



On Imprecise Investment Recommendations

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Abstract. The return rate is considered here as a fuzzy probabilistic set. Then the expected return is obtained as a fuzzy subset in the real line. This result is a theoretical foundation for new investment strategies. All considered strategies result of comparison profit fuzzy index and limit value. In this way we obtain an imprecise investment recommendation. Financial equilibrium criteria are a special case of comparison of the profit index and the limit value. The following criteria are generalized here: the Sharpe's Ratio, the Jensen's Alpha and the Treynor's Ratio. Moreover, the safety-first criteria are generalized here for the fuzzy case. The Roy Criterion, the Kataoka Criterion and the Telser Criterion are also generalized. Obtained results show that proposed theory is useful for the investment applications.

1. Research problem

In the interest theory, the present value (PV) is defined as discounted cash flow. In general case, PV may be defined as the utility of cash flow (Piasecki, 2012). Behavioral reasons cause subjective nature of this utility (Dacey, Zielonka, 2005). Imprecise estimation of PV is a result of the subjective approach to security valuation (Barberis et al., 1998). This is the main reason for the description of present value as a fuzzy number.

The suggestion to present the financial statements as fuzzy sets derives from (Ward, 1985), (Calzi, 1990). Ward (1995) defines fuzzy PV as discounted fuzzy cash flow. The fuzzy cash flow used here is interpreted as an imprecise forecast of future crisp cash flow. The Ward's definition is generalized to the case of fuzzy duration in (Greenhut et al., 1995). Sheen (1995) generalizes Ward's definition to the case of fuzzy interest rate. Some problems connected with the application of fuzzy arithmetic for calculating fuzzy PV are discussed in (Buckley, 1987, 1992), (Gutierrez, 1989), (Kuchta, 2000) and (Lesage, 2001). Huang (2007) generalizes Ward's definition for the case when future cash flow is given as a fuzzy random variable (Kwaker-naak, 1978, 1979). A more general definition of fuzzy PV is proposed by

Tsao (2005) who assumes that future cash flow is a fuzzy probabilistic set (Hiroto, 1981). In (Piasecki, 2011a, b) fuzzy PV is defined as a fuzzy number dependent on current market price, equilibrium price and sentiment index (Edwards, 1968), (Barberis et al., 1998). Next, in (Piasecki, 2013) the last model was generalized to the case when PV is described by the fuzzy number dependent only on the market price.

In agreement with uncertainty thesis (Mises, 1962), (Kaplan et al., 1967), each future cash flow is under uncertainty. Then future value (FV) is usually defined as a random variable. Therefore, if PV is a fuzzy number then return rate is the fuzzy probabilistic set (Hiroto, 1981). For this case, expected return rate is given as fuzzy subset in the real line \mathbb{R} (Piasecki, 2011b, c). This means that the financial market states are described imprecisely. It implies that investment goals are formulated imprecisely (Fang Yong; et al., 2008). Each security with fuzzy probabilistic return is at imprecision risk and at uncertainty risk. The knightian uncertainty (Knight, 1921) is excluded of course.

The main goal of this article is consideration of investment recommendations given for securities with fuzzy probabilistic return. We will consider the financial equilibrium criteria and the safety-first criteria. In this way, when setting investment goal we will be able to take into account the imprecision risk.

2. Fuzzy probabilistic return

Let us assume that the time horizon $t > 0$ of an investment is fixed. Then, any security is determined by two values:

- anticipated future value $V_t \in \mathbb{R}^+$,
- assessed present value $V_0 \in \mathbb{R}^+$.

The basic characteristic of benefits from owning this instrument is a rate of return $r \in \mathbb{R}$ given by the relationship

$$r = r(V_0, V_t). \tag{1}$$

In the general case, the function $r : \mathbb{R}^+ \times \mathbb{R}_0^+ \rightarrow \mathbb{R}$ is strictly decreasing function of present value. It implies that for any fixed future value V_t we can determine inverse function $r_0^{-1}(\cdot, V_t) : \mathbb{R} \rightarrow \mathbb{R}^+$.

