if α is a formula of $D2$, then $\sim \alpha$ is a formula of $D2$

(iii) if α and β are formulas of $D2$, then $\alpha \bullet \beta$ are formulas of $D2$,

where $\bullet \in \{\vee, \wedge_d, \rightarrow_d \leftrightarrow_d\}$ and the symbols $\sim$, $\vee$, $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$ denote negation, disjunction, discursive conjunction, discursive implication and discursive equivalence, respectively. The language of the discursive logic will be denoted by L_d.

Jaśkowski did not propose any axiomatization nor a direct semantics for $D2$, but defined his system through an interpretation of the language of $S5$ of Lewis; to be more precise:

$$D2 = \{\alpha \in L_d : \ulcorner \Diamond tr(\alpha) \urcorner \in S5\},$$

where $\Diamond$ is the connective of $S5$-possibility and tr denotes a translation function of the language of $D2$ into the language of $S5$ of Lewis, $tr : L_d \longrightarrow L_{S5}$, such that:

(i) $tr(p_i) = p_i$ if $p_i \in var$ and $i \in N$

(ii) $tr(\sim \alpha) = \ulcorner \sim tr(\alpha) \urcorner$

(iii) $tr(\alpha \vee \beta) = \ulcorner tr(\alpha) \vee tr(\beta) \urcorner$

(iv) $tr(\alpha \wedge_d \beta) = \ulcorner tr(\alpha) \wedge \Diamond tr(\beta) \urcorner$

(v) $tr(\alpha \rightarrow_d \beta) = \ulcorner \Diamond tr(\alpha) \rightarrow tr(\beta) \urcorner$

(vi) $tr(\alpha \leftrightarrow_d \beta) = \ulcorner (\Diamond tr(\alpha) \rightarrow tr(\beta)) \wedge \Diamond(\Diamond tr(\beta) \rightarrow tr(\alpha)) \urcorner$.

By placing the symbol $\Diamond$ in front of every translated formula, we protect $D2$ from collapsing into deductive insufficiency. 'Can a discussive system be based on ordinary two-valued logic?' – Jaśkowski asked – 'It can easily be seen that it is not so. Even such an elementary form of reasoning as the rule of modus ponens fails. If the implication is interpreted so as it is done in two-valued logic, then out of the two theses one of which is $\alpha \rightarrow \beta$, and thus states: *it is possible that if* α*, then* β, and the other is α, and thus states: *it is possible that* α, it does not follow that *it is possible that* β, so that the thesis β, does not follow intuitively, as the rule of modus ponens requires.' (*Ibid.*, p. 43.)

One might be curious about why Jaśkowski did not define the discursive conjunction as (iv)‡ $tr(\alpha \wedge_d^\ddagger \beta) = \ulcorner \Diamond tr(\alpha) \wedge \Diamond tr(\beta) \urcorner$ nor the discursive implication as follows: (v)‡ $tr(\alpha \rightarrow_d^\ddagger \beta) = \ulcorner \Diamond tr(\alpha) \rightarrow \Diamond tr(\beta) \urcorner$. It is because their application would lead to some unwanted consequences, that is, the acceptance of the formula $(\alpha \wedge_d^\ddagger \beta) \rightarrow_d (\sim (\alpha \wedge_d^\ddagger \beta) \rightarrow_d \gamma))$ or $(\alpha \rightarrow_d^\ddagger \beta) \rightarrow_d^\ddagger (\sim (\alpha \rightarrow_d^\ddagger \beta) \rightarrow_d^\ddagger \gamma))$ as a thesis of $D2$ (cf. Ciuciura, 2008, pp. 145–146).

So, metaphorically speaking, we do need diamonds and the two-step translation method.

Here is an example by way of illustration. Consider the formula $p_1 \rightarrow_d ((p_2 \wedge_d (p_2 \rightarrow_d p_1)) \vee p_1)$. Apply the definition of $D2$, to obtain: $\Diamond tr(p_1 \rightarrow_d ((p_2 \wedge_d (p_2 \rightarrow_d p_1)) \vee p_1))$. By (v) and (iv) of the recurrent definition of the function tr, we receive: $\Diamond(\Diamond p_1 \rightarrow ((p_2 \wedge \Diamond(\Diamond p_2 \rightarrow p_1)) \vee p_1))$.

Proposition 1.1 (Nasieniewski & Pietruszczak, 2012, p. 220)

For any formulas $\alpha_1, \alpha_2, \ldots \alpha_k \in L_d$ and $k \in N \cup \{0\}$:

$$\ulcorner \alpha_1 \rightarrow_d (\ldots \rightarrow_d (\alpha_k \rightarrow_d \beta)\ldots) \urcorner \in D2 \textit{ iff } \ulcorner (\alpha_1 \wedge_d \ldots \wedge_d \alpha_k) \rightarrow_d \beta \urcorner \in D2.$$

Proof. By induction on k, definition of tr, the rule (MP) for $\rightarrow$ and the fact that the formulas (1) $\ulcorner \Diamond(\Diamond\alpha \rightarrow \beta) \leftrightarrow (\Diamond\alpha \rightarrow \Diamond\beta) \urcorner$, (2) $\ulcorner (\Diamond\alpha_1 \rightarrow (\ldots \rightarrow (\Diamond\alpha_k \rightarrow \Diamond\beta)\ldots)) \leftrightarrow ((\Diamond\alpha_1 \wedge \ldots \wedge \Diamond\alpha_k) \rightarrow \Diamond\beta) \urcorner$, where $k \in N \cup \{0\}$, and (3)$\ulcorner \Diamond(\alpha \wedge \Diamond\beta) \leftrightarrow (\Diamond\alpha \wedge \Diamond\beta) \urcorner \in S5$. □

Jaśkowski's paraconsistent logic has been treated so far as a set of (discursive) formulas. Now we define a consequence relation ($D2$-consequence) on the set of all discursive formulas in the following way:

$$\Gamma \vdash_{D2} \alpha \quad \textit{iff} \quad \{\Diamond tr(\beta) : \beta \in \Gamma\} \vdash_{S5} \Diamond tr(\alpha),$$

for any $\Gamma \subseteq L_d$ and every $\alpha \in L_d$; where $\vdash_{S5}$ is an $S5$-consequence.

We say that the formula α is a thesis of $D2$, $\vdash_{D2} \alpha$ in symbols, if $\emptyset \vdash_{D2} \alpha$.

It is noteworthy to mention at this point that there are several axiomatizations of $D2$ with discursive connectives taken as primitive symbols. Unfortunately, some of them are not complete with respect to the semantics discussed here. For instance, one can easily check that $\ulcorner \alpha \rightarrow_d \sim\sim \alpha \urcorner \in D2$, but the formula is not provable in the axiomatic system that is presented in (Ciuciura, 2008).[3]

Observe that for any $\alpha \in L_d$, we have (cf. Nasieniewski & Pietruszczak, 2012, p. 222):

$$\emptyset \vdash_{D2} \alpha \;\; \textit{iff} \;\; \alpha \in D2 \;\; \textit{iff} \;\; \text{for every } \Gamma\text{, such that } \Gamma \subseteq L_d,\, \Gamma \vdash_{D2} \alpha.$$

A question arises: Is it possible to redefine $D2$ in such a way to eliminate the translation method and give a direct semantics for $D2$? We answer the question in the affirmative.[4]

Definition 1.1

A $D2$-frame is a pair $\langle W, R\rangle$, where W is a non-empty set of points and R is an equivalence relation on W. A $D2$-model is a triple $\langle W, R, v\rangle$, where v is a mapping from propositional variables to sets of points, $v : var \longrightarrow 2^W$. The satisfaction relation $\models_m$ is inductively defined as follows:

(var) $x \models_m p_i$ *iff* $x \in v(p_i)$ and $i \in N$

($\sim$) $x \models_m \sim \alpha$ *iff* $x \not\models_m \alpha$

($\vee$) $x \models_m \alpha \vee \beta$ *iff* $x \models_m \alpha$ or $x \models_m \beta$

($\wedge_d$) $x \models_m \alpha \wedge_d \beta$ *iff* $x \models_m \alpha$ and $\exists y \in W(xRy$ and $y \models_m \beta)$

($\rightarrow_d$) $x \models_m \alpha \rightarrow_d \beta$ *iff* if $\exists y \in W(xRy$ and $y \models_m \alpha)$ then $x \models_m \beta$.

