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HOW ARE CONCEPTS OF INFINITY ACQUIRED?

Abstract. Concepts of infinity have been subjects of dispute since antiquity. The main problems of this paper are: is the mind able to acquire a concept of infinity? and: how are concepts of infinity acquired? The aim of this paper is neither to say what the meanings of the word "infinity" are nor what infinity is and whether it exists. However, those questions will be mentioned, but only in necessary extent.

Keywords: concept of infinity.

In re mathematica ars proponendi quaestionem pluris facienda est quam solvendi.

Georg Cantor[1]

1. Introduction

The term "infinite"[2] is used in different contexts. We say that:

1. we have experienced something infinitely many times,
2. there are infinitely many numbers,
3. God is infinite in perfection.

In the first two cases "infinity" concerns quantity (countable nouns). In the third, the last case, "infinity" has a qualitative meaning (uncountable noun). In the first case "infinity" is used as in everyday talking; in the second, in scientific language; in the third, in philosophical or theological language. In everyday language "infinity" means, e.g.: "so many times that I do not remember", or "it is not possible for me to count how many times". This psychological meaning of the word will not be the subject of our considerations.

Infinity is inherently filled with paradoxes and contradictions. Though the concept of (potential) infinity is common and clear in such ordinary phrases as "and so on" or "and so forth", it is capable of giving rise to vertiginous bewilderment:

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> which may lead, on the one hand, to the mystical multiplication of contradictions, as also, on the other hand, to that voluntary curtailment of our talk and thought on certain matters, which is as ruinous to our ordered thinking. A notion which is at once so tantalizing and so ordinary plainly deserves the perpetual notice of philosophers. (Findlay, Lewy & Körner 1953, p. 21)

Infinity is linked with unending distance, space, time, and God. Philosophers and theologians were always fascinated with the concept of infinity, but mathematicians tried to avoid it or even demonstrated hostility throughout most of the history of mathematics. Nevertheless infinity, in its various manifestations, has played and still plays a key role in the conceptual development of mathematics. Only within the past century or so have mathematicians accepted infinity as a number – albeit many ideas that we find intuitive when working with normal numbers don't work anymore with infinite numbers, and instead there are countless apparent paradoxes. Is there a largest number? Is there anything bigger than infinity? What is infinity plus one? What is infinity plus infinity? Is the whole greater than a part?[3] Two critical periods in the foundation of mathematics originated around the problem of infinity: that which culminated in Euclid's *The Elements* and that which has its source in Cantor's works. Euclid (323–283 B.C.) is credited for proving that there are infinitely many prime numbers. He formulated his discovery in line with Aristotle's belief in potential infinity: Prime numbers are more than any assigned magnitude of prime numbers. It is likely that infinity, with its possibilities and paradoxes, will continue to be a source of theory and discussion.

In order to discuss infinity one must have a concept (or concepts) of infinity. Without it, it would not be possible to say anything about infinity, even that it does not exist. Without acquirement of a concept we would not be able to tell if it is, e.g. self-contradictory. (Frege & Austin 1980, pp. 105–106):

> A concept is still admissible even though its defining characteristics do contain a contradiction: all that we are forbidden to do, is to presuppose that something falls under it. But even if a concept contains no contradiction, we still cannot infer that for that reason something falls under it. If such concepts were not admissible, how could we are prove that a concept does not contain any contradiction? It is by no means always obvious; it does not follow that because we see no contradiction there is none there, nor does a clear and full definition afford any guarantee against it.

The question arises: how do we acquire concepts of infinity?

2. Disputes over the concepts of infinity

Throughout history philosophers and theologians, scientists, mathematicians, and logicians have dedicated vast amounts of work to unraveling the mysteries of infinity. David Hilbert (1862–1943) (1926, p. 163) said:

> Das Unendliche hat wie keine andere Frage von jeher so tief das *Gemüt* der Menschen bewegt; das Uendliche hat wie kaum eine andere *Idee* auf den Verstand so anregend und fruchtbar gewirkt; das Unendliche ist aber auch wie kein anderer *Begriff* so der Aufklärung bedürftig.
>
> The infinite! No other question has ever moved so profoundly the spirit of man; no other idea has so fruitfully stimulated his intellect; yet no other concept stands in greater need of clarification than that of the infinite.[4]

At first, infinity was ascribed simply to very long time intervals or distances, or to very large sets. The problem of infinity was known already to the ancient Greeks, though the concept of infinity is – which the history of the disputes confirm – fundamentally very hard to acquire by a human mind or even – as some maintain – beyond the human ability to comprehend. It would not be easy to point out a philosopher who did not make at least some remarks on infinity. Let us remember some of them.

Anaximander (610–546 B.C.) introduced the concept of *Apeiron*,[5] the 'basic stuff' of the universe, an infinite (or limitless) and infinitely divisible primordial form of matter that is the origin of all finite things:

> Infinity is the origin of all things.

Since the ancient period, numerous dilemmas have arisen from infinity. Zeno of Elea (490–430 B.C.) is mostly remembered for his extremely sophisticated aporiae of infinity.

Philosophical dispute over infinity is not possible without reference to mathematics and vice versa; one permeates the other.

> The interaction of philosophy and mathematics is seldom revealed so clearly as in the study of the infinite among the ancient Greeks. (Knorr 1982)

In this short review of the most important ideas of infinity, special attention should be devoted to Aristotle (384–322 B.C.), not because today philosophers of various stripes continue to look to him for guidance and inspiration in many different areas, but because his conception of infinity still plays an important role in contemporary disputes.

Aristotle described two ways of looking at an infinite series. Two distinct senses stand out: actually infinite (complete infinity, completed infinity) – one conceives of the series as completed, and potentially infinite – the series is never completed, but it is considered infinite because a next item in the series is always possible. The concept of actual infinity treats the infinite as timeless and complete, as something that exists wholly at one time. The concept of potential infinity treats the infinite as a non-terminating, never-ending process developing over time:

> For generally the infinite has this mode of existence: one thing is always being taken after another, and each thing that is taken is always finite, but always different.
>
> Aristotle, *Physics*, III.6.

For Aristotle actual infinity does not exist:

> *Infinitum actu non datur.*

In Aristotle's words:

> being infinite is a privation, not a perfection but the absence of a limit.
>
> Aristotle, *Physics*, III.7.

Moreover, for Aristotle a completed infinity cannot exist even as an idea in the mind of a human.

The idea introduced by Aristotle of the two kinds of infinity, actual and potential, would dominate thinking for two and half millenia and, even today, remains a persuasive argument. The existence of an actual infinity is an important shared concern within the realms of mathematics, philosophy, and theology. The set of natural numbers is potentially infinite, since there is no largest natural number, but the set of natural numbers as a whole is actually infinite.[6] A Turing machine tape is potentially infinite in both directions. In projective geometry "infinity" (actual) is the point where two parallel lines meet.[7] A human being is potentially infinitely perfect and God is actually infinitely perfect (*Matthew 5:48*):

> Be perfect, [...], as your heavenly Father is perfect.

The relation of the perfectness of human being to the perfectness of God is quite similar to the case of the algebraic function $y = \frac{1}{x}$ which approaches infinity as x approaches 0. The expression y could never actually be infinite, since x could never be zero, because an expression of division by zero has no meaning in algebra.

Many argue that Archimedes of Syracuse (c. 287 – c. 212 B.C.) had already discovered the mathematics of infinity.

Augustine (354–430) denied human beings the ability to recognize the infinite on their own. Only God is able to do so due to His infinite nature. Since Augustine, the word "infinity" has changed its ancient negative understanding. In the medieval period the word had come to mean endless, unlimited, and immeasurable, but not necessarily chaotic as was the case in the ancient period. The question of its intelligibility and conceivability by humans was disputed. Thomas Aquinas (1225–1274) allowed the existence of the potential infinity of material things. For him only God can actually be infinite (making God the only perfect being). God himself is both limitless and perfect (*Summa Theologie* Ia7.1). But[8]

> All creatures are finite or limited. For creatures receive their being and their perfections, and whatever is received is measured and limited by the giver or by the capacity of the receiver.

