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# KNOWLEDGE AND INTUITIONISTIC TENSE LOGIC

**Abstract**. In this paper we describe the system of intuitionistic tense logic and consider the possibility of using this system to represent the changing of knowledge over time.

**Keywords**: knowledge, tense logic, intuitionistic logic, epistemic logic.

Most of the systems used to formalize knowledge changing in time are constructed over classical propositional logic. It seems natural to ask whether this is the only possible way to construct logical systems for this task. We will show that this task can be accomplished even when as a basis we use intuitionistic propositional logic instead of classical propositional logic.

We will describe the intuitionistic tense logic system and demonstrate that this system can be used for formal description of knowledge changing over time, although in this system there are no epistemic operators. The representation of knowledge in our system is not realized at a syntactic level, but due to the properties of intuitionistic logic, knowledge is represented at the semantic level. This approach is the result of a proposed semantics for intuitionistic logic in which are used notions like *proof*[1], *information*, and *knowledge*.

For the considered system of intuitionistic temporal logic, Kripke-style semantics is proposed. This type of semantics is also the proper semantics for intuitionistic propositional logic. Kripke models [3] for intuitionistic propositional logic are similar to Kripke models for modal logic constructed over classical propositional logic. In these models we have a collection of the worlds $W$ and the relation of accessibility $R$ while, in the case of intuitionistic logic, the elements of the set $W$ we consider rather as states of information, knowledge, etc, than as possible worlds. The $R$ relation between elements $w$ and $v$ ($wRv$) is interpreted as *$w$ has access to $v$*, which

[1] The semantics proposed by Kolmogorov.

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means that the information state $v$ is an available extension of the information state $w$. The crucial difference between Kripke's models for intuitionistic logic and Kripke models for modal logic constructed over classical propositional logic is in the fact that in the case of modal logic constructed over classical propositional logic the relation $R$ is used only to interpret the modal operators; for intuitionistic logic, this relation is used to interpret intuitionistic connectives: negation and implication.

A formula $\neg\varphi$ is true[2] in a certain information state $w$ if there is no information state available from the state $w$ such that $\varphi$ is true at this state. In other words, the formula $\neg\varphi$ is true in the state $w$ if there is no possibility that $\varphi$ is true in any information state available from the state $w$.

We have the same in the case of intuitionistic implication. The formula $\varphi \rightarrow \psi$ is true in the information state $w$ when in any information state available from $w$, the truth of $\varphi$ implies the truth of $\psi$. Moreover, in Kripke models for intuitionistic logic is built a condition of monotonicity. The true (forced) formula in a given information state remains truthfulness in any extension of this state.

Modality in intuitionistic logic, we can see, for example, in the syntactic definition of intuitionistic negation. The formula $\neg\varphi$ is syntactically equivalent to the formula $\varphi \rightarrow \bot$. Thus, the intuitionistic negation can be seen as a kind of impossibility operator.

Kripke-style semantics for intuitionistic propositional logic is as follows:

## 1. Kripke semantics for intuitionistic propositional logic

The intuitionistic model is a triple $\mathfrak{M} = \langle W, \leq, V \rangle$. The truth of the formula $\varphi$ in a model $\mathfrak{M}$, in a state $s$ we define as follows:

$$\mathfrak{M}, s \models \quad \equiv \quad s \in V(p), \text{ where } p \text{ is a propositional letter,}$$

$$\mathfrak{M}, s \models \neg\varphi \quad \equiv \quad \underset{s \leq s'}{\forall} \mathfrak{M}, s' \not\models \varphi,$$

$$\mathfrak{M}, s \models \varphi \wedge \psi \quad \equiv \quad \mathfrak{M}, s \models \varphi \text{ and } \mathfrak{M}, s \models \psi,$$

$$\mathfrak{M}, s \models \varphi \vee \psi \quad \equiv \quad \mathfrak{M}, s \models \varphi \text{ or } \mathfrak{M}, s \models \psi,$$

$$\mathfrak{M}, s \models \varphi \rightarrow \psi \quad \equiv \quad \underset{s \leq s'}{\forall} (\text{if } \mathfrak{M}, s' \models \varphi, \text{ then } \mathfrak{M}, s' \models \psi).$$

Let us note that in intuitionistic logic, if a formula $\neg\varphi$ is true at some information state, then we know not only that in the current information

[2] Intuitionistic logic also uses the term *forced*.