For any formula $\alpha \in L_d$: α is valid in $D2$, $\models \alpha$ in symbols, *iff* for any $D2$-model $\langle W, R, v\rangle$, $\forall x \in W$, $\exists y \in W$ $(xRy$ and $y \models_m \alpha)$.

Proposition 1.2

For every formula $\alpha \in L_d$:

$$\alpha \text{ is valid in } D2 \textit{ iff } \alpha \in D2 \;\; (\textit{iff } \ulcorner \Diamond tr(\alpha) \urcorner \in S5).$$

Proof. The proof is analogous to the proof given in (Ciuciura, 2005, pp. 239-240). □

It follows from Proposition 1.2 that the translation procedure becomes rendered redundant. Moreover, $D2$ can be alternatively characterized by models where the accessibility relation is *universal*, that is, the accessibility relation includes every pair of points (or every point is accessible from any other). This results in some simplifications:

Definition 1.2

A $D2$-model is a pair $\langle W, v\rangle$, where W is a non-empty set of points and v is a function that each pair consisting of a formula and a point assigns an element of $\{1, 0\}$, $v : L_d \times W \longrightarrow \{1, 0\}$, defined as follows:

$(\sim)$ $v(\sim \alpha, x) = 1$ *iff* $v(\alpha, x) = 0$

$(\vee)$ $v(\alpha \vee \beta, x) = 1$ *iff* $v(\alpha, x) = 1$ or $v(\beta, x) = 1$

$(\wedge_d)$ $v(\alpha \wedge_d \beta, x) = 1$ *iff* $v(\alpha, x) = 1$ and $\exists y \in W(v(\beta, y) = 1)$

$(\rightarrow_d)$ $v(\alpha \rightarrow_d \beta, x) = 1$ *iff* $\forall y \in W(v(\alpha, y) = 0)$ or $v(\beta, x) = 1$.

For any formula $\alpha \in L_d$: α is valid in $D2$, $\models \alpha$ in symbols, *iff* for any $D2$-model $\langle W, v\rangle$, there exists $y \in W$ such that $v(\alpha, y) = 1$.

In 1968, da Costa and Dubikajtis presented the first algebraic semantics for Jaśkowski's discursive logic. The algebraic approach was further developed by Kotas. The aim of the next sections is to present an algebraic approach to Jaśkowski's discursive logic.

2. $D2$-discursive algebra

In this section, we present an infinite matrix for $D2$. Before going into details, let us introduce an algebraic semantics for the system $S5$.

Definition 2.1

A Henle's algebra $\mathbf{H} = \langle A, -, \vee, I\rangle$ is a Boolean algebra $\mathbf{A} = \langle A, -, \vee, \rangle$ with an additional unary operation I (interior) over A such that

$$Ia = \begin{cases} 1, & \text{if } a = 1 \\ 0, & \text{if } a \neq 1 \end{cases}$$

In addition to the operation I, we can define the closure operation C over A as follows:

$$Ca = \begin{cases} 1, & \text{if } a \neq 0 \\ 0, & \text{if } a = 0. \end{cases}$$

Informally speaking, the operation C (resp. I) is an algebraic counterpart of the $S5$-possibility (resp. $S5$-necessity) connective.

Definition 2.2

An algebra $\mathbf{A} = \langle A, -, \vee, \wedge_d, \rightarrow_d, 0, 1\rangle$ is called $D2$-discursive *iff* the following conditions hold:
(1) the structure $A = \langle A, -, \vee, 0, 1\rangle$ is a Boolean algebra
for every elements $a, b \in A$:

(2) $a \wedge_d b = \begin{cases} 0, & \text{if } b = 0 \\ a, & \text{if } b \neq 0 \end{cases}$

(3) $a \rightarrow_d b = \begin{cases} 1, & \text{if } a = 0 \\ b, & \text{if } a \neq 0. \end{cases}$

Let K_{D2} denote a class of all discursive matrices of the form $K_{D2} =^{df} \{\langle \mathbf{A}, A - \{0\}\rangle : \mathbf{A}$ is a $D2$-discursive algebra$\}$. Let $h : L_d \rightarrow \mathbf{A}$ be a homomorphism of the language L_d into $D2$-discursive algebra and $Hom(L_d, \mathbf{A})$ be a class of all homomorphism of L_d into $\mathbf{A}$.

Theorem 2.1 (cf. da Costa & Dubikajtis, 1968)

For every formula $\alpha \in L_d$ the following holds:

α is valid in $D2$ *iff* for every matrix $m \in K_{D2}$ and any $h \in Hom(L_d, \mathbf{A})$, $h\alpha \neq 0$.

Proof. By definition of $D2$, translation function tr and Henle's algebra.[5] □

3. Congruence difficulties, classical implicative lattice and Jaśkowski's metatheorems

Definition 3.1

A given binary relation $\approx$ on a set X is said to be an *equivalence relation* on the set *iff* it is reflexive, symmetric and transitive on X, i.e. for all a, b and $c \in X$:

(i) $a \approx a$

(ii) if $a \approx b$ then $b \approx a$

(iii) if $a \approx b$ and $b \approx c$ then $a \approx c$.

The *equivalence class* of a under $\approx$, $[a]$ for short, is defined in the usual way, *viz.* $[a] = \{b \in X : a \approx b\}$.

Proposition 3.1 (Kotas, 1975a, p. 156)

(1) A binary relation $\approx$ defined as

$\alpha \approx \beta$ *iff* $\ulcorner \alpha \to_d \beta \urcorner, \ulcorner \beta \to_d \alpha \urcorner \in D2$, for any formulas $\alpha, \beta \in L_d$,

is an equivalence relation on the set of all formulas of $D2$.

For any formulas $\alpha, \beta, \gamma, \delta \in L_d$:

(2) if $\alpha \approx \beta$ and $\gamma \approx \delta$ then

(i) $\alpha \vee \gamma \approx \beta \vee \delta$

(ii) $\alpha \wedge_d \gamma \approx \beta \wedge_d \delta$

(iii) $\alpha \to_d \gamma \approx \beta \to_d \delta$

(iv) $\alpha \leftrightarrow_d \gamma \approx \beta \leftrightarrow_d \delta$

(3) if $\alpha \approx \beta$ then it is not true that $\sim \alpha \approx \sim \beta$.

Proof. (1) By Definition 3.1, $\ulcorner(\alpha \to_d \alpha)\urcorner$, $\ulcorner(\alpha \to_d \beta) \to_d ((\beta \to_d \gamma) \to_d (\alpha \to_d \gamma))\urcorner \in D2$ and the rule of detachment (MP) $\alpha, \alpha \to_d \beta \;/\; \beta$.
(2) By Definition 3.1, $\ulcorner(\alpha \to_d \beta) \to_d ((\gamma \to_d \delta) \to_d ((\alpha * \gamma) \to_d (\beta * \delta)))\urcorner$, $\ulcorner(\beta \to_d \alpha) \to_d ((\delta \to_d \gamma) \to_d ((\beta * \delta) \to_d (\alpha * \gamma)))\urcorner \in D2$, where $* \in \{\wedge_d, \vee, \to_d, \leftrightarrow_d\}$, and (MP).
(3) By Definition 3.1, Proposition 1.2 and the translation method. □

It follows from Proposition 3.1 that the equivalence relation $\approx$ is not a congruence on the set of all formulas of $D2$. Before solving the problem, let us define the set of all *positive* formulas of the discursive logic $D2$:

(i) if $\alpha \in var$, then α is a positive formula of $D2$

(ii) if α and and β are formulas of $D2$, then $\alpha \bullet \beta$ are positive formulas of $D2$, where $\bullet \in \{\vee, \wedge_d, \to_d \leftrightarrow_d\}$.