God is not able to create an infinite entity for it would be against His rationality[9]

> For bodily infinity would be infinity in size, and size is always measurable; that is, size is always finite. Even a mathematical body must be thought of as contained within its lines and surfaces.

Though God is omnipotent, He is not able to create a rock so heavy that he cannot lift it. Omnipotence is the ability to bring about any logically possible state of affairs. Omnipotence is limited by logical possibility.

For Galileo Galilei (1564–1642), who advanced greatly human understanding of infinity, infinity by its very nature was incomprehensible to human minds. In *Discorsi e dimostrazioni matematiche, intorno á due nuove scienze* (1638) he maintained:[10]

> Infinity and indivisibles transcend our finite understanding, the former on account of their magnitude, the latter because of their smallness.

René Descartes (1596–1650) denied the existence of the actually infinite, but he did believe in indefiniteness. In *The Principles of Philosophy* we read (Descartes 1978):[11]

> we should regard things in which we observe no limits as indefinite – for example the extension of the world, the divisibility of the parts of matter, the number of the stars, etc. [...]

> 27. The difference between the indefinite and the infinite.
>
> There are two reasons for calling these things 'indefinite' rather than 'infinite'. The first is in order to reserve the word 'infinite' for God alone, because in him alone and in every respect, we do not merely fail to recognise any limits, but we also understand positively that there are none. The second is that, in the case of everything else, we do not have the same sort of positive understanding of their lacking limits in some respect. We merely, in a negative way, admit that we cannot discover their limits – if they have any.

In the 17th century mathematicians began to deal with abstract objects that did not match anything in nature. Employing the concept of infinitesimal numbers, numbers which are infinitely small, mathematics was able to describe much better the behaviour of nature. Isaac Newton's (1642–1727) efforts, using the infinite divisions of mathematical curves in his fluxion calculus, were rewarded by his discovery of fundamental laws of physics.

Gottfried Leibniz (1646–1716) is one of the major figures involved in the mathematical revolution of the 17th century. He saw the world as being composed of infinitely many indivisible monads but rejected the idea of an infinite totality (whole) of things. In *New Essays on Human Understanding* we read (1982, § 158):[12]

> It is perfectly correct to say that there is an infinity of things, i.e. there are always more of them than can be assigned. But it is easy to demonstrate that there is no infinite number, nor any infinite line or other infinite quantity, if these are taken to be genuine wholes. The true infinite, strictly speaking, is only the absolute, which precedes all composition and is not formed by the addition of parts.

Since the 17th century, infinity has had to tolerate quite diverse treatments in the sciences, mathematics, and philosophy. Bernard Bolzano (1781–1841) was the first who defended the role of the concept of infinity in mathematics.[13] For him the world is full of actual infinities, so that it does not make any sense to exclude them from mathematics.[14] The human mind, according to Bolzano, is capable of imagining an infinity as a whole. He developed the calculation of infinite quantities without inconsistency.

Before Georg Cantor (1845–1918), for two and half millenia, infinity was the subject of confused thinking and invalid argumentation. Built up by Cantor, a mathematical theory fully rehabilitated the actual infinity, resolved the old paradoxes, but involved new antinomies.[15] Cantor (Rucker 2007, p. 7):

> in the late 1800s, finally created a theory of the actual infinite which by its apparent consistency, demolished the Aristotelian and scholastic "proofs" that no such theory could be found.

Cantor was undoubtedly the first who realised that there are different kinds and different sizes of infinity. 700 years earlier, infinity, both as an endless collection of discrete items as well as a continuum, was suggested in about 1280 in the Kabbalah, the writings of Jewish mysticism.

Cantor himself was so surprised by his own discovery that he wrote:

> Ich see es, aber ich glaube es nicht.
>
> I see it, but I don't believe it!

Before him infinity had been a taboo subject in mathematics. For some, e.g. Johann Carl Friedrich Gauss (1777–1855), the Princeps mathematicorum, the greatest mathematician since antiquity, the term "infinity" should only be used as "a way of speaking", a *facon de parler* in which one properly speaks of limits, but it does not refer to any mathematical value. In a letter to Schumacher, 12 July 1831, Gauss wrote:[16]

> I protest against the use of infinite magnitude as something completed, which is never permissible in mathematics. Infinity is merely a way of speaking, the true meaning being a limit which certain ratios approach indefinitely close, while others are permitted to increase without restriction.

Cantor considered infinite sets not as merely going on forever but as completed entities; that is, having an actual though infinite number of members. Moreover, Cantor, a deeply religious philosopher, viewed his discoveries in a very philosophical way and believed that his study of infinity could explain the absolute infinity of God. Cantor deemed himself a kind of prophet empowered by God to reveal the truth of Transfinite numbers. He equated Him with the Absolute Infinite[17] (Hallett 1986, p. 13), (Dauben 1977, p. 86), (Dauben 1979, pp. 120, 143). Cantor – as it is suggested (Dauben 2004, pp. 8, 11, 12–13) – believed his theory of transfinite numbers had been communicated to him by God. New ideas gained him numerous enemies. To Henri Poincare is (mis)attributed the statement that Cantor's set theory would be considered by future generations as:[18]

> a disease from which one has recovered.

He held that "most of the ideas of Cantorian set theory should be banished from mathematics once and for all."(Dauben 1979, p. 266) Leopold

Kronecker, the uncrowned king of the German mathematical world of those days, attacked Cantor, his former student, personally, calling him a “scientific charlatan,” a “renegade,” and a “corrupter of youth.” He said (Tall 2013, p. 363):

> I don’t know what predominates in Cantor’s theory – philosophy or theology – but I am sure that there is no mathematics there.

For Hermann Weyl (1885–1955), a great and versatile mathematician of the 20th century, set theory is a “house built on sand”.[19] He also wrote (2012, p. 141):

> classical logic was abstracted from the mathematics of finite sets and their subsets [...]. Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and original sin of [Cantor’s] set theory.

Finally, Wittgenstein’s attacks were finitist: he believed that Cantor’s diagonal argument conflated the intension of a set of cardinal or real numbers with its extension, thus conflating the concept of rules for generating a set with an actual set. (Rodych 2007) Cantor was well aware of the opposition his ideas were encountering. For him the fear of infinity is a form of myopia (1932, p. 374). In 1883 he wrote:[20]

> I realize that in this undertaking I place myself in a certain opposition to views widely held concerning the mathematical infinite and to opinions frequently defended on the nature of numbers.

Some, such as Karl Weierstrass and long-time friend Richard Dedekind supported Cantor’s ideas. Hilbert (1983, p. 188) described Cantor’s work as:

> the finest product of mathematical genius and one of the supreme achievements of purely intellectual human activity.

Today, Cantor’s work is widely accepted by the mathematical and philosophical communities. It is not by accident that almost every basic mathematical concept relates to Cantor either directly or indirectly. We can be sure that in mathematics (Hilbert 1925):

> No one shall drive us from the paradise that Cantor created for us.[21]

3. Is the human mind able to acquire the concept of infinity?

> Either mathematics is too big for the human mind or the human mind is more than a machine.
>
> Kurt Gödel

3.1. The concept of concept

In order to consider the main problem of the paper, if at all and how the human mind acquires the concept(s) of infinity, some reflection on the concept of concept would be useful. The question what is a concept, is one of the most important problems of philosophy, psychology, and cognitive science. The concept of concept depends on the philosophical school. In cognitive science, concepts are mental representations. They are entities that exist in the brain.

There are no standard definitions of empirical and abstract concepts. Empirical concepts are being abstracted from individual perceptions, and they are intended to refer to concrete objects. Abstract concepts have a different nature and, at least not immediately, are not the product of individual perceptions. They are intended to refer to abstract objects. An object is abstract if it is neither an embedder of nor embedded in a network of casual relations.[22] There are several alternative conceptions of abstract and concrete objects. Perhaps the most common conception of abstract objects is that of non-spatiotemporal and causally inert objects.[23] Abstract objects have a different relation to the human mind than concrete objects. The main difference there is in the way of its cognition, and thus there is a difference between acquirement of abstract concepts and acquirement of empirical concepts. Cognitive sciences (in particular psychology and neurology) find many such differences.[24] The concept of chair is an empirical concept. The concept of infinity is an abstract concept. No theory of conceptual representation is complete without an explicit account of how abstract concepts are acquired.