The future value is at risk of uncertainty. A formal uncertainty model is the presentation of future value V_t as a random variable $\tilde{V}_t : \Omega \rightarrow \mathbb{R}^+$, where $\Omega = \{\omega\}$ is a set of financial market states. In the classical approach to the problem of return rate determination, security PV is identified with

the current market price \check{C} . Then the return rate is a random variable which is at uncertainty risk. This random variable is determined by the identity

$$\tilde{r}(\omega) = r(\check{C}, \tilde{V}_t(\omega)). \quad (2)$$

In practice of financial markets analysis, the uncertainty risk is usually described by probability distribution of return rates. Let us assume that this probability distribution is given by cumulative distribution function $F : \mathbb{R} \rightarrow [0; 1]$. In the case of Warsaw Stock Exchange, the normal inverse Gaussian (NIG) distribution is the best matching probability distribution of logarithmic returns (Piasecki, Tomasik, 2013). For any random variable with NIG distribution its variance exists (Bølviken, Benth, 2000). On the other hand, any return rate is increasing function of logarithmic return (Piasecki, 2013). Thus we can assume that return variance σ^2 exists. Then, the uncertainty risk may be characterized by this variance.

The probability distribution of FV is described by conditional cumulative distribution function $F_V(\cdot|\check{C}) : \mathbb{R}^+ \rightarrow [0; 1]$ determined in the following way

$$F_V(x|\check{C}) = F(r(\check{C}, x)). \quad (3)$$

The PV of security is approximately equal to security market price \check{C} . Thus it may be at imprecision risk. Then the security PV is described by fuzzy number (Dubois, Prade 1979) dependent on the market price \check{C} . Each PV membership function $\mu(\cdot|\check{C}) : \mathbb{R}^+ \rightarrow [0; 1]$ fulfils the following properties

$$\mu(\check{C}|\check{C}) = 1, \quad (4)$$

$$\forall_{x,y,z \in \mathbb{R}^+} : x \leq y \leq z \Rightarrow \mu(y|\check{C}) \geq \min\{\mu(z|\check{C}), \mu(x|\check{C})\}. \quad (5)$$

Each investor takes into account the lowest possible market price and the highest one. The security PV should be greater than the lowest possible price. Also, the security PV should be less than the highest possible price. Therefore, about any PV membership function $\mu(\cdot|\check{C}) : \mathbb{R}^+ \rightarrow [0; 1]$ we assume additionally that it fulfils the following condition

$$\begin{aligned} \forall_{\check{C} \in \mathbb{R}^+} \exists_{\check{C}_{\min}, \check{C}_{\max} \in \mathbb{R}^+} : \check{C}_{\min} < \check{C} < \check{C}_{\max} \wedge \\ \wedge \mu(\check{C}_{\min}|\check{C}) = \mu(\check{C}_{\max}|\check{C}) = 0. \end{aligned} \quad (6)$$

Immediately from (5) and (6) we obtain

$$\forall_{\check{C} \in \mathbb{R}^+} \forall_{x \in \mathbb{R}^+} : x \notin (\check{C}_{\min}, \check{C}_{\max}) \Rightarrow \mu(x|\check{C}) = 0 \quad (7)$$

The return rate is then at risk of coincidence uncertainty and imprecision. According to the Zadeh extension principle, for each fixed elementary state $\omega \in \Omega$ of financial market, membership function $\rho(\cdot, \omega | \check{C}) : \mathbb{R} \rightarrow [0; 1]$ of return rate is determined by the identity

$$\begin{aligned} \rho(r, \omega | \check{C}) &= \max\{\mu(y | \check{C}) : y \in \mathbb{R}^+, r = r(y, \tilde{V}_t(\omega))\} = \\ &= \mu(r_0^{-1}(r, \tilde{V}_t(\omega)) | \check{C}). \end{aligned} \tag{8}$$

It means that the return rate considered here is represented by fuzzy probabilistic set (Hiroto, 1981). For this reason, this return rate is called fuzzy probabilistic return.