Proposition 3.2

The formulas

(1) $\alpha \to_d (\beta \to_d \alpha)$

(2) $(\alpha \vee \beta) \to_d (\beta \vee \alpha)$

(3) $(\alpha \wedge_d \beta) \to_d (\beta \wedge_d \alpha)$

(4) $(\alpha \vee (\beta \vee \gamma)) \to_d ((\alpha \vee \beta) \vee \gamma)$

(5) $(\alpha \wedge_d (\beta \wedge_d \gamma)) \to_d ((\alpha \wedge_d \beta) \wedge_d \gamma)$

(6) $(\alpha \vee \alpha) \to_d \alpha$

(7) $\alpha \to_d (\alpha \vee \alpha)$

(8) $(\alpha \wedge_d \alpha) \to_d \alpha$

(9) $\alpha \to_d (\alpha \wedge_d \alpha)$
(10) $(\alpha \vee (\alpha \wedge_d \beta)) \to_d \alpha$
(11) $\alpha \to_d (\alpha \vee (\alpha \wedge_d \beta))$
(12) $(\alpha \wedge_d (\alpha \vee \beta)) \to_d \alpha$
(13) $\alpha \to_d (\alpha \wedge_d (\alpha \vee \beta))$
(14) $((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta) \to_d \beta$
(15) $\beta \to_d ((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta)$
(16) $((\alpha \vee (\alpha \to_d \beta)) \to_d (\alpha \to_d \alpha)$
(17) $(\alpha \to_d \alpha) \to_d ((\alpha \vee (\alpha \to_d \beta))$
(18) $(((\alpha \to_d \beta) \to_d \alpha) \vee \alpha) \to_d \alpha$
(19) $\alpha \to_d (((\alpha \to_d \beta) \to_d \alpha) \vee \alpha)$

are valid in $D2$.

Proof. Apply the direct semantics for $D2$ (or, alternatively, the translation method). As an example, we prove that $\models ((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta) \to_d \beta$.

Assume, for contradiction, that the formula $((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta) \to_d \beta$ is not valid in $D2$. There is a $D2$-model $\langle W, v\rangle$ such that for every $y \in W(v(((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta) \to_d \beta, y) = 0)$. By $(\to_d)$, we obtain: $\exists z \in W(v((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta, z) = 1)$ and $v(\beta, y) = 0$. Then, $v(\alpha \wedge_d (\alpha \to_d \beta), z) = 1$ or $v(\beta, z) = 1$ by $(\vee)$.

Case 1. Let $v(\alpha \wedge_d (\alpha \to_d \beta), z) = 1$, then $v(\alpha, z) = 1$ and $\exists u \in W$ $(v(\alpha \to_d \beta, u) = 1)$, by $(\wedge_d)$. And, consequently, $\forall w \in W(v(\alpha, w) = 0)$ or $v(\beta, u) = 1$, by $(\to_d)$.

Subcase 1a. If $\forall w \in W(v(\alpha, w) = 0)$, then, in particular: $v(\alpha, z) = 0$. But $v(\alpha, z) = 1$. A contradiction.

Subcase 1b. If $v(\beta, u) = 1$, then, in particular: $v(((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta) \to_d \beta, u) = 0$ and, as a result, $\exists t \in W(v((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta, t) = 1)$ and $v(\beta, u) = 0$, by $(\to_d)$. A contradiction.

Case 2. Let $v(\beta, z) = 1$. So, in particular: $v(((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta) \to_d \beta, z) = 0$. And then $\exists s \in W(v((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta, s) = 1)$ and $v(\beta, z) = 0$, by $(\to_d)$. A contradiction.

Consequently, there is no $D2$-model $\langle W, v\rangle$ such that for every $y \in W(v(((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta) \to_d \beta, y) = 0)$ and the formula $((\alpha \wedge_d (\alpha \to_d \beta)) \vee \beta) \to_d \beta$ is valid in $D2$. □

Proposition 3.3

$D2$ is closed under the rule:

$$((\alpha \wedge_d \gamma) \vee \beta) \leftrightarrow_d \beta \;/\; (\gamma \vee (\alpha \to_d \beta)) \leftrightarrow_d (\alpha \to_d \beta).$$

Proof. Assume that (1) $\ulcorner((\alpha \wedge_d \gamma) \vee \beta) \leftrightarrow_d \beta\urcorner \in D2$. Then, by the definition of $(\leftrightarrow_d)$, we obtain: (2) $\ulcorner(((\alpha \wedge_d \gamma) \vee \beta) \rightarrow_d \beta) \wedge_d (\beta \rightarrow_d ((\alpha \wedge_d \gamma) \vee \beta))\urcorner \in D2$. Apply the definition of $D2$ and the recurrent definition of the function tr, to receive (3): $\ulcorner\Diamond\{(\Diamond((\alpha \wedge \Diamond\gamma) \vee \beta) \rightarrow \beta) \wedge \Diamond(\Diamond\beta \rightarrow ((\alpha \wedge \Diamond\gamma) \vee \beta))\}\urcorner \in S5$.[6] Since (T1) $\ulcorner\Diamond(\phi \wedge \Diamond\psi) \rightarrow (\Diamond\phi \wedge \Diamond\psi)\urcorner \in S5$ then (4) $\ulcorner\Diamond(\Diamond((\alpha \wedge \Diamond\gamma) \vee \beta) \rightarrow \beta) \wedge \Diamond(\Diamond\beta \rightarrow ((\alpha \wedge \Diamond\gamma) \vee \beta))\urcorner \in S5$ by (T1), (3) and the rule of detachment for material implication.[7] Note that $S5$ is closed under the rule (R1): if $\phi \wedge \psi$ then ϕ, so we receive: (5) $\ulcorner\Diamond(\Diamond((\alpha \wedge \Diamond\gamma) \vee \beta) \rightarrow \beta)\urcorner \in S5$, by (R1) and (3). Since (T2) $\ulcorner\Diamond(\phi \rightarrow \Diamond\psi) \rightarrow (\Diamond\phi \rightarrow \Diamond\psi)\urcorner \in S5$, then (6) $\ulcorner\Diamond((\alpha \wedge \Diamond\gamma) \vee \beta) \rightarrow \Diamond\beta\urcorner \in S5$ by (T2), (5) and the rule of detachment. Similarly, since (T3) $\ulcorner(\Diamond(\phi \vee \psi) \rightarrow \Diamond\chi) \rightarrow ((\Diamond\phi \vee \Diamond\psi) \rightarrow \Diamond\chi)\urcorner \in S5$, then (7) $\ulcorner(\Diamond(\alpha \wedge \Diamond\gamma) \vee \Diamond\beta) \rightarrow \Diamond\beta\urcorner \in S5$ by (T3), (6) and the rule of detachment. Observe that (T4) $\ulcorner((\Diamond(\phi \wedge \Diamond\psi) \vee \Diamond\chi) \rightarrow \Diamond\chi) \rightarrow (((\Diamond\phi \wedge \Diamond\psi) \vee \Diamond\chi) \rightarrow \Diamond\chi)\urcorner \in S5$. So, (8) $\ulcorner((\Diamond\alpha \wedge \Diamond\gamma) \vee \Diamond\beta) \rightarrow \Diamond\beta\urcorner \in S5$ by (T4), (7) and the rule of detachment. Notice that (T5) $\ulcorner\Diamond\phi \rightarrow ((\Diamond\psi \wedge \Diamond\chi) \vee \Diamond\phi)\urcorner \in S5$ and $S5$ is closed under the rule of adjunction (AR): if ϕ and ψ then $\phi \wedge \psi$. As a result, we have: (9) $\ulcorner(((\Diamond\alpha \wedge \Diamond\gamma) \vee \Diamond\beta) \rightarrow \Diamond\beta) \wedge (\Diamond\beta \rightarrow ((\Diamond\alpha \wedge \Diamond\gamma) \vee \Diamond\beta))\urcorner \in S5$ by (AR), (8) and (T5). By the definition of $(\leftrightarrow)$ and (9), we get: (10) $\ulcorner((\Diamond\alpha \wedge \Diamond\gamma) \vee \Diamond\beta) \leftrightarrow \Diamond\beta\urcorner \in S5$. Now, since (T6) $\ulcorner(((\Diamond\phi \wedge \Diamond\chi) \vee \Diamond\psi) \leftrightarrow \Diamond\psi) \rightarrow ((\Diamond\chi \vee (\Diamond\phi \rightarrow \Diamond\psi)) \leftrightarrow (\Diamond\phi \rightarrow \Diamond\psi))\urcorner \in S5$, then (11) $\ulcorner(\Diamond\gamma \vee (\Diamond\alpha \rightarrow \Diamond\beta)) \leftrightarrow (\Diamond\alpha \rightarrow \Diamond\beta)\urcorner \in S5$ by (T6), (10) and the rule of detachment. Apply the definition of $(\leftrightarrow)$, (R1) and (R1)': if $\phi \wedge \psi$ then ψ, to obtain: (12) $\ulcorner(\Diamond\gamma \vee (\Diamond\alpha \rightarrow \Diamond\beta)) \rightarrow (\Diamond\alpha \rightarrow \Diamond\beta)\urcorner \in S5$ and (13) $\ulcorner(\Diamond\alpha \rightarrow \Diamond\beta) \rightarrow (\Diamond\gamma \vee (\Diamond\alpha \rightarrow \Diamond\beta))\urcorner \in S5$. Since (T7) $\ulcorner((\Diamond\chi \vee (\Diamond\phi \rightarrow \Diamond\psi)) \rightarrow (\Diamond\phi \rightarrow \Diamond\psi)) \rightarrow (((\Diamond\chi \vee \Diamond(\Diamond\phi \rightarrow \psi)) \rightarrow \Diamond(\Diamond\phi \rightarrow \psi)))\urcorner \in S5$, then (14)$\ulcorner(\Diamond\gamma \vee \Diamond(\Diamond\alpha \rightarrow \beta)) \rightarrow \Diamond(\Diamond\alpha \rightarrow \beta)\urcorner \in S5$ by (T7), (12) and the rule of detachment. Similarly, (15) $\ulcorner\Diamond(\gamma \vee (\Diamond\alpha \rightarrow \beta)) \rightarrow \Diamond(\Diamond\alpha \rightarrow \beta)\urcorner \in S5$ by (T8) $\ulcorner((\Diamond\phi \vee \Diamond\psi) \rightarrow \Diamond\psi) \rightarrow (\Diamond(\phi \vee \psi) \rightarrow \Diamond\psi)\urcorner \in S5$, (14) and the rule of detachment; (16) $\ulcorner\Diamond(\Diamond(\gamma \vee (\Diamond\alpha \rightarrow \beta)) \rightarrow (\Diamond\alpha \rightarrow \beta))\urcorner \in S5$ by (T9) $\ulcorner(\Diamond\phi \rightarrow \Diamond\psi) \rightarrow \Diamond(\Diamond\phi \rightarrow \psi)\urcorner \in S5$, (15) and the rule of detachment.