The three main ideas of acquiring a concept can be pointed out:

1. a concept is acquired by an individual human mind iff the individual human mind has mentally constructed an object for that the concept refers to;
2. a concept is acquired iff the concept can be subject of formal characterization as given by a formal axiom system;
3. the acquirement of a concept by a human mind is a result of a cognitive process determined by such factors as intuition, experience, and knowledge acquired by the human mind.

There are three possibilities for disputes over infinity.

1. the name "infinity" (and any other word) does not have a meaning, is not intelligible, unless there is a mental construction of an object (a concept) to that the word refers to;
2. the word "infinity" is intelligible, but the logical possibility of the existence of infinity should be proved;
3. an object, in particular infinity, exists independently of knowledge about it. To maintain that it exists, it is enough to prove its existence.

The important difference between the last two possibilities is in

- conception of proof;
- mode of existence: in case 2 we talk about (logically) possible existence but in case 3 about actual existence.

In case 1, acquirement of the concept of infinity depends on individual mind construction, in case 2 it depends on construction of a formal theory of infinity. In the last case, case 3, the acquirement of the concept of infinity is a result of the cognitive process.

3.2. Brouwer and the intuitionists

> Nos mathematici sumus isti veri poetae sed quod fingimus nos et probare decet.
>
> Leopold Kronecker[25]

Intuitionism as a school in the philosophy of mathematics was founded by the Dutch mathematician Luitzen Egbertus Jan Brouwer (1881–1966). He devoted a large part of his life to the development of mathematics on this new basis. Some traces of intuitionism are already in the mathematics of Antiquity. Brouwer's intuitionism is essentially more radical and a future step toward pure constructivism (Weyl 2012, p. 140). His intuitionism stands apart from other philosophies of mathematics.

Intuitionists and their followers say that – roughly speaking – it is only abstract objects that are constructed.[26] Intuitionism is a form of constructivism in the wider sense, since many contemporary constructivists do not accept all the principles that intuitionists believe to be true. According to intuitionist doctrine the mathematical universe is man-made. Moreover, for Brouwer even the outer world does not have a-priori existence; it is a fruit of the activity of the subject.

For intuitionists, mental processes are the only proper way to construct of abstract objects. The construction excludes impredicative definitions, which are the source of the fault of the vicious circular. The existence of an abstract object is equivalent to the possibility of its construction by an individual human mind (Heyting 1956, p. 1). Mathematical statement corresponds to a mental construction, and a mathematician can assert the truth of a statement only by verifying the validity of that construction by intuition. The activity of the mind has its origin in the perception of the movement of time. Time – in the Kantian sense – is the only an a priori notion. Awareness of time is indispensable in the process of mathematical cognition, or – what is equivalent – the process of creation. The mental construction of one natural number after the other needs an awareness of time within us. The perception (Brouwer 1981, pp. 4–5)

> of a move of time may be described as the falling apart of a life moment into two distinct things, one of which gives way to the other, but is retained by memory. If the twoity thus born is divested of all quality, it passes into the empty form of the common substratum of all twoities. And it is this common substratum, this empty form, which is the basic intuition of mathematics.

It is the First Act of Intuitionism.[27]

The Second Act of Intuitionism admits

> two ways of creating new mathematical entities: firstly in the shape of more or less freely proceeding infinite sequences of mathematical entities previously acquired (so that, for decimal fractions having neither exact values, nor any guarantee of ever getting exact values admitted); secondly, in the shape of mathematical species, i.e. properties supposable for mathematical entities previously acquired, satisfying the condition that if they hold for a certain mathematical entity, they also hold for all mathematical entities which have been defined to be 'equal' to it, definitions of equality having to satisfy the conditions of symmetry, reflexivity, and transitivity.

Mathematics is the intrinsically subjective activity of an individual mathematician's intuition.[28] The communication between mathematicians only serves as a means to create the same mental process in different minds.[29] The process of construction essentially depends on time. Any object has its date of "birth". It starts to exist after the process of construction is completed. To say "I have finished the mental construction of an object" it is like (Snapper 1984, p. 187):

> a bricklayer saying, "I have finished that wall," which he can say only when he has laid every stone in place.

For intuitionists, truth has a temporal aspect: a statement is true after its proof is constructed and false before the proof.

Brouwer rejected the use of formal systems entirely. They are irrelevant for mathematical reasoning. The role of language as a tool of mathematical cognition – indispensable in the case of the formalists – was questioned. Constructions are independent from language. Mathematics is a languageless creation of the mind. In (Brouwer 1933, p. 443)[30] we read:

> the languageless constructions originating by self-unfolding of the primordial intuition are, by virtue of their presence in memory alone, exact and correct; [...] the human power of memory, however, which has to survey these constructions, even when it summons the assistance of linguistic signs, by its very nature is limited and fallible. For a human mind equipped with an unlimited memory, a pure mathematics which is practiced in solitude and without the use of linguistic signs would be exact; this exactness, however, would again be lost in an exchange between human beings with unlimited memory, since they remain committed to language as a means of communication.

Such is the very nature of language which – according to Brouwer – is an irreparable imperfect means of communication and inevitably inadequate as a mode of description of mathematical constructions (Brouwer 1952, p. 510).[31]

Distinguishing discursive and intuitive[32] knowledge we see that, for Brouwer and his followers, mathematics is an intuitive knowledge based on the intuition of time whereas for the other traditional schools of philosophy it is a discursive knowledge.[33] In *Consciousness, Philosophy and Mathematics* (1948, p. 1240), Brouwer more or less repeats the message that "there is no exchange of thought either":[34]

> Thoughts are inseparably bound up with the subject. So-called communication-of-thoughts to somebody means influencing his actions. Agreeing with somebody means being contented with his cooperative acts or having entered into an alliance. Dispelling misunderstanding means repairing the wire-netting of will-transmission of some cooperation. By so-called exchange of thought with another being the subject only touches the outer wall of an automaton.

An intuitive intellect grasps objects of cognition immediately without the need for conceptualization and without the need of being affected by an object. In Kant's terminology, it is an archetypal or creative rather than echtypal intellect (Showler 2008). Objects of the creative intellect are created in the act of intuition. The act of intuition of an object is the same as the act of creation of this object. In such a case, to acquire a concept is the same as the creation of the object to which the concept is intended for.

Immanuel Kant answers "no" the question whether the world would exist if there were no intelligence capable of conceiving its existence. Similarly, there is no infinite collection of integers since we can never enumerate more than a finite number.

The concept of infinity as an expression of reality is itself disallowed in intuitionism, since the human mind cannot intuitively construct an infinite set (Snapper 1979). Strictly speaking, intuitionists should not discuss theses about the existence of actual infinity. For intuitionists any expression with words that do not signify any concept should not be understandable; in particular, this is the case with expressions in which the word "infinity" occurs. Any text with "infinity" is a meaningless combination of words. Intelligibility is not the property of a text in general.

Brouwer rejected the possibility of acquiring the concept of actual infinity by a human mind, but admitted the idea of potential infinity. Only arbitrarily long but finite initial segments of a sequence of objects are allowed. Closed sets of e.g., all natural numbers are not allowed. According to Weyl[35] (1946):

> Brouwer made it clear, as I think beyond any doubt, that there is no evidence supporting the belief in the existential character of the totality of all natural numbers [...] the sequence of numbers which grows beyond any stage already reached by passing to the next number, is a manifold of possibilities open towards infinity; it remains forever in the status of creation, but is not a closed realm of things existing in themselves. That we blindly converted one into the other is the true source of our difficulties, including the antinomies – a source of more fundamental nature than Russell's *vicious circle principle* indicated. Brouwer opened our eyes and made us see how far classical mathematics, nourished by a belief in the 'absolute' that transcends all human possibilities of realization, goes beyond such statements as can claim real meaning and truth founded on evidence. According to this view and reading of history, classical logic was abstracted from the mathematics of finite sets and their subsets. (The word *finite* is here to be taken in the precise sense that the members of such sets are explicitly exhibited one by one.) Forgetful of this limited origin, one afterwards mistook that logic for something above and prior to all mathematics, and finally applied it, without justification, to the mathematics of infinite sets. This is the Fall and Original sin of set theory even if no paradoxes result from it. Not that contradictions showed up is surprising, but that they showed up at such a late stage of the game!