state $\varphi$ is not satisfied (this information we obtain in the case of classical logic), but furthermore we know that the formula $\varphi$ will never be satisfied. Our *never* refers to all available extensions of the current information state. Aside from information given explicitly in intuitionistic logic, we have additional information. This feature of intuitionistic logic van Benthem called *implicit knowledge* [1]. For expression of this kind of knowledge in the language of intuitionistic logic, we do not need any additional specific operators. Although similar semantics, this feature strongly differentiates intuitionistic logic from epistemic logic based on classical logic. In the language of epistemic logic we can express *explicit knowledge*, and we use for it the epistemic operator $K$. The language of intuitionistic logic allows us to express interesting notions without referring to epistemic operators. For example from the truth of the formula $\neg\neg\varphi$ we can conclude that for each state there is such an extension, in which $\varphi$ is true. This statement is, omitting details, close to *we know that $\varphi$ must be true.*

In Kripke's semantics for epistemic logic based on classical propositional logic, the satisfiability of the formula $K\varphi$ in model $\mathfrak{M}$, in the information state $s$ we define as follows:

$$\mathfrak{M}, s \models K\varphi \qquad \equiv \qquad \underset{s \leq s'}{\forall}, \mathfrak{M}, s' \models \varphi.$$

Let us consider the truth of the formula $K\neg\varphi$ in the model $\mathfrak{M}$, in the state $s$. According to the definition of satisfiability of $K$ operator we have:

$$\mathfrak{M}, s \models K\neg\varphi \qquad \equiv \qquad \underset{s \leq s'}{\forall}, \mathfrak{M}, s' \models \neg\varphi.$$

If we regard the definition of satisfiability for negation in epistemic logic we have:

$$\mathfrak{M}, s \models K\neg\varphi \qquad \equiv \qquad \underset{s \leq s'}{\forall}, \mathfrak{M}, s' \not\models \varphi.$$

From the definition of satisfiability of negation in intuitionistic logic

$$\mathfrak{M}, s \models \neg\varphi \qquad \equiv \qquad \underset{s \leq s'}{\forall}, \mathfrak{M}, s' \not\models \varphi.$$

we see that the intuitionistic negation ($\neg$) may, in some sense, be regarded as a combination of the $K$ operator and classical negation ($K\neg$). We can make a similar reasoning for intuitionistic implication and show that the intuitionistic formula $\varphi \to \psi$ can be, ignoring the details, regarded as a *modalized implication* or a combination of epistemic operator $K$ and classical implications: $K(\varphi \to \psi)$.

## 2. $IK'_t$ - intuitionistic tense logic

Now we describe the system of intuitionistic tense logic. This system is a simpler version of Ewald's system [2], described in [4], [5], [6].

### 2.1. Syntax

An alphabet $\mathcal{A}$ of the language of $\mathfrak{L}_{IK'_t}$:

- a countable set of propositional letters $\mathcal{AP}$,
- unary connective: $\neg$,
- binary connectives: $\wedge$, $\vee$, $\rightarrow$, $\leftrightarrow$,
- tense operators: $G, H, F, P$,
- parentheses: $), ($.

**Definition 2.1**

The set $FOR(\mathfrak{L}_{IK'_t})$ is the smallest set of finite sequences of elements of the alphabet $\mathcal{A}$ such that:

- $\mathcal{AP} \subseteq FOR(\mathfrak{L}_{IK'_t})$,
- if $\varphi, \psi \in FOR(\mathfrak{L}_{IK'_t})$, then $\neg\varphi, G\varphi, F\varphi, H\varphi, P\varphi, \varphi \wedge \psi, \varphi \vee \psi, \varphi \rightarrow \psi, \varphi \leftrightarrow \psi \in FOR(\mathfrak{L}_{IK'_t})$.

Tense operators are interpreted as usual:

$G\varphi$ – always in the future $\varphi$,

$F\varphi$ – sometime in the future $\varphi$,

$H\varphi$ – always in the past $\varphi$,

$P\varphi$ – sometime in the past $\varphi$.

### 2.2. Semantics

We use $IK'_t$ system to describe changing states of knowledge. Changing knowledge in $IK'_t$ is understood as a transition to the next state of knowledge. It is assumed that whole knowledge of the current state of knowledge is available in every state of knowledge which is no less than the current state. We assume monotonicity of the process of acquiring knowledge. A bigger state of knowledge we reach by enriching knowledge of an agent with new facts. This may occur in some cases.

We can enrich the knowledge through research when we make a description of past events which took place at time points that were not known in the state of knowledge. In the given state of knowledge we had no information about these events. In this case, the structure of time of the new state of knowledge expands in the past and is a superset of the time structure of the new state.

We can know new elements of the future time. In this case, the structure of time of a new state of knowledge expands in the future, and the relation of inclusion of time structure is similar to the first case.

We can learn new moments of time from the past and from the future. In this case, the time structure of the not lesser state of knowledge expands, both in the past and in the future.