For any fuzzy probabilistic return we define the expected return distribution as follows

$$\varrho(r | \check{C}) = \int_{-\infty}^{+\infty} \mu(r_0^{-1}(r, x) | \check{C}) dF_V(x | \check{C}). \tag{9}$$

The function $r_0^{-1}(r, \cdot) : \mathbb{R}_0^+ \rightarrow \mathbb{R}^+$ is increasing one. Therefore, the condition (7) implies that

$$\begin{aligned} \forall \check{C} \in \mathbb{R}^+ \exists x_{\min}, x_{\max} \in \mathbb{R}_0^+ \forall x \in \mathbb{R}_0^+ : x \notin (x_{\min}, x_{\max}) \Rightarrow \\ \Rightarrow \wedge \mu(r_0^{-1}(r, x) | \check{C}) = 0 \end{aligned} \tag{10}$$

It proves that expected return distribution always exists. Distribution of expected return $\varrho(\cdot | \check{C}) : \mathbb{R} \rightarrow [0; 1]$ is a membership function of fuzzy subset \tilde{R} in the real line. This subset is imprecise assessment of the expected return value. Therefore, it is called fuzzy expected return. In general case, fuzzy expected return is not any fuzzy number. In this way, any security with fuzzy probabilistic return is represented by pair $(\varrho(\cdot | \check{C}), \sigma^2)$.

3. Investment recommendations dependent on expected return

The investment recommendation is the counsel given by the advisers to the investor. For convenience, these recommendations may be expressed by means of standardized ratings. Here we will consider the following advisers vocabulary:

- **Buy** suggesting that evaluated security is significantly undervalued,
- **Accumulate** suggesting that evaluated security is undervalued,
- **Hold** suggesting that evaluated security is fairly valued,
- **Reduce** suggesting that evaluated security is overvalued,
- **Sell** suggesting that evaluated security is significantly overvalued.

Many advisers use a different terminology and different number of words forming advice vocabulary.¹ In this article we will concentrate on the five-element adviser's vocabulary. Recommendations mentioned above form the set

$$\mathbb{A} = \{A^{++}, A^+, A^0, A^-, A^{--}\} \quad (11)$$

called advisers' vocabulary, where

- A^{++} denotes the advice **Buy**,
- A^+ denotes the advice **Accumulate**,
- A^0 denotes the advice **Hold**,
- A^- denotes the advice **Reduce**,
- A^{--} denotes the advice **Sell**.

Let us take into account fixed security \check{S} represented by the pair (r_s, σ_s^2) where:

- r_s denotes expected return on \check{S} ,
- σ_s^2 denotes variance of return on \check{S} .

In such case adviser's counsel depends on expected return and return variance. Then the criterion for competent choice of advice can be presented as a comparison of the values $g(\bar{r}|\sigma_s^2)$ and \hat{G} defined as follows:

- $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ is an increasing function in substantially justified form,
- \hat{G} denotes the substantially justified limit value.

The function $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ serves as a profit index. Using this criterion we define the advice choice function $\Lambda : \mathbb{R}^2 \rightarrow 2^{\mathbb{A}}$ as follows

$$\left\{ \begin{array}{l} A^{++} \in \Lambda(r_s, \check{G}) \Leftrightarrow g(r_s|\sigma_s) > \hat{G} \Leftrightarrow g(r_s|\sigma_s) \geq \hat{G} \wedge \neg g(r_s|\sigma_s) \leq \hat{G}, \\ A^+ \in \Lambda(r_s, \check{G}) \Leftrightarrow g(r_s|\sigma_s) \geq \hat{G}, \\ A^0 \in \Lambda(r_s, \check{G}) \Leftrightarrow g(r_s|\sigma_s) = \hat{G} \Leftrightarrow g(r_s|\sigma_s) \geq \hat{G} \wedge g(r_s|\sigma_s) \leq \hat{G}, \\ A^- \in \Lambda(r_s, \check{G}) \Leftrightarrow g(r_s|\sigma_s) \leq \hat{G}, \\ A^{--} \in \Lambda(r_s, \check{G}) \Leftrightarrow g(r_s|\sigma_s) < \hat{G} \Leftrightarrow \neg g(r_s|\sigma_s) \geq \hat{G} \wedge g(r_s|\sigma_s) \leq \hat{G}. \end{array} \right. \quad (12)$$

In this way we assign the advice subset to the security \check{S} . The value $\Lambda(r_s, \check{G})$ is called the investment recommendation. This recommendation may be used as valuable starting point for equity portfolio strategies. On the other hand, the weak point of the proposed choice function is omitting the fundamental analysis result and the behavioral factors impact. When analyzing the above choice function it is easy to see lack of strong boundary between the advice **Buy** and **Accumulate** and between the advice **Reduce** and **Sell**. Justification for distinguishing between these recommendations we can search on the basis of fundamental analysis and between the behavioral aspects of the investment process.