Why are we not able to acquire the concept of infinity? The answer is: because we are not able to construct – if the construction follows the intuitionistic conditions – any infinite object, in particular any infinite set of natural numbers. The human mind is unable to construct an infinite collection as an existing totality (Placek 1999, p. 119):

> our inability to complete an infinite sequence is not a mere medical impossibility, as Russell once said, but it is the essential feature of infinity, which cannot be neglected in mathematical practice.

The potential character of infinity and the freedom of generation of choice sequences justify the rejection of an actual, completed infinity.

1. The properties of, e.g. numbers in a choice sequence (potentially infinite) do not determine properties of the sequence of the number as a finished totality. A choice sequence is an incomplete object, for it is never finished. Properties of elements do not decide about properties of its totality[36]
2. The Second Act allows freedom in the construction of infinite sequences.[37] A variety of a choice sequence depends on how much freedom is allowed by its constructor. There are two extreme cases (van Atten & van Dalen, 2002, pp. 517–518):
   - the lawless sequence, where there is no restriction whatsoever on future choices,
   - the lawlike sequence, where the future choice is generated by a law or algorithm.

   Due to the freedom of choices, properties of a longer sequence are undetermined and remain undefined. Something is completely finished only if it is definite.

Casual sequences could be divided with respect to a degree of egocity (van Atten & van Dalen 2002, pp. 212–213):

- Non-egoic sequences, such as physical objects, lend themselves to all the operations we are used to.
- The highly egoic-sequence, strongly depending on the will of the subject, can be expected to have its own peculiarities.

Formalists and Platonists view intuitionism as the most harmful philosophy of mathematics.

Platonism differs from intuitionism in admitting of the existence of a mathematical realm outside of the human mind and independently of it.

3.3. Hilbert and the formalist school

> God exists since mathematics is consistent, and the devil exists since we cannot prove it.
>
> André Weil[38]

The formalistic school legitimates classical mathematics allowing infinitistic concepts as characterized procedurally by a formal system.

To maintain that a concept contains a contradiction, the concept has to be intelligible. We suppose that a self-contradictory concept does not refer to any object. Thus to prove that there is not a certain object, it suffices to prove that the concept of the object, or the description of the object, is self-contradictory. E.g. there is no square circle, since the concept of a square circle is self-contradictory – one understands what "square circle" means and there is a proof that the concept of a square circle is self-contradictory. Somebody who does not understand the expression "square circle" should ask what the expression means rather than say that no object is a square and a circle at the same time.

To prove that there is no infinity, it is enough to prove that the concept of infinity is self-contradictory. If a concept is not self-contradictory, the existence of the object to which the concept refers (the concept of this object) is not excluded but not confirmed. We say that such an object is logically possible. The distinction between logically possible and logically impossible objects has to do with the idea of self-contradiction. An object, the description of which is self-contradictory, is a logically impossible object. All other objects are logically possible.

A proof of consistency is necessary to solve the question of the logical possibility of infinity. The only concepts of proof that are acceptable should not assume explicitly or implicitly the existence of infinity. In another case the proof of the consistency of the concept of infinity would be viciously circular. The proof of consistency of the concept of infinity should be finitistic. Using infinitistic methods of proving would be circular. Only finitistic logic is allowed. In a proof *regressus ad infinitum* should be avoided. Something should be taken as intuitively undoubted.

The last condition fulfills contentual elementary number theory, which relies only on a purely intuitive basis of concrete signs. It is a privileged part of mathematics. There is a realm of (Hilbert 1922, p. 202):[39]

> extra-logical discrete objects, which exist intuitively as immediate experience before all thought. If logical inference is to be certain, then these objects must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something which cannot be reduced to something else.

This finitistic mathematics is incontestable in every respect. The study of how these strings of symbols can be manipulated comprises what Hilbert termed *proof theory*. This was Hilbert's royal road to certainty.[40]

On top of this finitistic mathematics, the rest of classical mathematics is built. All mathematics should be placed on a completely reliable computational foundation. Infinite mathematics and infinity themselves could not be reduced to finitistic mathematics, but texts about classical mathematics (in general, any text with finite expressions) could be. In order to do so, a formal language is needed. Any sentence of classical mathematics should be convertible into formulas of the language. The intuitive-contentual operations with signs forms the basis of Hilbert's metamathematics. Just as contentual number theory operates with sequences of strokes, so metamathematics operates with sequences of symbols (formulas, proofs). Formulas and proofs can be syntactically manipulated according to certain rules comprehensible without recourse to any meaning of the formulas. To do so, mathematics should be presentable as a formal system.[41]

The properties and relationships of formulas and proofs are similarly based in a logic-free intuitive capacity which guarantees certainty of knowledge about formulas and proofs arrived at by such syntactic operations.[42] Proofs of mathematics conceived as (1930):

> a sequence of formulae each of which is either an axiom or follows from earlier formulae by a rule of inference

are subject to metamathematical, contentual investigation.[43] If we regard mathematics as a formal game, like a game of chess played with symbols, we should be able to show that the game is consistent. Weyl (1925), Hilbert's former student converted to intuitionism, described this project as replacing meaningful mathematics by a meaningless game of formulas. The goal of Hilbert's program is then to give a contentual, metamathematical proof that there can be no derivation of a contradiction, i.e., no formal derivations of a formula ϕ and of its negation $\neg\phi$. Hilbert aimed (1996):

> to eliminate once and for all the questions [...] by turning every mathematical proposition into a formula that can be concretely exhibited and strictly derived, thus recasting mathematical definitions and inferences in such a way that they are unshakable and yet provide an adequate picture of the whole science.

Hilbert's program[44] was abolished by Gödel. He proved that using only finitistic means, no proof of the non-contradiction of infinitistic mathematics is possible. Gödel (1931) in *Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme*[45] proved that Hilbert's program cannot be carried through. Nevertheless Gödel's theorem

> did not destroy the fundamental idea of formalism, but it did demonstrate that any system would have to be more comprehensive than that envisaged by Hilbert.[46]

Hilbert's program survived in some form, under the name of Proof Theory. The lack of the finistic proof of the consistency of infinitistic mathematics does not imply that the concept of infinity is self-contradictory or that infinity does not exist. The statement of the program was an immense contribution to our understanding of infinitistic mathematics. Infinitistic mathematics has been developed, even by mathematicians, who accept the formalistic point of view. The Gödel Incompleteness theorem does not nullify the hope that this infinitistic mathematics is consistent. This belief is justified by the fact that all contradictions that were encountered so far have been solved. Mathematicians of today acknowledge formalism as the basis for their work.

Knowledge could be divided into two basic parts: declarative (conceptual) and procedural, or representational and computational. The representational, contentual part of the knowledge encoded in an expression contains information about the representations to be manipulated. The procedural (computational) part encodes information about how to manipulate this expression. The first one, conceptual knowledge, provides an abstract understanding, the second one, enables problem solving. Procedural knowledge is the knowledge exercised in the accomplishment of a task. It comprises knowledge of formal language or symbolic representation, knowledge of rules, algorithms, and procedures. It is one type of knowledge that can be possessed by an intelligent agent. Justifications for these distinctions have been developed in both strictly linguistic and more broadly cognitive terms.[47] The relation between both is based on the distinction between knowing and doing.[48]

Due to the axiomatic system, as conceived by Hilbert and his formalist school, the procedural concepts of infinitistic mathematics, and in particular procedural concepts of infinity, are acquired. Due to this, it is knowable how to operate concepts of infinity in mathematics. Is the procedural part of the concepts of infinity complete? Gödel proved (in 1929) the completeness of system of first-order logic, that first-order logical consequence relation is captured by finite provability.[49] The rules of first-order logic account for all correct reasoning in first-order logic. Thus the logical concepts of first-order logic are completely procedurally characterized by any (complete) system of first-order logic. There is no ambiguity in the rules of deduction. But as a consequence of his Incompleteness Theorem, we know that no formal axiom system is, if consistent, such that all the truth of infinitistic mathematics are provable in it. Moreover, it is the subject matter of the Löwenheim-Skolem Theorem,[50] if a countable first-order theory has an infinite model, then it has a model of higher cardinality. No first-order axiom system can characterize a unique infinite model. Every first-order theory expression is either contradictory or satisfiable in a denumerable infinite domain (Skolem 1970). The fact was hard to accept by Skolem himself (Cohen 2005, p. 2417):

> Skolem, in his papers, was so struck by the existence of non-isomorphic models of all but the most trivial axiom systems that he was led to doubt the relevance of any mathematical axiom system to the philosophical questions concerning the foundations of mathematics.