Now, we revert to (13) and obtain: (17) $\ulcorner\Diamond(\Diamond\alpha \rightarrow \beta) \rightarrow (\Diamond\gamma \vee \Diamond(\Diamond\alpha \rightarrow \beta))\urcorner \in S5$ by (T10) $\ulcorner((\Diamond\phi \rightarrow \Diamond\psi) \rightarrow (\Diamond\chi \vee (\Diamond\phi \rightarrow \Diamond\psi)) \rightarrow (\Diamond(\Diamond\phi \rightarrow \psi) \rightarrow (\Diamond\chi \vee \Diamond(\Diamond\phi \rightarrow \psi)))\urcorner \in S5$, (13) and the rule of detachment. Since (T11) $\ulcorner(\Diamond\phi \rightarrow (\Diamond\psi \vee \Diamond\phi)) \rightarrow (\Diamond\phi \rightarrow \Diamond(\psi \vee \phi))\urcorner \in S5$, then (18) $\ulcorner\Diamond(\Diamond\alpha \rightarrow \beta) \rightarrow \Diamond(\gamma \vee (\Diamond\alpha \rightarrow \beta))\urcorner \in S5$ by (T11), (17) and the rule of detachment. Use (T9), (18) and the rule of detachment, to receive: (19) $\ulcorner\Diamond(\Diamond(\Diamond\alpha \rightarrow \beta) \rightarrow (\gamma \vee (\Diamond\alpha \rightarrow \beta)))\urcorner \in S5$. Apply the rule of adjunction (AR) to (16) and (19), to have: (20) $\ulcorner\Diamond(\Diamond(\gamma \vee (\Diamond\alpha \rightarrow \beta)) \rightarrow (\Diamond\alpha \rightarrow \beta)) \wedge \Diamond(\Diamond(\Diamond\alpha \rightarrow \beta) \rightarrow (\gamma \vee (\Diamond\alpha \rightarrow \beta)))\urcorner \in S5$. But, since (T1)' $\ulcorner(\Diamond\phi \wedge \Diamond\psi) \rightarrow \Diamond(\phi \wedge \Diamond\psi)\urcorner \in S5$, then

(21) $\ulcorner\Diamond((\Diamond(\gamma \vee (\Diamond\alpha \to \beta)) \to (\Diamond\alpha \to \beta)) \wedge \Diamond(\Diamond(\Diamond\alpha \to \beta) \to (\gamma \vee (\Diamond\alpha \to \beta))))\urcorner \in S5$. Now apply the definitions of $D2$, the translation function tr and the definition of $(\leftrightarrow_d)$, to finally get: $\ulcorner(\gamma \vee (\alpha \to_d \beta)) \leftrightarrow_d (\alpha \to_d \beta)\urcorner \in D2$. □

Let L_d^+ be a set of all positive formulas of $D2$. Obviously, the relation $\approx$ is a congruence of the algebra $A^+ = \langle L_d^+, \vee, \wedge_d, \to_d\rangle$. Let $\mathbf{A}^+ = A^+/\approx$ be a quotient algebra of A^+ by $\approx$, and $\cup$, $\cap_d$, $\preceq$ be operations in $\mathbf{A}^+$ introduced by connectives $\vee$, $\wedge_d$, $\to_d$ as follows: $[\alpha] \cup [\beta] = [\alpha \vee \beta]$, $[\alpha] \cap_d [\beta] = [\alpha \wedge_d \beta]$, $[\alpha] \preceq [\beta] = [\alpha \to_d \beta]$ and, additionally, $1 = [\alpha \to_d \alpha]$.

Definition 3.2

A binary relation $\leq$ defined on a nonempty set X is a *partial order* on the set X if it is reflexive, antisymmetric and transitive on X.

Let $D2^+$ denote the set of all positive (negation-free) formulas of $D2$.

Proposition 3.4

(1) The relation $\leq$ defined on $A^+/\approx$ as $[\alpha] \leq [\beta]$ *iff* $\ulcorner\alpha \to_d \beta\urcorner \in D2^+$ is a partial order on $A^+/\approx$;

(2) for any formula $\beta \in L_d^+$: $[\beta] = D2^+$ *iff* $\beta \in D2^+$.