4. Investment recommendations dependent on fuzzy expected return

Let us assume that considered security is represented by the pair $(\varrho_s(\cdot|\check{C}), \sigma_s^2)$, where $\varrho_s(\cdot|\check{C}) \in [0; 1]^{\mathbb{R}}$ is the expected return distribution. This distribution describes fuzzy expected return \tilde{R} determined for current market price \check{C} . Fuzzy expected return may be dependent on fundamental and behavioural factors.² Therefore, there shall be proposed advisor's counsels dependent on fuzzy expected return and return variance.

In the first step, we extend the profit index domain \mathbb{R}^2 to the set $[0; 1]^{\mathbb{R}} \times \mathbb{R}$. According to Zadeh's Extension Principle, the profit index value $\tilde{g}(\tilde{R}|\check{C})$ is described by its membership function $\gamma : \mathbb{R} \rightarrow [0; 1]$ determined as follows

$$\gamma(x|\check{C}) = \sup\{\varrho_s(r|\check{C}) : x = g(r|\sigma_s)\} = \varrho_s(g^{-1}(r|\sigma_s)|\check{C}). \quad (13)$$

In this way we define profit fuzzy index $\tilde{g}(\cdot|\check{C}) : [0; 1]^{\mathbb{R}} \times \mathbb{R} \rightarrow [0; 1]^{\mathbb{R}}$.

In the second step, we extend the advice choice function domain \mathbb{R}^2 to the set $[0; 1]^{\mathbb{R}} \times \mathbb{R}$. Under Zadeh's Extension Principle, the advice choice function value $\tilde{\Lambda}(\tilde{R}, \check{G})$ is described by its membership function $\lambda : \mathbb{A} \rightarrow [0; 1]$ which due to the definition (12) is determined in following way

$$\lambda(A^{++}) = \sup\{\gamma(x|\check{C}) : x \geq \check{G}\} \wedge (1 - \sup\{\gamma(x|\check{C}) : x \leq \check{G}\}), \quad (14)$$

$$\lambda(A^+) = \sup\{\gamma(x|\check{C}) : x \geq \check{G}\}, \quad (15)$$

$$\lambda(A^0) = \sup\{\gamma(x|\check{C}) : x \geq \check{G}\} \wedge \sup\{\gamma(x|\check{C}) : x \leq \check{G}\}, \quad (16)$$

$$\lambda(A^-) = \sup\{\gamma(x|\check{C}) : x \leq \check{G}\}, \quad (17)$$

$$\lambda(A^{--}) = \sup\{\gamma(x|\check{C}) : x \leq \check{G}\} \wedge (1 - \sup\{\gamma(x|\check{C}) : x \geq \check{G}\}), \quad (18)$$

In this way we define advice fuzzy choice function $\tilde{\Lambda} : [0; 1]^{\mathbb{R}} \times \mathbb{R} \rightarrow [0; 1]^{\mathbb{A}}$. The value $\tilde{\Lambda}(\tilde{R}, \check{G})$ is called imprecise investment recommendation. Moreover, the value $\lambda(A)$ may be interpreted as a degree in which the advice $A \in \mathbb{A}$ was chosen.

The expected return is replaced above by the fuzzy return which also takes into account fundamental behavioral aspects of decision making in finance. However, such an increase in cognitive value has a price. This price is imprecise formulation of investment recommendations. When we use imprecise image of security, we cannot precisely indicate the recommended action out of a set of advices. Each advice is thus recommended to some extent. The investor shifts some of the responsibility to advisers. For this reason,

the investor restricts his or her choice of investment decisions to advice recommended to the greatest degree. In this way, the investor minimizes his or her individual responsibility for financial decision making. However, final investment decision will be made by the investor. Guided by own knowledge and intuition, the investor can choose the advice which is recommended in the lower degree. This problem was widely discussed in (Piasecki, 1990).

Let us note that we have here

$$\lambda(A^{++}) = \lambda(A^+) \wedge (1 - \lambda(A^-)), \quad (19)$$

$$\lambda(A^0) = \lambda(A^+) \wedge \lambda(A^-), \quad (20)$$

$$\lambda(A^{--}) = \lambda(A^-) \wedge (1 - \lambda(A^+)), \quad (21)$$

This shows that the functions values $\lambda(A^+)$ and $\lambda(A^-)$ are sufficient to determine the imprecise investment recommendation. For this reason, when considering the specific method of advice choice, we will only determine these values. Let us note that the use of imprecisely estimated return will allow determining the differences between the advice **Buy** and **Accumulate** and between the advice **Reduce** and **Sell**.