We know how to operate concepts of infinite mathematics, but a formal axiom system does not provide the contentual part of concepts of infinitistic mathematics. This fact is expressed by a famous quote[51] of Hilbert's:

> One must be able to say at all times – instead of points, straight lines, and planes – tables, chairs, and beer mugs.

3.4. Cantor's idea of acquirement of concepts of infinity

The essence of mathematics lies in its freedom.
Georg Cantor[52]

Intuitionists reject the possibility of acquiring the concept of infinity by a human mind:[53] A necessary condition of acquirement of any concept is the creation by a human individual mind of the object for which the concept is intended; but the human individual mind is not able to create any infinite object. For formalists, concepts of infinity could be acquired only in its procedural aspects and as implicated, involved in an axiom system. The pragmatic argument is the most important reason for introducing the notion of infinity to mathematics – in Hilbert's words – to create a paradise of mathematicians. The third and final idea that we will discuss allows the acquiring of concepts of infinity by a human mind. This position is typical for supporters of the existence of (actual) infinity. Georg Cantor was the originator of this position and the most distinguished adherent of the idea of the possibility of acquiring the concept of infinity by a human mind. In Russell's opinion (Russell 1903, p. 307):

> The mathematical theory of infinity may almost be said to begin with Cantor. [...] Cantor has established a new branch of Mathematics, in which, by mere correctness of deduction, it is shown that the supposed contradictions of infinity all depend upon extending, to the infinite, results which, while they can be proved concerning finite numbers, are in no sense necessarily true of all numbers.

Mittag-Leffler wrote to Cantor that his transfinites were as revolutionary as non-Euclidean geometry (Dauben 1979, p. 138). Joseph Dauben (1992, pp. 59–60), the biographer of Cantor, argues that Cantor's theory of infinity was a revolution[54] not only in mathematics, but in metaphysics, too. Due to Cantor's efforts to vindicate the concept of actual infinity, the Aristotelian and scholastic tradition seems to have been overcome.

We ask how the human mind is able to acquire concepts of infinity.

3.4.1. Symbols for infinities

The sign "∞" (sometimes called the "lemniscate") is usually used as the name of potential infinity. The symbol is credited to English mathematician and theologian John Wallis (1616–1703)[55] It first appeared in his paper *Tractatus de sectionibus conicis* (1659; Tract on Conic Sections)[56] We read (p. 4):

> quorum quidem singulorum altitudo sit totius altitudinis $1/\infty$, sive alicuota pars infinite parva; (esto enim ∞ nota numeri infiniti;) adeo/q; omnium simul altitude aequalis altitudini figurae.

The "lazy eight" was used later in his more influential work *Arithmetica infinitorum*[57]; for instance, he wrote (p. 70):

> Cum enim primus terminus in serie Primanorum sit 0, primus terminus in serie reciproca erit ∞ vel infinitus: (sicut, in divisione, si diviso sit 0, quotiens erit infinitus).

Wallis never explained why "his" symbol for infinity was shaped as such. Nevertheless let us remark that it is a curve that can be traced out infinitely many times.

In *From Zero to Lazy Eight* (Humaez, Humez & MaGuire 1994) we read:

> Wallis was a classical scholar and it is possible that he derived the symbol for infinity from the old Roman sign for 1,000, **CIↃ**, also written **M** – though it is also possible that he got the idea from the lowercase omega, omega being the last letter of the Greek alphabet and thus a metaphor of long standing for the upper limit, the end.

Cajori (1993, vol. 2, p. 44) attributes this conjecture to Wilhelm Wattenbach (1819–1897) (1872, Appendix: p. 41). It seems to have been tacitly accepted by Karl Menninger when he remarked (1969, Fig. 73):

> For 1000 the Romans could also use the curious form ∞, which ever since the English mathematician Wallis proposed it in 1655 has been accepted as the mathematical symbol for infinity.

This conjecture is lent credence by the labels inscribed on a Roman hand abacus stored at the Bibliothèque Nationale in Paris. A plaster cast of this abacus is shown in a photo on page 305 of the English translation of Karl Menninger's *Number Words and Number Symbols* (1969); at the time, the cast was held in the Cabinet des Médailles in Paris. The photo reveals that the column devoted to 1000 on this abacus is inscribed with a symbol quite close in shape to the lemniscate symbol, and which Menninger shows could easily have evolved into the symbol **M**, the eventual Roman symbol for 1000.[58]

The symbol "∞" and lower case omega were used by Cantor. "Lazy eight" we can find in *Über unendliche, lineare Punktmannigfaltigkeiten* (1880, p. 357). About two years later, instead of "∞" Cantor introduced his lower case omega (1883*b*, p. 577):

> Um diese Verwechselung von vornherein auszuschließen, bezeichne ich die kleinste transfinite Zahl mit einem von dem gewóhnlichen, den Sinne des Uneigentlich-unendlichen entsprechenden Zeichen ∞ *verschiedenen* Zeichen, nämlich mit ω.

This shift in notation meant a remarkable conceptual change.

The aleph null symbol was conceived by Georg Cantor around 1893, and became widely known after the publishing of *Beiträge zur Begründung der transfiniten Mengenlehre* (1895, p. 492):

> wir nennen die ihr zukommende Cardinalzahl, in Zeichen, $\aleph_0$.
>
> We call the cardinal number related to that (set); in symbol, $\aleph_0$.

Cantor did not want to use the Roman or Greek alphabets, because they were already widely used. Aleph is the first letter of the Hebrew alphabet. According to Dauben (1979, p. 179):

> His new numbers deserved something unique. [...] Not wishing to invent a new symbol himself, he chose the aleph, the first letter of the Hebrew alphabet [...] the aleph could be taken to represent new beginnings.

Aleph is also the first letter of the Hebrew word "Einsof," which means infinity; the Kabbalists use "einsof" for the Godhead. The aleph numbers differ from the ∞: alephs measure the sizes of sets; ∞ is an extreme limit of the real number line.[59]

3.4.2. Meanings of "infinity"

Cantor considered various notions of infinity.[60] He aimed to precise definitions of meanings of "infinity"; first of all to determine the most important difference – that is, between potential and actual infinities (Dewender 2002, p. 124).

In *Grundlagen einer allgemeinen Mannigfaltigkeitslehre: Ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen* (1883*a*) Cantor distinguishes between Uneigentlich-Unendliche (improper-infinity) and Eigentlich-Unendliche (proper-infinity) (1932, p. 165–166): The improper-infinity is in fact a changeable finite and not any definite value (Cantor 1887):

> The potential infinite means nothing other than an undetermined, variable quantity, always remaining finite, which has to assume values that either become smaller than any finite limit no matter how small, or greater than any finite limit no matter how great.

Contrary to this, the proper-infinity appears in definite form. In the name of potential infinity it would be better to abandon the word "infinity" (Cantor 1932, p. 404).

For Cantor (1932, Über die verschiedenen Standpunkte in Bezug auf das Aktual-Unendliche, pp. 370–377, p. 374):

> Trotz wesentlicher Verschiedenheit der Begriffe als *potentialen* und *aktualen* Unendlichen, indem ersteres eine *veränderliche* endliche, über all endlich Grenzen hinaus *wachsende* Größe, letzteres ein *in sich festes, konstantes*, jedoch jenseits aller endlichen Größen liegendes Quantum bedeutet, tritt doch leider nur zu oft der Fall ein, daß das eine mit dem anderen verwechselt wird.