Proof. (1) By $\ulcorner\alpha \to_d \alpha\urcorner$, $\ulcorner(\alpha \to_d \beta) \to_d ((\beta \to_d \gamma) \to_d (\alpha \to_d \gamma))\urcorner$, $\ulcorner(\alpha \to_d \beta) \to_d ((\beta \to_d \alpha) \to_d (\alpha \leftrightarrow_d \beta))\urcorner \in D2^+$, the rule (MP) α, $\alpha \to_d \beta$ / β and definition of $\leq$.
(2) (if-then) Obvious.
(2) (then-if) Let $\beta \in D2^+$ and for any formula γ: $\gamma \in [\beta]$, then $\beta \approx \gamma$. But this means that $\ulcorner\beta \to_d \gamma\urcorner$, $\ulcorner\gamma \to_d \beta\urcorner \in D2^+$ and, consequently, $\gamma \in D2^+$. Now for the reverse inclusion observe that if $\gamma \in D2^+$, then $\ulcorner\beta \to_d (\gamma \to_d \beta)\urcorner$, $\ulcorner\gamma \to_d (\beta \to_d \gamma)\urcorner \in D2^+$ and $D2^+$ is closed under (MP), so $\ulcorner\gamma \to_d \beta\urcorner$, $\ulcorner\beta \to_d \gamma\urcorner \in D2^+$, $\beta \approx \gamma$ and, finally, $\gamma \in [\beta]$. □

Definition 3.3

A nonempty set X together with two binary operations $\cap$ (join) and $\cup$ (meet) on X is called a *lattice* if the following axiomatic identities hold for all elements a, b, c of X:

(Commutativity) $a \cup b = b \cup a$, $a \cap b = b \cap a$

(Associativity) $a \cup (b \cup c) = (a \cup b) \cup c$, $a \cap (b \cap c) = (a \cap b) \cap c$

(Idempotence) $a \cup a = a$, $a \cap a = a$

(Absorption) $a = a \cup (a \cap b)$, $a = a \cap (a \cup b)$.

Theorem 3.1 (Kotas, 1975a, p. 156)

(1) Algebra $\langle \mathbf{A}^+, \cup, \cap_d \rangle$ is a lattice;

(2) for every elements a, b, $x \in \mathbf{A}^+$:

(i) $(a \cap_d (a \preceq b)) \cup b = b$

(ii) if $(a \cap_d x) \cup b = b$ then $x \cup (a \preceq b) = a \preceq b$

(iii) $a \cup (a \preceq b) = a \preceq a$

(iv) $((a \preceq b) \preceq a) \cup a = a$.

Proof. By Definition 3.3, propositions 3.2-3.4 and the definitions of $\cap_d$ $\cup$, $\preceq$. □

Notice that $\langle \mathbf{A}^+, \cup, \cap_d, \preceq \rangle$ is a *classical* implicative lattice (cf. Kotas, 1975a, p. 156) and the set of $D2^+$ coincides with the positive part of the classical propositional calculus. In this regard, we may read off the validity of any negation-free discursive formula directly from a classical true-value analysis.

Jaśkowski was aware of this fact. He formulated two theorems of connection linking $D2$ to the classical propositional calculus (cf. Ciuciura, 2008; Jaśkowski, 1948; Jaśkowski, 1949):

Theorem 3.2

Suppose that a formula α contains, besides variables, at most the connectives $\wedge$, $\rightarrow$, $\leftrightarrow$ and $\vee$. If α is valid in the classical propositional calculus, then α_d is valid in $D2$, where α_d is obtained from α by replacing $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$ with $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$, respectively.

Proof. See (Jaśkowski, 1999a, pp. 45-46) and (Jaśkowski, 1999a, pp. 57-58). □

Theorem 3.3

Let a formula α include, besides variables, at most the connectives $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$ and $\vee$. If α is valid in $D2$, then α_{cpc} is valid in classical propositional calculus, where α_{cpc} is obtained from α by replacing $\wedge_d$, $\rightarrow_d$, $\leftrightarrow_d$, $\vee$ with $\wedge$, $\rightarrow$, $\leftrightarrow$, $\vee$, respectively.

Proof. See (Jaśkowski, 1999a, p. 49). □

4. Lindenbaum–Tarski algebra for $D2$ and $D2$-matrices

As seen in the previous section, the standard approach fails to construct the Lindenbaum algebra for Jaśkowski's discursive logic. The equivalence relation $\approx$ is not a congruence on the set of all formulas of $D2$. This, however, can be resolved. It suffices to suppose that the connectives of $\sim$, $\vee$ and $\wedge_d$ are taken as primitives. Let L_d^* denote the resulting language.

The connective of discursive implication is introduced through the definition:

Definition 4.1

$\alpha \rightarrow_d \beta =^{df} \sim (\top \wedge_d \alpha) \vee \beta$, where $\top = p_1 \vee \sim p_1$ and $p_1 \in var$.

To describe the relevant results, we need one more definition:

Definition 4.2 (Kotas, 1975a, p. 157)

$\alpha \Rightarrow \beta =^{df} \sim (\top \wedge_d \sim (\sim \alpha \vee \beta))$, where $\top = p_1 \vee \sim p_1$ and $p_1 \in var$.

Definition 4.2 is of special interest because it shows that strict implication may be expressed in terms of negation, disjunction and discursive conjunction. It brings about the following:

Proposition 4.1 (cf. *ibid.*)

The rule of strict detachment

(SD) $\alpha, \alpha \Rightarrow \beta \ / \ \beta$

is an admissible rule in $D2$.

Proof. Assume that (1) $\alpha \in D2$ and (2) $\ulcorner \alpha \Rightarrow \beta \urcorner \in D2$, then, by Definition 4.2 and (2), we have: (3) $\ulcorner \sim (\top \wedge_d \sim (\sim \alpha \vee \beta))\urcorner \in D2$. Apply the definition of $D2$ and the recurrent definition of the function tr, to get: (4) $\ulcorner \Diamond \alpha \urcorner \in S5$ and (5) $\ulcorner \Diamond \sim (\top \wedge \Diamond \sim (\sim \alpha \vee \beta))\urcorner \in S5$. Observe that (T1) $\ulcorner \Diamond \sim (\phi \wedge \Diamond \psi) \rightarrow \Diamond(\sim \phi \vee \sim \Diamond \psi)\urcorner \in S5$. Hence, by (T1), (5) and the rule of detachment for material implication, we receive: (6) $\ulcorner \Diamond(\sim \top \vee \sim \Diamond \sim (\sim \alpha \vee \beta))\urcorner \in S5$. Since (T2) $\ulcorner \Diamond(\sim \phi \vee \sim \Diamond \psi) \rightarrow (\Diamond \sim \phi \vee \sim \Diamond \psi)\urcorner \in S5$, then (7) $\ulcorner \Diamond \sim \top \vee \sim \Diamond \sim (\sim \alpha \vee \beta)\urcorner \in S5$ by (T2), (6) and the rule of detachment. Again, since (T3) $\ulcorner (\Diamond \sim \top \vee \sim \Diamond \sim \phi) \rightarrow \sim \Diamond \sim \phi \urcorner \in S5$, then (8) $\ulcorner \sim \Diamond \sim (\sim \alpha \vee \beta)\urcorner \in S5$ by (T3), (7) and the rule of detachment. Apply the rule of adjunction (AR) to (4) and (8), to receive: (9) $\ulcorner \Diamond \alpha \wedge \sim \Diamond \sim (\sim \alpha \vee \beta)\urcorner \in S5$. Since (T4) $\ulcorner (\Diamond \phi \wedge \sim \Diamond \sim (\sim \phi \vee \psi)) \rightarrow \Diamond \psi \urcorner \in S5$, then (10) $\ulcorner \Diamond \beta \urcorner \in S5$ by

(T4), (9) and the rule of detachment. But if $\ulcorner \Diamond\beta \urcorner \in S5$, then $\beta \in D2$ by the definition of $D2$. □

Proposition 4.2 (cf. *ibid.*, pp. 157-158)

The formulas

(1) $\alpha \Rightarrow \alpha$

(2) $(\alpha \Rightarrow \beta) \rightarrow_d ((\beta \Rightarrow \gamma) \rightarrow_d (\alpha \Rightarrow \gamma)$

(3) $(\alpha \Rightarrow \beta) \rightarrow_d ((\gamma \Rightarrow \delta) \rightarrow_d ((\alpha \vee \gamma) \Rightarrow (\beta \vee \delta)))$

(4) $(\alpha \Rightarrow \beta) \rightarrow_d ((\gamma \Rightarrow \delta) \rightarrow_d ((\alpha \wedge_d \gamma) \Rightarrow (\beta \wedge_d \delta)))$

(5) $(\beta \Rightarrow \alpha) \rightarrow_d ((\gamma \Rightarrow \delta) \rightarrow_d ((\alpha \rightarrow_d \gamma) \Rightarrow (\beta \rightarrow_d \delta)))$

(6) $(\delta \Rightarrow \gamma) \rightarrow_d ((\alpha \Rightarrow \beta) \rightarrow_d ((\beta \rightarrow_d \delta) \Rightarrow (\alpha \rightarrow_d \gamma)))$

(7) $(\alpha \Rightarrow \beta) \rightarrow_d ((\beta \Rightarrow \alpha) \rightarrow_d ((\gamma \Rightarrow \delta) \rightarrow_d ((\delta \Rightarrow \gamma) \rightarrow_d ((\alpha \leftrightarrow_d \gamma) \Rightarrow (\beta \leftrightarrow_d \delta)))))$

(8) $(\alpha \Rightarrow \beta) \rightarrow_d (\sim \beta \Rightarrow \sim \alpha)$

are valid in $D2$.