In the next two sections some examples of methodologies for imprecise recommendations will be presented. There we limit our discussion to the case when the security return is described by means of a simple return rate given as follows

$$\tilde{r}(\omega) = \frac{\tilde{V}_t(\omega) - \check{C}}{\check{C}}, \quad (22)$$

where all used symbols are defined in Section 2. All methodologies presented below are extensions of well know classical methodologies to the fuzzy case.

5. Financial equilibrium criteria

Each financial equilibrium model is given as the comparison of expected return from considered security and expected return of the market portfolio. We will consider fixed security \check{S} .

5.1. Sharpe's Ratio

We assume that there exists the risk free bond instrument represented by the pair $(r_0, 0)$ and the market portfolio represented by the pair (r_M, σ_M^2) . If the security \check{S} is represented by the pair (r_s, σ_s^2) , then Sharpe (1966) defines the profit index $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ and the limit value \hat{G} as follows

$$g(r|\sigma_s) = \frac{r - r_0}{\sigma_s}, \tag{23}$$

$$\hat{G} = \frac{r_M - r_0}{\sigma_M}. \tag{24}$$

Sharpe's profit index estimates amount of the premium per overall risk unit. Sharpe's limit value is equal to the unit premium of the market portfolio risk.

If the security \check{S} is represented by the pair $(\varrho_s(\cdot|\check{C}), \sigma_s^2)$ then profit fuzzy index is described by its membership function $\gamma : \mathbb{R} \rightarrow [0; 1]$ determined as follows

$$\gamma(x|\check{C}) = \sup \left\{ \varrho_s(r|\check{C}) : x = \frac{r - r_0}{\sigma_s} \right\} = \varrho_s(\sigma_s \cdot x + r_0|\check{C}). \tag{25}$$

In accordance with (15) and (17) we have here

$$\begin{aligned} \lambda(A^+) &= \sup \left\{ \gamma(x|\check{C}) : x \geq \frac{r_M - r_0}{\sigma_M} \right\} = \\ &= \sup \left\{ \varrho_s(r|\check{C}) : \frac{r - r_0}{\sigma_s} \geq \frac{r_M - r_0}{\sigma_M} \right\}, \end{aligned} \tag{26}$$

$$\begin{aligned} \lambda(A^-) &= \sup \left\{ \gamma(x|\check{C}) : x \leq \frac{r_M - r_0}{\sigma_M} \right\} = \\ &= \sup \left\{ \varrho_s(r|\check{C}) : \frac{r - r_0}{\sigma_s} \leq \frac{r_M - r_0}{\sigma_M} \right\}. \end{aligned} \tag{27}$$

The values $\lambda(A^{++})$, $\lambda(A^0)$, $\lambda(A^{--})$ are determined respectively by relationships (19), (20) and (21).

5.2. Jensen's Alpha

On the capital market we observe the risk-free return r_0 and the expected market return r_M . The security \check{S} is represented by the pair (r_s, β_s) , where β_s is the directional factor of the CAPM model assigned to this instrument. Jensen (1969) defines the profit index $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ and the limit value \hat{G} as follows

$$g(r|\sigma_s) = r - \beta_s \cdot (r_M - r_0), \tag{28}$$

$$\hat{G} = r_0. \tag{29}$$

Let us note that the above profit index $g(\cdot|\sigma_s)$ does not depend on the standard deviation σ_s . Jensen's profit index estimates amount of the premium

of the market portfolio risk. Jensen's limit value is equal to the risk-free return.