Cantor distinguished also two notions of actual infinity: transfinite and absolute (1932, Über die verschiedenen Standpunkte in Bezug auf das Aktual-Unendliche, pp. 370–377, p. 375):

> Eine *andere* häufige *Verwechselung* geschieht mit beiden Formen des *aktualen* Unedlichen, indem nämlich das *Transfinite* mit dem *Absoluten* vermengt wird, während doch diese Begriffe streng geschieden sind, insofern ersteres ein *zwar Unendliches*, aber doch *noch Vermehbares*, das letztere aber wesentlich als *unvermehrbar* und daher mathematisch *undeterminierbar* zu denken ist; [...]

The distinctions are subject of further explanation (Cantor 1932, Mitteilungen zur Lehre vom Transfiniten, pp. 378–439, p. 378):

> Es wurde das A.-U.[61] nach *drei* Beziehungen unterschieden: *erstens* sofern es in der höchsten Vollkommenheit, im völlig unabhängingen, außerwetlichen Sein, *in Deo* realisiert ist, wo ich es *Absolutunendliches* oder Kurzweg *Absolutes* nenne; *zweitens* sofern es in der abhängigen, kreatürlichen Welt vertreten

> ist; *dritens* sofern es als mathematische Größ, Zahl oder Ordnungstypus vom Denken *in abstracto* aufgefaßt werden kann. In den *beiden* letzten Beziehungen, wo es offenbar als beschränktes, noch weiter Vermehrung fähiges und *insofern dem Endlichen verwandtes* A.-U. sich darstellt, nenne ich es *Transfinitum* und setze es dem *Absoluten* strengstens entgegen.

Let us remark that "Absolute Infinity"[62] is conceived by Cantor as a quantitative concept (not as qualitative). It differs from the transfinite only in that it is non-augmentable (vermehbar).[63]

In investigation of Cantor's considerations it should not be omitted that (1979, p. 291):

> The theological side of Cantor's set theory, though perhaps irrelevant for understanding its mathematical content, is nevertheless essential for the full understanding of his theory and the developments he gave it.

The absolute infinity, the source of all other infinities, leads directly to God, being just the divine nature of this absolute infinitude which makes it inconsistent for human minds (Cantor 1883*a*).

For Cantor (1887, 1888):[64]

> the potential infinite is only an auxiliary or relative (or relational) concept, and always indicates an underlying transfinite without which it can neither be nor be thought.

The notion of potential infinity is important in mathematics, for instance it enabled mathematicians to develop the concept of a limit.[65]

The three basic meanings of "infinity" are different and should not be confused, though they are, even the "theological" meaning (Neidhart 2008, p. 623), in mutual close dependence on one another.

Cantor was a realist: abstract entities, in particular sets, are there for us to discover, not to be constructed.

The name "infinity" is not empty in any of its meanings. There are three levels of existences:

1. *in Intellectu Divino* (in the mind of God);
2. *in abstracto* (in the mind of man); and,
3. *in concreto* (in the physical universe).

The Absolute Infinite exists only in the mind of God. Cantor maintains that also numbers, transfinite and cardinal, exist in the highest level of reality because of eternity as Ideen *in intellectu Divino* (Tapp 2005, Letter to Jeiler, 13.10.1895, p. 427). But what about infinity in *in natura creata*?

In *Mitteilungen zur Lehre vom Transfiniten* (1887) we read (Cantor 1932, p. 405):

> Das *Transfinite* mit seiner Fülle von Gestaltungen und Gestalten weist mit Nothwendigkeit auf ein *Absolutes* hin, auf das "wahrhaft Unendliche", an dessen Größe keinerlei Hinzufügung oder Abnahme statthaben kann und welches daher quantitativ als *absolutes* Maximum anzusehen ist. Letzteres übersteigt gewissermassen die menschliche Fassungskraft und entzieht sich namentlich mathematischer Determination; wogegen das *Transfinite* nicht nur das weite Gebiet des Möglichen in Gottes Erkenntnis erfüllt, sondern auch ein reiches, stets zunehmendes Feld idealer Forschung darbietet und meiner Überzeugung nach auch in der Welt des Geschaffenen bis zu einem gewissen Grade und in verschiedenen: Beziehungen zur Wirklichkeit und Existenz gelangt, um die Herrlichkeit des Schöpfers, nach dessen absolut freiem Rathschluß, stärker zum Ausdrucke zu bringen, als es durch eine bloß "endliche Welt" hätte geschehen können. Dies wird aber auf allgemeine Anerkennung noch lange zu warten haben, zumal bei den Theologen, so werthvoll auch diese Erkenntnis als Hilfsmittel zur Förderung der von ihnen vertretenen Sache (der Religion) sich erweisen würde.

Numbers exist in their intra-subjective reality in the human mind (*in abstracto*) as well as in their trans-subjective reality (*in concreto*). The concrete infinite is found in nature[66] and the abstract infinite is found in mathematics.

Cantor (1932, p. 404)[67] maintains that:

> in truth the potentially infinite has only a borrowed reality, insofar as a potentially infinite concept always points towards a logically prior actually infinite concept whose existence it depends on.

Each potential infinite presupposes the existence of an actual infinity (Hallet 1984, p. 25), (Moore 2001, p. 117). The thesis that potential infinity is ontologically dependent on actual infinity is justified by the fact that potential infinity is a variable (Hallet 1984, p. 25):

> In order for there to be a *variable* quantity in some mathematical study, the 'domain' of its variability must strictly speaking be known beforehand through a definition. However, this 'domain' cannot itself be something variable, since otherwise each fixed support for the study would collapse. Thus this domain is a definite, actually infinite set of values.
>
> Hence each potential infinite, if it is rigorously applicable mathematically, presupposes an actual infinite.

3.4.3. Acquiring the concept of infinity

Cantor was aware of the problem of acquiring of concepts (1991, p. 6):

> in Cantors späteren Briefen finden wir Äußerungen von höher erkenntnistheoretischer Bedeutung. So mußte er sich immer wieder mit dem Einwand auseinandersetzen, daß seine "transfiniten Zahlen" ganz andere Eigenschaften haben als die natürlichen. Cantor räumte das ein; aber er wußte auch hinzuzufügen, daß man bei einer Verallgemeinerung von Begriffsbildungen nicht erwarten könne, daß bei Erweiterung von Definitionsbereichen alles beim Alten bleiben würde. Damit sprach er eine Erkenntnis aus, die für die vielen Erweiterungen mathematischer Begriffsbildungen im zwanzigsten Jahrhundert wichtig werden sollte.

Cantor believed that the transfinites had been revealed to him. Moreover he felt a calling to spread the truth about God's creation of infinities for the benefit of both the Church and the world (1979, p. 291). In a letter (1991, pp. 523–527, facsimile) to a Dominican priest Esser, February 15, 1896, Cantor wrote (1991, p. 526):

> Von mir wird der christlichen Philosophie zum ersten Mal die wahre Lehre vom Unendlichen in ihren Anfängen dargeboten.
>
> From me, Christian philosophy will be offered for the first time the true theory of the infinite.[68]

There are two ways to consider objects: successively or simultaneously. The first leads to the concept of number,[69] and the second to the concept of set – the human mind has the ability to recognize groups, natural or abstract. Infinite concepts cannot be acquired by finite means and on the basis of finite experiences: all things we have been capable of observing and measuring are finite.

To acquire the concept of potential infinity seems to suffice the experience of lack of limits: for any number there is a number greater than it, for any moment of time there is a moment later than it, for any segment there is a segment smaller than it, etc. Is this experience sufficient to acquire the concept of actual infinity?

If cognitive dependence is the same as ontological dependence, potential infinity is cognitively dependent on actual infinity, since potential infinity is ontologically dependent on actual infinity. Thus the experience of potential infinity cannot be the means of acquiring the concept of actual infinity. Conversely, the concept of actual infinity is necessary to acquire the concept of potential infinity.