Proof. By Definition 4.2 and the direct semantics for $D2$. As an example, we show that $\models (\alpha \Rightarrow \beta) \rightarrow_d (\sim \beta \Rightarrow \sim \alpha)$.

Suppose, for contradiction, that the formula $(\alpha \Rightarrow \beta) \rightarrow_d (\sim \beta \Rightarrow \sim \alpha)$, i.e. $\sim (\top \wedge_d \sim (\sim \alpha \vee \beta)) \rightarrow_d \sim (\top \wedge_d \sim (\sim\sim \beta \vee \sim \alpha))$, is not valid in $D2$. There is a $D2$-model $\langle W, v \rangle$ such that for every $y \in W(v(\sim (\top \wedge_d \sim (\sim \alpha \vee \beta)) \rightarrow_d \sim (\top \wedge_d \sim (\sim\sim \beta \vee \sim \alpha)), y) = 0)$. By $(\rightarrow_d)$, we obtain: $\exists z \in W$ $(v(\sim (\top \wedge_d \sim (\sim \alpha \vee \beta)), z) = 1)$ and $v(\sim (\top \wedge_d \sim (\sim\sim \beta \vee \sim \alpha)), y) = 0$. By $(\sim)$, we receive: $v(\top \wedge_d \sim (\sim \alpha \vee \beta), z) = 0$ and $v(\top \wedge_d \sim (\sim\sim \beta \vee \sim \alpha), y) = 1$. By $(\wedge_d)$, we get: $(v(\top, z) = 0$ or $\forall u \in W(v(\sim (\sim \alpha \vee \beta), u) = 0))$ and $(v(\top, z) = 1$ and $\exists s \in W(v(\sim (\sim\sim \beta \vee \sim \alpha), s) = 1))$.

Case 1. If $v(\top, z) = 0$, then $v(p_i \vee \sim p_i, z) = 0$; $v(p_i, z) = 0$ and $v(\sim p_i, z) = 0$ by $(\vee)$; $v(p_i, z) = 1$ by $(\sim)$. But $v(p_i, z) = 0$. A contradiction.

Case 2. If $\forall u \in W(v(\sim (\sim \alpha \vee \beta), u) = 0)$, then in particular: $v(\sim (\sim \alpha \vee \beta), s) = 0$ and, as a result, $v(\sim \alpha \vee \beta, s) = 1$ by $(\sim)$. By $(\vee)$, we have: $v(\sim \alpha, s) = 1$ or $v(\beta, s) = 1$.

Subcase 2a. Let $v(\sim \alpha, s) = 1$. Observe that $v(\sim (\sim\sim \beta \vee \sim \alpha), s) = 1$. So $v(\sim\sim \beta \vee \sim \alpha, s) = 0$ by $(\sim)$; $v(\sim\sim \beta, s) = 0$ and $v(\sim \alpha, s) = 0$ by $(\vee)$. But $v(\sim \alpha, s) = 1$. A contradiction.

Subcase 2b. Let $v(\beta, s) = 1$. As before, $v(\sim (\sim\sim \beta \vee \sim \alpha), s) = 1$. So, $v(\sim\sim \beta \vee \sim \alpha, s) = 0$ by $(\sim)$; $v(\sim\sim \beta, s) = 0$ and $v(\sim \alpha, s) = 0$ by $(\vee)$; $v(\beta, s) = 0$ by $(\sim)$ twice. But $v(\beta, s) = 1$. A contradiction.

Consequently, there is no $D2$-model $\langle W, v\rangle$ such that for every $y \in W(v(\sim(\top\wedge_d \sim(\sim\alpha \vee \beta)) \rightarrow_d \sim(\top\wedge_d \sim(\sim\sim\beta\vee\sim\alpha)), y) = 0)$ and the formula $(\alpha \Rightarrow \beta) \rightarrow_d (\sim\beta \Rightarrow \sim\alpha)$ is valid in $D2$. □

Definition 4.3 (Kotas, 1975a, p. 158)

For any formula α, $\beta \in L^*_d$: $\alpha \approx^* \beta$ *iff* $\ulcorner\alpha \Rightarrow \beta\urcorner$, $\ulcorner\beta \Rightarrow \alpha\urcorner \in D2$.

Proposition 4.3 *(ibid.)*

(1) $\approx^*$ is an equivalence relation on the set of all formulas of $D2$.

For any formulas α, β, γ, $\delta \in L^*_d$:

(2) if $\alpha \approx^* \beta$ and $\gamma \approx^* \delta$ then

(i) $\alpha \vee \gamma \approx^* \beta \vee \delta$

(ii) $\alpha \wedge_d \gamma \approx^* \beta \wedge_d \delta$

(iii) $\alpha \rightarrow_d \gamma \approx^* \beta \rightarrow_d \delta$

(iv) $\alpha \leftrightarrow_d \gamma \approx^* \beta \leftrightarrow_d \delta$

(3) if $\alpha \approx^* \beta$ then $\sim\alpha \approx^* \sim\beta$.

Proof. (1), (2), (3): By Definition 4.3 and propositions 4.1–4.2. □

Proposition 4.4

Let $\top = p_1 \vee \sim p_1$, where $p_1 \in var$, and $\bot = \sim\top$. The formulas

(1) $\alpha \Rightarrow (\alpha \vee \alpha)$

(2) $(\alpha \vee \alpha) \Rightarrow \alpha$

(3) $(\alpha \vee \beta) \Rightarrow (\beta \vee \alpha)$

(4) $(\alpha \vee (\beta \vee \gamma)) \Rightarrow ((\alpha \vee \beta) \vee \gamma)$

(5) $(\alpha \vee \bot) \Rightarrow \alpha$

(6) $\alpha \Rightarrow (\alpha \vee \bot)$

(7) $(\alpha \vee \top) \Rightarrow \alpha$

(8) $\alpha \Rightarrow (\alpha \vee \top)$

(9) $(\alpha \vee \sim\alpha) \Rightarrow \top$

(10) $\top \Rightarrow (\alpha \vee \sim\alpha)$

(11) $(\alpha \wedge_d (\beta \wedge_d \gamma)) \Rightarrow ((\alpha \wedge_d \beta) \wedge_d \gamma)$

(12) $((\alpha \wedge_d \beta) \vee \alpha) \Rightarrow (\alpha \wedge_d (\beta \vee \alpha))$

(13) $(\alpha \wedge_d (\beta \vee \alpha)) \Rightarrow ((\alpha \wedge_d \beta) \vee \alpha)$

(14) $(\alpha \wedge_d (\beta \vee \alpha)) \Rightarrow \alpha$

(15) $\alpha \Rightarrow (\alpha \wedge_d (\beta \vee \alpha))$

(16) $(\alpha \wedge_d \beta \wedge_d \alpha) \Rightarrow (\alpha \wedge_d \beta)$
(17) $(\alpha \wedge_d \beta) \Rightarrow (\alpha \wedge_d \beta \wedge_d \alpha)$
(18) $(\alpha \wedge_d (\beta \vee \gamma)) \Rightarrow ((\alpha \wedge_d \beta) \vee (\alpha \wedge_d \gamma))$
(19) $((\alpha \wedge_d \beta) \vee (\alpha \wedge_d \gamma)) \Rightarrow (\alpha \wedge_d (\beta \vee \gamma))$
(20) $((\alpha \vee \beta) \wedge_d \gamma) \Rightarrow ((\alpha \wedge_d \gamma) \vee (\beta \wedge_d \gamma))$
(21) $((\alpha \wedge_d \gamma) \vee (\beta \wedge_d \gamma)) \Rightarrow ((\alpha \vee \beta) \wedge_d \gamma)$
(22) $(\alpha \wedge_d \beta) \Rightarrow \sim (\sim \alpha \vee \sim (\top \wedge_d \beta))$
(23) $\sim (\sim \alpha \vee \sim (\top \wedge_d \beta)) \Rightarrow (\alpha \wedge_d \beta)$
(24) $(\top \wedge_d \bot) \Rightarrow \bot$
(25) $\bot \Rightarrow (\top \wedge_d \bot)$
(26) $\alpha \Rightarrow (\top \wedge_d \alpha)$
are valid in $D2$.