If the security \check{S} is represented by the pair $(\varrho_s(\cdot|\check{C}), \beta_s)$ then profit fuzzy index is described by its membership function $\gamma : \mathbb{R} \rightarrow [0; 1]$ determined as follows

$$\begin{aligned} \gamma(x|\check{C}) &= \sup \left\{ \varrho_s(r|\check{C}) : x = r - \beta_s \cdot (r_M - r_0) \right\} = \\ &= \varrho_s(x + \beta_s \cdot (r_M - r_0)|\check{C}). \end{aligned} \quad (30)$$

In accordance with (15) and (17) we have here

$$\begin{aligned} \lambda(A^+) &= \sup \left\{ \gamma(x|\check{C}) : x \geq r_0 \right\} = \\ &= \sup \left\{ \varrho_s(r|\check{C}) : r - \beta_s \cdot (r_M - r_0) \geq r_0 \right\}, \end{aligned} \quad (31)$$

$$\begin{aligned} \lambda(A^-) &= \sup \left\{ \gamma(x|\check{C}) : x \leq r_0 \right\} = \\ &= \sup \left\{ \varrho_s(r|\check{C}) : r - \beta_s \cdot (r_M - r_0) \leq r_0 \right\}. \end{aligned} \quad (32)$$

The values $\lambda(A^{++})$, $\lambda(A^0)$, $\lambda(A^{--})$ are determined respectively by relationships (19), (20) and (21).

5.3. Treynor's Ratio

Let us assume that the assumptions are the same as the assumptions used in subsection 5.2. Additionally we assume that the security return is positively correlated with the market portfolio return. Treynor (1965) defines the profit index $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow \mathbb{R}$ and the limit value \hat{G} as follows

$$g(r|\sigma_s) = \frac{r - r_0}{\beta_s}, \quad (33)$$

$$\hat{G} = r_M - r_0. \quad (34)$$

Treynor's profit index estimates amount of the premium per market risk unit. Treynor's limit value is equal to the premium of the market risk.

If the security \check{S} is represented by the pair $(\varrho_s(\cdot|\check{C}), \sigma_s^2)$ then profit fuzzy index is described by its membership function $\gamma : \mathbb{R} \rightarrow [0; 1]$ determined as follows

$$\gamma(x|\check{C}) = \sup \left\{ \varrho_s(r|\check{C}) : x = \frac{r - r_0}{\beta_s} \right\} = \varrho_s(\beta_s \cdot x + r_0|\check{C}). \quad (35)$$

In accordance with (15) and (17) we have here

$$\begin{aligned}\lambda(A^+) &= \sup \left\{ \gamma(x|\check{C}) : x \geq r_M - r_0 \right\} = \\ &= \sup \left\{ \varrho_s(r|\check{C}) : \frac{r - r_0}{\beta_s} \geq r_M - r_0 \right\},\end{aligned}\tag{36}$$

$$\begin{aligned}\lambda(A^-) &= \sup \left\{ \gamma(x|\check{C}) : x \leq r_M - r_0 \right\} = \\ &= \sup \left\{ \varrho_s(r|\check{C}) : \frac{r - r_0}{\beta_s} \leq r_M - r_0 \right\}.\end{aligned}\tag{37}$$

The values $\lambda(A^{++})$, $\lambda(A^0)$, $\lambda(A^{--})$ are determined respectively by relationships (19), (20) and (21).

Investment recommendation made by means of the Treynor Ratio is identical with investment recommendation made by Jensen's Alpha.

6. The Safety-First Criteria

We will consider the simple return rate $\tilde{r}(\omega)$ on fixed security. For each assumed value $r \in \mathbb{R}$ of expected simple return rate the probability distribution of this return is given by the cumulative distribution function $F(\cdot|r) : \mathbb{R} \rightarrow [0; 1]$ which is strictly increasing and continuous. Then the Safety Condition (Roy, 1952) is given in following way

$$F(L|r) = \varepsilon,\tag{38}$$

where:

- L denotes minimum acceptable return rate;
- ε is equal to probability realization of return below the minimum acceptable rate.

The realization of return below the minimum acceptable rate is identified with a loss. Therefore, the variable ε denotes the loss probability.

Additionally, let us note that the function $G_l : \mathbb{R} \rightarrow [0; 1]$ defined for fixed $l \in R$ as follows

$$G_l(r) = F(l|r)\tag{39}$$

is strictly decreasing.

Let us take into account the security \check{S} with expected return r_s . The distribution of this return is described by its cumulative distribution function $F_s(\cdot|r_s) : \mathbb{R} \rightarrow [0; 1]$. It is obvious that the cumulative distribution function $F_s(\cdot|r_s)$ depends on the standard deviation σ_s .