The human mind cannot embrace actual infinity. For instance, it can embrace (at most) any rational number (a pair of natural numbers). There are $\aleph_0$ rational numbers. But what with the real numbers that are not rational?[70] The name of a real number as an infinite sequence of digits cannot be embraced by a human mind. Some real numbers can be described finitely by means of an algorithm, but there are such numbers for which it cannot be done (Chaitin 1966, Chaitin 1987, Chaitin 1997, Chaitin 2004, Trzęsicki 2006). If real numbers which are not finitely described exist, then real numbers are not a human mind-dependent entity. Thus the human mental experience could not be the source of acquiring the actual infinity (of real numbers).

Are there any experiences due to which we are able to acquire concepts of infinity?

Cantor introduced into mathematics the notion of a completed set. Integers can be considered together as a set in themselves. The first infinity, the first transfinite number, which Cantor denoted by a lower case omega is a result of conceiving integers as a whole entity (as a Ding für sich). But how are we able to have knowledge about infinite objects?

Cantor maintains that even God is unable to have direct knowledge about infinity. He quotes Augustine's argument that[71]

> all infinity is in some ineffable way made finite to God, for it is comprehended by his knowledge.

Is the human mind, similarly to God, able to experience the infinite as finite? Cantor argues that God instilled the concept of number, both finite and transfinite, into the mind of man (Cantor 1991, p. 275f):

> sowohl getrennt als auch in ihrer aktual unendlichen Totalität als ewige Ideen *in intellectu Divino* im höchsten Grad der Realität existieren.

This means that transfinite numbers exist in the mind of man, as eternal ideas exist in the mind of God (Dauben 1979, *Cantor's Correspondence with Hermite Concerning the Nature and Meaning of the Transfinite Numbers*, pp. 228–232). God put them into man's mind to reflect his own perfection (Dauben 1979, p. 126). Infinite concepts are innate.

Conclusions

The main aim of Cantor's philosophical work was to provide arguments to support the legitimate mathematical employment of transfinite numbers

(Hallet 1984, p. 124). He emphasized plainly and constantly that all transfinite objects of his set theory are based on actual infinity. Cantor's explanation of acquiring infinite concepts is irrelevant for mathematicians. They solved the problem in axiomatic set theory. The existence of sets is assumed axiomatically; in particular, there are axioms that 'create' infinite sets, e.g.: the axiom of infinity, the axiom of a power set. Nevertheless (Cohen 2005, p. 2410):

> A recurring concern has been whether set theory, which speaks of infinite sets, refers to an existing reality, and if so how does one 'know' which axioms to accept. It is here that the greatest disparity of opinion exists (and the greatest possibility of using different consistent axiom systems).

The question how an individual human mind acquires infinite concepts – if there are any – is still an open problem of cognitive science.

NOTES

[1] *In mathematics the art of asking questions is more commonly applied than that of solving problems.* Quoted in (Ulam 1991). It is the third of the theses which Cantor defended in oral examination to obtain the degree of doctor in 1867. The first two were:

- In arithmetica methodi mere arithmeticae analyticis longe praestant.
- Num spatii ac temporis realitas absoluta sit, proper ipsam controversiae questionem pluris facienda est quam solvendi.

Cantor's dissertation was entitled *De aequationibus secundi gradus indeterminatis*, pp. 26. The dissertation was described as "disertatio docta at ingeniosa" and in the oral examination he got the remark "magne cum laude".(Prasad 1993, chapter 13)

[2] The English word "infinity" derives from Latin "infinitas", meaning "being without finish", and which can be translated as "unboundedness", itself calqued from the Greek word "apeiros", meaning "endless". See http://en.wikipedia.org/wiki/Infinity, http://www.etymonline.com/index.php?allowed_in_frame=0&search=infinity&searchmode=none. Retrieved 2014–07–30.

[3] I.e., is the old proposition "Totum est majus sua parte" true?

[4] Quoted in (Newman 1956, Maor 1987).

[5] It was a negative, even a pejorative, word. The idea of *apeiron* has linked contemporary physicists (Simonyi 2012, p. 546):

> Heisenberg thus arrived at the idea that the elementary particles are to be seen as different manifestations, different quantum states, of one and the same "primordial substance." The elementary particles, it would follow, are the only possible manifestations of matter. Because of its similarity to the primordial substance hypothesized by Anaximander, Born called this substance *apeiron*.

[6] Set is conceived by Cantor (1932, p. 204) as

> a Many that allows itself to be thought of as a One.

In 1895 the definition was restated (1932, p. 282):

> By a 'set' we mean any gathering into a whole M of distinct perceptual or mental objects m (which are called the 'elements' of M).

[7] Into geometry the concept of this infinite point was introduced in the 17th century by Gérard Desargues. It was already known in art.

[8] See: A Tour of the Summa by Paul J. Glenn. The Infinity of God http://www.catholic theology.info/summa-theologica/summa-part1.php?q=25, 30.062014.

[9] See: A Tour of the Summa by Paul J. Glenn. The Infinity of God http://www.catholic theology.info/summa-theologica/summa-part1.php?q=25, 30.062014.

[10] Quoted in (Maor 1987, p. 179).

[11] See http://www.faculty.umb.edu/gary_zabel/Courses/Phil%20100-08/Immortality %20and%20Philosophy%20of%20Mind/Principles%20of%20Philosophy.htm.

[12] See also (Jacquette 2000, pp. 193–194).

[13] His texts on it were published posthumously under the title *Paradoxien des Unendlichen* (*Paradoxes of Infinity*) (1851).

[14] See (Waldegg 2005).

[15] Cantor's theory of sets posed a difficult problem, not for the concept of infinity, but for the fundamental concept of set.

[16] Cited in (Bell 2014, p. 556). See (Gauss & Schumacher 1860–1865, Bd. II, p. 268):

> so protestiere ich zuvörderst gegen den Gebrauch einer unendlichen Größe als einer vollendeten, welcher in der Mathematik niemals erlaubt ist. Das Unendliche ist nur eine *facon de parler*, indem man eigentlich von Grenzen spricht, denen gewisse Verhältnisse so nahe kommen als man will, während anderen ohne Einschränkung zu wachsen gestattet ist.

[17] Cantor had a set of numinous feelings about the infinite. He equated the concept of Absolute Infinite with God. Cf. (1932, p. 378)

> The actual infinite arises in three contexts: *first* when it is realized in the most complete form, in a fully independent otherworldly being, in Deo, where I call it the Absolute Infinite or simply Absolute; *second* when it occurs in the contingent, created world; *third* when the mind grasps it *in abstracto* as a mathematical magnitude, number or order type.

The notion of Absolute Infinity was introduced as early as 1882 along with the ordinal theory of cardinality. Quoted in (Rucker 2007, p. 9).

[18] In (Bell 2014, p. 558) we read:

> At the International Mathematical Congress of 1918 at Rome, the satiated physician delivered himself of this prognosis: "Late generations will regard *Mengenlehre* as a disease from which one has recovered.

[19] In the introduction to *Das Kontinuum* (1918, p. III) we read:

> Hier wird vielmehr die Meinung vertreten, daß jenes Haus zu einem wesentlichen Teil auf Sand gebaut ist.

[20] Quote in (Dauben 1979, p. 96).

[21] Ludwig Wittgenstein replied (1978, p. 264):

> if one person can see it as a paradise of mathematicians, why should not another see it as a joke?

[22] The physical universe as a whole is neither embedder nor is embedded in a network of casual relations.

[23] More for abstract objects see (Hale 1987).

[24] See e.g. (Kousta, Andrews, Vinson, Vigliocco & Del Campo 2012).

[25] Poets in truth are we in mathematics, but our creations also must be proved.

[26] The idea of construction is various in different schools. In constructive mathematics definitions, proofs and theorems should be entirely constructive. From constructive definitions and proofs algorithms should be extractable that compute the objects and simulate they construction.