Proof. By Definition 4.2 and the direct semantics for $D2$. As an example, we prove that $\models (\alpha \wedge_d \beta) \Rightarrow \sim (\sim \alpha \vee \sim (\top \wedge_d \beta))$.

Assume, for contradiction, that the formula $(\alpha \wedge_d \beta) \Rightarrow \sim (\sim \alpha \vee \sim (\top \wedge_d \beta))$, or to be precise: $\sim (\top \wedge_d \sim (\sim (\alpha \wedge_d \beta) \vee \sim (\sim \alpha \vee \sim (\top \wedge_d \beta))))$, is not valid in $D2$. There is a $D2$-model $\langle W, v\rangle$ such that for every $y \in W(v(\sim (\top \wedge_d \sim (\sim (\alpha \wedge_d \beta) \vee \sim (\sim \alpha \vee \sim (\top \wedge_d \beta)))), y) = 0)$. So, $v(\top \wedge_d \sim (\sim (\alpha \wedge_d \beta) \vee \sim (\sim \alpha \vee \sim (\top \wedge_d \beta))), y) = 1$, by $(\sim)$. But then $v(\top, y) = 1$ and $\exists z \in W(v(\sim (\sim (\alpha \wedge_d \beta) \vee \sim (\sim \alpha \vee \sim (\top \wedge_d \beta))), z) = 1)$ by $(\wedge_d)$. By $(\sim)$, we have: $v(\sim (\alpha \wedge_d \beta) \vee \sim (\sim \alpha \vee \sim (\top \wedge_d \beta)), z) = 0$. Then $v(\sim (\alpha \wedge_d \beta), z) = 0$ and $v(\sim (\sim \alpha \vee \sim (\top \wedge_d \beta)), z) = 0$ by $(\vee)$. So, $v(\alpha \wedge_d \beta, z) = 1$ and $v(\sim \alpha \vee \sim (\top \wedge_d \beta), z) = 1$ by $(\sim)$, and finally, $v(\alpha, z) = 1$ and $\exists u \in W(v(\beta, u) = 1)$, by $(\wedge_d)$. But if $v(\sim \alpha \vee \sim (\top \wedge_d \beta), z) = 1$, then $v(\sim \alpha, z) = 1$ or $v(\sim (\top \wedge_d \beta), z) = 1$, by $(\vee)$.

Case 1. If $v(\sim \alpha, z) = 1$, then $v(\alpha, z) = 0$ by $(\sim)$. But $v(\alpha, z) = 1$. A contradiction.

Case 2. If $v(\sim (\top \wedge_d \beta), z) = 1$, then $v(\top \wedge_d \beta, z) = 0$ by $(\sim)$ and, as a result, $v(\top, z) = 0$ or $\forall w \in W(v(\beta, w) = 0)$, by $(\wedge_d)$.

Subcase 2a. Let $v(\top, z) = 0$. By the definition of $\top$, we have: $v(p_i \vee \sim p_i, z) = 0$. Then $v(p_i, z) = 0$ and $v(\sim p_i, z) = 0$ by $(\vee)$ and, finally, $v(p_i, z) = 1$ by $(\sim)$. But $v(p_i, z) = 0$. A contradiction.

Subcase 2b. Let $\forall w \in W(v(\beta, w) = 0)$. So, in particular, $v(\beta, u) = 0$. But $v(\beta, u) = 1$. A contradiction.

Consequently, there is no $D2$-model $\langle W, v\rangle$ such that for every $y \in W(v(\sim (\top \wedge_d \sim (\sim (\alpha \wedge_d \beta) \vee \sim (\sim \alpha \vee \sim (\top \wedge_d \beta)))), y) = 0)$ and the formula $(\alpha \wedge_d \beta) \Rightarrow \sim (\sim \alpha \vee \sim (\top \wedge_d \beta))$ is valid in $D2$. □

Let $\mathbf{A}^* = A^*/\approx^*$ be a quotient algebra of A^* by $\approx^*$ (where $A^* = \langle L_d^*, \sim, \vee, \wedge_d \rangle$), and $\cup$, $\cap_d$, $-$ be operations in $\mathbf{A}^*$ introduced by connectives $\vee$, $\wedge_d$, $\sim$ as follows: $[\alpha] \cup [\beta] = [\alpha \vee \beta]$, $[\alpha] \cap_d [\beta] = [\alpha \wedge_d \beta]$, $[\sim \alpha] = -[\alpha]$ and, additionally, $1 = [\top]$ and $0 = [\bot]$.

Before going further we introduce the following definition:

Definition 4.4

A *skew lattice* is an algebra $\langle \mathbf{A}, \cup, \cap \rangle$ of type (2, 2), where both $\cup$ and $\cap$ are associative, idempotent binary operations on $\mathbf{A}$ which satisfy the absorption identities: $(a \cap b) \cup a = a = a \cap (b \cup a)$, for all elements a, $b \in \mathbf{A}$ (cf. Jordan, 1949). A skew lattice $\langle \mathbf{A}, \cup, \cap \rangle >$ is *distributive* if it satisfies the identities: $a \cap (b \cup c) = (a \cap b) \cup (a \cap c)$ and $(a \cup b) \cap c = (a \cap c) \cup (b \cap c)$, for any a, b, $c \in \mathbf{A}$. A skew lattice $\langle \mathbf{A}, \cup, \cap \rangle >$ is *left-handed* if it satisfies: $a \cap b \cap a = a \cap b$ and $a \cup b \cup a = b \cup a$, for all elements a, $b \in \mathbf{A}$.[8]

Theorem 4.1

(1) Algebra $\langle \mathbf{A}^*, \cup, -, 1, 0 \rangle$ is a Boolean algebra with 1 (as unit) and 0 (as zero)

(2) Algebra $\langle \mathbf{A}^*, \cup, \cap_d \rangle$ is a distributive left-handed skew lattice

(3) for any a, b, $c \in \mathbf{A}^*$:

(i) $a \cap_d b = -(-a \cup -(1 \cap_d b))$

(ii) $1 \cap_d 0 = 0$

(iii) $a \preceq^* 1 \cap_d a$, where $[\alpha] \preceq^* [\beta]$ *iff* $\ulcorner \alpha \Rightarrow \beta \urcorner \in D2$.

Proof. (1), (2), (3): Apply Definition 4.3, propositions 4.1 and 4.4, the definitions of $\cap_d$ $\cup$, $-$, 1 and 0. □

Proposition 4.5 (Kotas, 1975a, p. 159)

For every elements a, $b \in \mathbf{A}^*$:

(1) $1 \cap_d (1 \cap_d a) = 1 \cap_d a$

(2) $1 \cap_d -(-a \cup -(1 \cap_d b)) = -(-(1 \cap_d a) \cup -(1 \cap_d b))$.

Proof. (1): $1 \cap_d (1 \cap_d a) = (1 \cap_d 1) \cap_d a = 1 \cap_d a$.
(2): $1 \cap_d -(-a \cup -(1 \cap_d b)) = 1 \cap_d (a \cap_d b) = (1 \cap_d a) \cap_d b = -(-(1 \cap_d a) \cup -(1 \cap_d b))$. □

Theorem 4.2 (Kotas, 1975a, p. 159)

A formula α is a thesis of $D2$ (see Section 1) *iff* the equality $1 \cap_d a = 1$, where $a = [\alpha]$, is true in $\langle \mathbf{A}^*, \cup, \cap_d, -, 1, 0 \rangle$.