6.1. Roy’s Criterion

Roy’s Criterion (Roy, 1952) is that for fixed minimum acceptable return rate L the investor minimizes the loss probability. Additionally in order to ensure financial security, the investor assumes the maximum level ε^* of loss probability. Then the profit index $g(\cdot|\sigma_s) : \mathbb{R} \rightarrow [-1; 0]$ and the limit value \hat{G} are defined as follows

$$g(r|\sigma_s) = -G_L(r), \tag{40}$$

$$\hat{G} = -\varepsilon^*. \tag{41}$$

If the security \check{S} is represented by the pair $(\varrho_s(\cdot|\check{C}), \beta_s)$ then profit fuzzy index is described by its membership function $\gamma : [-1; 0] \rightarrow [0; 1]$ determined in following way

$$\gamma(x|\check{C}) = \sup \left\{ \varrho_s(r|\check{C}) : x = -G_L \right\} = \varrho_s(G_L^{-1}(-x)|\check{C}). \tag{42}$$

In accordance with (15) and (17) we have here

$$\lambda(A^+) = \sup \left\{ \gamma(x|\check{C}) : x \geq -\varepsilon^* \right\} = \sup \left\{ \varrho_s(r|\check{C}) : F_s(L|r) \leq \varepsilon^* \right\}, \tag{43}$$

$$\lambda(A^-) = \sup \left\{ \gamma(x|\check{C}) : x \leq -\varepsilon^* \right\} = \sup \left\{ \varrho_s(r|\check{C}) : F_s(L|r) \geq \varepsilon^* \right\}. \tag{44}$$

The values $\lambda(A^{++})$, $\lambda(A^0)$, $\lambda(A^{--})$ are determined respectively by relationships (19), (20) and (21).

6.2. Kataoka’s Criterion

Kataoka’s Criterion (Kataoka, 1963) is that, for fixed loss probability ε the investor maximizes the minimum acceptable return rate. In addition, in order to ensure interest yield, the investor assumes the minimum level L^* of return. Then the profit index $g(\cdot|\sigma_s) : [0; 1] \rightarrow \mathbb{R}$ and the limit value \hat{G} are defined in following way

$$g(r|\sigma_s) = F_s^{-1}(\varepsilon|r), \tag{45}$$

$$\hat{G} = L^*. \tag{46}$$

If the security \check{S} is represented by the pair $(\varrho_s(\cdot|\check{C}), \beta_s)$ then profit fuzzy index is described by its membership function $\gamma : \mathbb{R} \rightarrow [0; 1]$ determined as follows

$$\gamma(x|\check{C}) = \sup \left\{ \varrho_s(r|\check{C}) : x = F_s^{-1}(\varepsilon|r) \right\} = \varrho_s(G_L^{-1}(\varepsilon)|\check{C}). \tag{47}$$

In accordance with (15) and (17) we have here

$$\lambda(A^+) = \sup \left\{ \gamma(x|\check{C}) : x \geq L^* \right\} = \sup \left\{ \varrho_s(r|\check{C}) : F_s^{-1}(\varepsilon|r) \geq L^* \right\}, \quad (48)$$

$$\lambda(A^-) = \sup \left\{ \gamma(x|\check{C}) : x \leq L^* \right\} = \sup \left\{ \varrho_s(r|\check{C}) : F_s^{-1}(\varepsilon|r) \leq L^* \right\}. \quad (49)$$

The values $\lambda(A^{++})$, $\lambda(A^0)$, $\lambda(A^{--})$ are determined respectively by relationships (19), (20) and (21).

6.3. Telser's Criterion

Based on the safety and profitability of the investment, the investor assumes a minimum level L^* of acceptable return and the maximum level ε^* of loss probability. If for the security \check{S} the probability that return is not acceptable is less than acceptable loss probability then \check{S} is called safe-haven security.