[27] This text and the text describing the Second Act of Intuitionism are subjects of many quotations in very diverse contexts, as e.g., (Murk 2009, p. 60) *The Dreambook of Sklyer Dream* by Murk – an alchemical force of nature from New Mexico. Skyler Dread's experiments with dreams & mesmerism & his amber hookah full of potent hash fails to save the day, even though they were humanity's last hope; *In Defense of Intuition: A New Rationalist Manifesto* (Chapman, Ellis, Hanna, Hildebrand & Pickford 2013, p. 296); *What is Mathematics, Really?* (Hersh 1997, p. 154).

[28] The question of the role played by intuition in mathematics in the context of computer mathematics is deliberated, e.g. in (Barendregt & Wiedijk 2005).

[29] More about the question of intersubjectivity of intuitionism, see (Placek 1999).

[30] Quoted in (Tieszen 1994, p. 580).

[31] For more about the place of language in intuitionistic cognition, see (van Dalen, 1999).

[32] Intuition is not conceived as prior to reason or outside of reason. It is the activity of reason itself. More about rational intuition and its role in mathematics, logic, and philosophy, see (Chapman et al. 2013).

[33] It does not mean that intuition is not helpful in the discovery and development of mathematics. It plays a certain role at the initial stage of mathematical cognition.

[34] Quoted in (van Dalen 1998, p. 18).

[35] Cf. Kleene (1967, pp. 48–49).

[36] The fallacy of composition arises if the fact that all elements of a set (or all parts of an object) have a property P is the reason of prediction P to the set (or to the whole object). The fallacy of composition – according to intuitionists – take place if the law of the excluded middle is applied to infinite sets on the reason that it is applicable to finite sets. Let us remark that similarly formalists do not apply it to the set of theorems conceived as a set of (already) proved statements, even to the set of a provable statement (if the formal system is incomplete). The interpretation in terms of proofs was made by (Heyting 1934). Kolmogorov gave an interpretation in terms of problems and solutions.

[37] In this fact lies the essential difference between intuitionism and other constructive views of mathematics according to which mathematical objects and arguments should be computable.

[38] In this way André Weil, one of the leading mathematicians of XX century, characterized mathematics (Meschkowski 1938, p. 112), (Rosenbloom 1950, p. 72).

[39] Quoted in http://plato.stanford.edu/entries/hilbert-program/. The passage is repeated almost verbatim in (Hilbert 1926) (van Heijenoort 1967, p. 376), (Hilbert 1928, p. 464) and (Hilbert 1931, p. 267).

[40] For more about Hilbert's Program see (Zach 2006).

[41] The notion of formal system was given definitive form with the publication of Frege's epic work *Begriffschrift* (1879).

[42] Frege's *Begriffschrift* (1879) not only included a description of the language (which we might nowadays call the *machine language*), but also a description of the rules for manipulating this language.

[43] This definition of proof is important for metamathematical investigations. Mathematicians write rigorous proofs that are convincing arguments for mathematicians, in whose soundness the mathematical community has confidence. Such a proof, contrary to Hilbertian proof, the correctness of which can be readily automated-checked, is not secured to a fault. The history of mathematics is full of erroneous proofs. Attempts to create such a definition of formal proof that would be in agreement with intuitive rigorous argumentation of mathematicians gave origin to the natural systems of logic of Jaśkowski (Jaśkowski 1934) and Gentzen (Gentzen 1934). MIZAR is a proof-checker that checks the correctness of mathematical proofs written in an (almost) natural way http://www.mizar.org/.

[44] Hilbert's metamathematical program comprises three fundamental problems (Hilbert 1929):

1. Is mathematics complete; i.e. can every mathematical statement be either proved or disproved?
2. Is mathematics consistent; that is, is it true that statements such as "0 = 1" cannot be proved by valid methods?
3. Is mathematics decidable; that is, is there a mechanical method that can be applied to any mathematical assertion and (at least in principle) eventually tell whether that assertion is true or not?

This last question was called the *Entscheidungsproblem*.

[45] The Incompleteness Theorems ended a hundred years of attempts to put the whole of mathematics on an axiomatic basis.

[46] See http://www-gap.dcs.st-and.ac.uk/history/Biographies/Godel.html.

[47] More about distinguishing conceptual and procedural meaning/knowledge see. e.g., (Escandell-Vidal, Leonetti & Ahern 2011).

[48] See, eg., (Ryle 1949, Hiebert 1986).

[49] Some regard this theorem as implicit in Skolem's work (1922); but the result itself was not mentioned there.

[50] Löwenheim–Skolem Theorem was the first truly important discovery about formal systems in general, and it remains probably the most basic.

[51] More about this quotation, see: http://math.stackexchange.com/questions/56603/provenance-of-hilbert-quote-on-table-chair-beer-mug, 06.07.2014.

[52] See: (1932, p. 182).

[53] For intuitionists the phrase "concept of infinity" does not make any sense.

[54] In the sense of (Kuhn 1962).

[55] Before Isaac Newton, John Wallis is unquestionably the most important English mathematician.

[56] Available online http://books.google.pl/books/about/De_sectionibus_conicis_nova_methodo_expo.html?id=03M_AAAAcAAJ&redir_esc=y. Retrieved 09.08.2014.

[57] Available online https://archive.org/stream/ArithmeticaInfinitorum#page/n5/mode/2up. Retrieved 09.08.2014.

[58] Cf. http://www.kermitrose.com/math/history/infinity. Retrieved 30.07.2014.

[59] For more on the choice of symbols and associations with Jewish culture, see Aczel *The Mystery of the Aleph. Mathematics, the Kabbalah, and the Search for Infinity* (2000).

[60] Cf. (Décaillot 2011, Anhang 2, Die verschiedene Bedeutungen des Begriffes "unendlich" in der Mathematik).

[61] Aktual-Unendliche

[62] There are two notions of "absolute infinity": mathematical and theological. More about relations between the two notions see (Tapp 2012).

[63] Maybe Pesch's definition (Pesch 1883, ¶ 403) of infinity as "id, quo non sit maius nec esse possit" (that than which there is nothing bigger or could be) influenced Cantor to take non-augmentability as the characteristic property of absolute infinity (Tapp 2012, p. 5).

[64] Quoted in (Hallett 1986, p. 25).

[65] The concept of limit is basic for the theory of calculus.

[66] The infinite he identified with Spinoza's "natura naturata" (which inspired a controversy about the pantheism of Cantor's views). Following Leibniz, Cantor thought that there were a transfinite number of elementary units: corporeal (matter) and ethereal (ether) monads. Cantor's views are closer to Leibniz than Spinoza. For more see (Newstead 2009).

[67] Quoted in (Rucker 2007, p. 3).

[68] See (Meschkowski 1965). Quoted in *Cantor's Concept of Infinity: Implications of Infinity for Contingence* by the reverend Bruce A. Hedman, Ph.D. http://www.asa3.org/ASA/PSCF/1993/PSCF3-93Hedman.html. Retrieved 06.08.2014.

[69] This experience enables the acquiring of the concept of a natural number. Let us remember that for Kronecker, one of Cantor's most hard-fought opponents:

> Die ganzen Zahlen hat der liebe Gott gemacht, alles andere ist Menschenwerk
>
> God made natural numbers; all else is the work of man.

Quoted by Weber (Weber 1893, p. 19). He said this in a lecture for Berliner Naturforscher-Versammlung in 1886.

[70] There are $\aleph_0$ rational numbers. The set of all real numbers is much greater than the set of rational numbers. There are $\mathfrak{c}$, or – assuming continuum hypothesis – $2^{\aleph_0}(= \aleph_1)$ real numbers.

[71] In Augustine's *City of God*, book 12, chapter 18 *Against Those Who Assert that Things that are Infinite Cannot Be Comprehended by the Knowledge of God* we read (Schaff 1890, Book 12, chapter 18, p. 345–346):

> Far be it, then, from us to doubt that all number is known to Him "whose understanding," according to the Psalmist, "is infinite." The infinity of number, though there be no numbering of infinite numbers, is yet not incomprehensible by Him whose understanding is infinite. And thus, if everything which is comprehended is defined or made finite by the comprehension of him who knows it, then all infinity is in some ineffable way made finite to God, for it is comprehensible by His knowledge.

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