Let ∇ be a subset of $\mathbf{A}^*$ such that $\nabla = \{a \in \mathbf{A}^* : 1 \cap_d a = 1\}$.

Theorem 4.3 (*ibid.*)

∇ is a proper filter of the algebra $\langle \mathbf{A}^*, \cup, \cap_d, -, 1, 0\rangle$.

Proof. First, we have to show that ∇ is a filter of the algebra $\langle \mathbf{A}^*, \cup, \cap_d, -, 1, 0\rangle$, i.e. for every element a, $b \in \mathbf{A}^*$:

(1) $a \cap_d b \in \nabla$ *iff* $a \in \nabla$ and $b \in \nabla$

(2) if $a \in \nabla$ then $a \cup b \in \nabla$.

(1) (if-then) Let $a \cap_d b \in \nabla$, that is, $1 \cap_d a \cap_d b = 1$. By Theorem 4.1, i.e. $(a \cap_d b) \cup a = a \cap_d (b \cup a) = a$, we have $1 \cap_d a = (1 \cap_d a \cap_d b) \cup (1 \cap_d a)$. Since $1 \cap_d a \cap_d b = 1$ and $1 \cup a = 1$, then $(1 \cap_d a \cap_d b) \cup (1 \cap_d a) = 1 \cup (1 \cap_d a) = 1$ and, consequently, $a \in \nabla$. To show that $b \in \nabla$, we proceed analogously as for the case of $a \in \nabla$.
(1) (then-if) Let $a, b \in \nabla$. This means that $1 \cap_d a = 1$ and $1 \cap_d b = 1$. Apply Theorem 4.1, i.e. $a \cap_d (b \cap_d c) = (a \cap_d b) \cap_d c$, to obtain $1 \cap_d (a \cap_d b) = (1 \cap_d a) \cap_d b$. Since $(1 \cap_d a) \cap_d b = 1 \cap_d b = 1$ then $a \cap_d b \in \nabla$.
(2) Let $a \in \nabla$, that is, $1 \cap_d a = 1$. By Theorem 4.1, i.e. $a \cap_d (b \cup c) = (a \cap_d b) \cup (a \cap_d c)$, we have $1 \cap_d (a \cup b) = (1 \cap_d a) \cup (1 \cap_d b)$. Since $1 \cap_d a = 1$ and $1 \cup a = 1$ then $(1 \cap_d a) \cup (1 \cap_d b) = 1 \cup (1 \cap_d b) = 1$ and, finally, $a \cup b \in \nabla$.

Observe that $[\bot] \notin D2/\approx^*$. This implies that $D2/\approx^* = \{[\alpha] : \alpha \in D2\}$ is a proper filter of the algebra $\langle \mathbf{A}^*, \cup, \cap_d, -, 1, 0\rangle$. □

Let L be a logical system and $E(\mathbf{M})$ denote a set of all formulas which are valid in the matrix $\mathbf{M}$. A matrix $\mathbf{M}$ is called *L-matrix* if $L \subset E(\mathbf{M})$. A matrix $\mathbf{M}$ is called characteristic for L (or: A matrix $\mathbf{M}$ is adequate with respect to L) if $L = E(\mathbf{M})$.

Definition 4.5 (*ibid.*, p. 160)

A matrix $\mathbf{M} = \langle \mathbf{A}^*, B, \cup, \cap_d, -\rangle$ is $D2$-matrix if $D2 \subset E(\mathbf{M})$. A $D2$-matrix is adequate with respect to $D2$ if $D2 = E(\mathbf{M})$.

Theorem 4.4 (*ibid.*)

$\mathbf{M} = \langle \mathbf{A}^*, B, \cup, \cap_d, -\rangle$ is $D2$-matrix *iff*

(1) $\langle \mathbf{A}^*, \cup, -\rangle$ is a Boolean algebra

(2) $a \in B$ *iff* $1 \cap_d a = 1$

for any a, b, $c \in \mathbf{A}^*$:

(3) $a \cap_d b = -(-a \cup -(1 \cap_d b))$

(4) $1 \cap_d 0 = 0$

(5) $a \preceq^* 1 \cap_d a$

(6) $a \cap_d (b \cap_d c) = (a \cap_d b) \cap_d c$

(7) $a \cap_d (b \cup c) = (a \cap_d b) \cup (a \cap_d c)$.

Theorem 4.5

$D2$ is an infinitely valued system.

Proof. See (Kotas, 1975b). □

5. da Costa, Kotas and Dubukajtis' discursive logic

In 1977, da Costa and Dubikajtis presented an axiomatization of $D2$ (cf. da Costa & Dubikajtis, 1977). However, as it was shown in (Ciuciura, 2008) and (Ciuciura, 2005), they axiomatized a system that differed from Jaśkowski's discursive logic. Their axiomatics included, among others, a connective intended to be Jaśkowski's discursive conjunction, but, in fact, it did not correspond to any of Jaśkowski's connectives, *viz.* (iv)* $tr^\star(\alpha \wedge_d^\star \beta) = \ulcorner \Diamond tr^\star(\alpha) \wedge tr^\star(\beta) \urcorner$.

Let $D2^\star$ denote da Costa and Dubikajtis' system of discursive logic and $L_d^\star$ stand for a language of the system.

Definition 5.1

An algebra $\mathbf{A}^\star = \langle A, -, \vee, \wedge_d^\star, \rightarrow_d, 0, 1\rangle$ is said to be $D2^\star$-discursive *iff* the following conditions hold:

(1) the structure $A = \langle A, -, \vee, 0, 1\rangle$ is a Boolean algebra

for any $a, b \in A$:

(2)* $a \wedge_d^\star b = \begin{cases} 0, & \text{if } a = 0 \\ b, & \text{if } a \neq 0 \end{cases}$

(3) $a \rightarrow_d b = \begin{cases} 1, & \text{if } a = 0 \\ b, & \text{if } a \neq 0. \end{cases}$

Theorem 5.1

For any $\alpha \in L_d^\star$ the following holds:

α is valid in $D2^\star$ *iff* for every matrix $m \in K_{D2^\star}$ and any $h \in Hom(L_d^\star, \mathbf{A})$, $h\alpha \neq 0$.

Proof. An exact analogue of the proof of Theorem 2.1. □

Since the general idea behind $D2^\star$ is almost the same as in the case of $D2$, the only substantial difference is the use of one of the variants of the discursive conjunction.

6. Conclusion

The purpose of this paper was to present an algebraic approach to Jaśkowski's discursive logic. It was shown that the set of $D2^+$ corresponded to the classical sentential calculus without negation and we might read off the validity of any negation-free discursive formula directly from a classical true-value analysis. It was also proved that the equivalence relation $\approx$ was not a congruence on the set of all formulas of $D2$. As a result, the standard approach fails to construct the Lindenbaum algebra for Jaśkowski's discursive logic. This problem was solved with the help of $\sim$, $\vee$ and $\wedge_d$ taken as primitives.

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NOTES

[1]. See, e.g., (Arieli, Avron, & Zamansky, 2011) and (Béziau, 2006).

[2] A reader can find more information about the systems of modal logic in (Chellas, 1980) and (Hughes & Cresswell, 1996).

[3] Cf. Omori, H., & Alama, J. Axiomatizing Jaśkowski's Discursive Logic D2. Submitted to *Studia Logica*.

[4] See (Ciuciura, 2008) for details.

[5] See (Hawranek, 1980) for details on a completeness result for $S5$ with respect to the class of Henle algebras.

[6] We have omitted the symbols of the translation function *tr* and parentheses for the sake of simplicity.

[7] Those formulas which are valid in $S5$ will be hereafter denoted by (Tn), where n $\in N$. The notation will be frequently used here (and in the proof of Proposition 4.1) as a way of making proofs more readable.

[8] See (Cvetko-Vah, 2011) and (Leech, 1989) for details.

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