Let the return on security \check{S} be given as the fuzzy probabilistic set outlined in (8) by its membership function $\rho(\cdot|\check{C}) : \mathbb{R} \times \Omega \rightarrow [0; 1]$. Then, for any minimum acceptable minimum return $L \in \mathbb{R}$ the loss is the fuzzy probabilistic set described by membership function $\eta(\cdot|L, \check{C}) : \Omega \rightarrow [0; 1]$ determined as follows

$$\begin{aligned} \eta(\omega|L, \check{C}) &= \sup \left\{ \rho(r, \omega|\check{C}) : r < L \right\} = \\ &= \sup \left\{ \omega(\tilde{V}_t(\omega) \cdot (1+r)^{-1}|\check{C}) : r < L \right\}. \end{aligned} \quad (50)$$

In accordance with (3), for expected return r_s on \check{S} the probability distribution of future value is described by conditional cumulative distribution function $F_{V,S}(\cdot|r_s, \check{C}) : \mathbb{R}^+ \rightarrow [0; 1]$ determined in the following way

$$F_{V,S}(x|r_s, \check{C}) = F_s \left(\frac{x - \check{C}}{\check{C}} \middle| r_s \right). \quad (51)$$

In this case, due (4), (6) and (7), the loss probability $P_s(L)$ may be calculated as follows

$$\begin{aligned} P_s(L) &= \int_{-\infty}^{+\infty} \sup \left\{ \mu(x \cdot (1+r)^{-1}|\check{C}) : r < L \right\} dF_{V,S}(x|r_s, \check{C}) = \\ &= \int_{-\infty}^{+\infty} \sup \left\{ \mu(\check{C} \cdot (1+t) \cdot (1+r)^{-1}|\check{C}) : r < L \right\} dF_s(t|r_s) = \\ &= \int_{-\infty}^L dF_s(t|r_s) + \int_L^{+\infty} \mu(\check{C} \cdot (1+t) \cdot (1+L)^{-1}|\check{C}) dF_s(t|r_s) = \\ &= F_s(L|r_s) + \int_L^T \mu(\check{C} \cdot (1+t) \cdot (1+L)^{-1}|\check{C}) dF_s(t|r_s), \end{aligned} \quad (52)$$

where

$$T = \frac{\check{C}_{\max}}{\check{C}} \cdot (1 + L) - 1. \tag{53}$$

The security \check{S} is safe-haven one iff the following condition is fulfilled

$$P_s(L^*) \leq \varepsilon^*. \tag{54}$$

Telsers's Criterion (Telser, 1955) is that the investor maximizes the return on safe-haven securities. In addition, due to the profitability investment, the investor takes into account the equilibrium rate $r^* > L^*$. Then the profit index $g(\cdot|\sigma_s) : [0; 1] \rightarrow \mathbb{R}$ and the limit value \hat{G} are defined in the following way

$$g(r|\sigma_s) = r, \tag{55}$$

$$\hat{G} = r^*. \tag{56}$$

If the security \check{S} is represented by the pair $(\varrho_s(\cdot|\check{C}), \beta_s)$ then profit fuzzy index is described by its membership function $\gamma : \mathbb{R} \rightarrow [0; 1]$ determined as follows

$$\gamma(x|\check{C}) = \varrho_s(x|\check{C}). \tag{57}$$

In accordance with (15) and (17) we have here

$$\lambda(A^+) = \sup \left\{ \varrho_s(r|\check{C}) : r \geq r^* \right\}, \tag{58}$$

$$\lambda(A^-) = \sup \left\{ \varrho_s(r|\check{C}) : r \leq r^* \right\}. \tag{59}$$

The values $\lambda(A^{++})$, $\lambda(A^0)$, $\lambda(A^{--})$ are determined respectively by relationships (19), (20) and (21).

7. Summary

Imprecision is relevant to the investment process. It is shown above that imprecise estimate of the present value may be considered as a sufficient basis for determining the investment recommendation. The results so obtained may be applied in behavioural finance theory as a normative model of investment's decisions. These results may provide theoretical foundations for constructing an investment decision support system.

On the other hand imprecise estimate of the expected return could be a consequence of taking into account behavioral aspects of the investment

process. Thus, we have shown here that the behavioral premises can influence investment recommendations in a controlled manner.

Applications of the above normative models cause several difficulties. The main difficulty is the high formal and computational complexity of the tasks involved in determining the membership function for the imprecise recommendations. The computational complexity of these models is the price we pay for the lack of detailed assumptions about the return rate. On the other hand, the low logical complexity is an important attribute of each formal model.

In this paper, the main cognitive result is to propose general methodology for imprecise investments recommendations. Moreover, the paper also offers original generalization of the financial equilibrium criteria and of the safety-first criteria to the fuzzy case.

N O T E S

¹ Can find <http://www.marketwatch.com/tools/guide.asp> [access 01.09.2013].

² For example (Piasecki, 2011 a, b).

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