In each case under consideration the increase in the level of knowledge induces appropriate inclusions of temporary structures. The expansion of the structure of time (regardless of the direction in which it occurs) changes the relation of temporal succession. In the new state of knowledge we therefore have to consider the changed relation.

Another possible option to achieve a new state of knowledge is a case when the set of moments of time is not changing, but the power of sets of events mapped to moments of time is increasing.

The state of knowledge in the proposed semantics is conceived as consisting of a set of facts, which are semantic correlates of sentences, a collection of moments of time and the relation of temporal succession. A subset of a particular set of facts mapped to a moment of time is conceived as a set of facts known at this moment.

We want to construct a language which we use to describe the states of knowledge changing over time. Assume therefore, that the state of knowledge, let's call it $m$, is not a total state of knowledge, so there are unknown facts in this state. This implies the possibility of multiple states of knowledge, different from $m$. In these states of knowledge the agent knows all facts which are known in the state $m$; additionally, there are also known new facts. The main possible differences between initial state $m$ and new states of knowledge are: a bigger set of known facts, a set of new moments of time, and a set of new moments of time which are in the relation *before-after*. Such states of knowledge we call states reachable from the state $m$. We also accept that the state $m$ is reached from $m$. This means that the relationship is reflexive. If this assumption is not accepted, we reject the possibility the state $m$ is not total. We do not want, however, logic $IK_t'$ to resolve it. The logic is to be independent in this respect.

In the states of knowledge reachable from $m$ there are no less known facts than in the state $m$. We say that these states of knowledge are not lesser than $m$, or that they are states of knowledge in which the level of knowledge is not lesser than the level of knowledge in the state $m$.

A state of $m'$, reachable from a state $m$ may also not be a total state of knowledge. From the point of view of the state of $m'$ reachable can be state $m''$ such that in $m''$ is known everything that it is known in the state of $m'$ and also new events are known which are not known in the state $m'$. Since everything that is known in the state $m$ is also known in the state $m'$ and everything that is known in the state $m'$ is known in the state $m''$, therefore everything that is known in the state of $m$ is also known in the state $m''$. State $m''$ is therefore reachable from state $m$. So we conclude that the relation*before-after* is transitive. Since the relation *before-after* is reflexive and transitive, then it is a part-ordering relation.

We say that the state $m''$ has a not lesser level of knowledge than the state $m'$, if $m''$ meets the following conditions:

1. A set of moments of time of state $m'$ is included in the set of moments of time of the state $m''$. (Changing the number of moments of time changes the state of knowledge.)
2. In the state of $m''$ are preserved - occurring between time moments - connections *before-after*, which existed in the state $m'$. Moreover, in the state $m''$ may occur before-after connections, which did not hold in the state $m'$.
3. All events which are known in the state $m'$ are known in the state $m''$. (Everything that is known does not cease to be known, even if there are new events known.) Moreover, in the moments of time of state $m''$ may be known events, which are not known in the equivalents of those moments in the state $m'$.

Between the conditions 1), 2) and 3) there are certain connections.

The fulfillment of condition 1) implies condition 2), because in our discussion we omit cases where *new* moments of time are not connected in the relation *before-after* with other moments of time. The change of the set of moments of time involves a change in the relation before-after. However, the change of the relation of temporal succession does not involve change of the set of moments of time. In a state of knowledge with no lesser level of knowledge, new connections may occur between moments of time existing in the state of knowledge with a lesser level of knowledge. Condition 2) does not therefore fulfill condition 1). Similarly, condition 3) does not imply condition 1) or 2), as new events can be known without the existence of new moments of time or new connections *before-after*.

Each moment of time is assigned to a non-empty set of known events. If there are new moments of time, there are also new events known. Condition 1) implies therefore condition 3).

The existence of new connections *before-after* implies the existence of new known events in those moments of time in which new connections take place. Thus, condition 2) implies condition 3).

In our system we have two kinds of time. First, is the time that is assigned to a state of knowledge. This is a structure consisting of a collection of moments which are in the relation of temporal succession of a given state of knowledge. Second, is the time that is not relativized to any state of knowledge. This time is the sum of the time assigned to all possible states of knowledge.

These intuitions we describe in a formal way.

- $I$ is a nonempty set.
- $T_i$ $(i \in I)$ is a nonempty set.
- $R_i (\subseteq T_i \times T_i)$.
- $\mathcal{T}_i = \langle T_i, R_i \rangle$.
- $T = \bigcup_{i \in I} T_i$.
- $R = \bigcup_{i \in I} R_i$.
- $\mathcal{T} = \langle T, R \rangle$.
- $V_i \subseteq T_i \times 2^{\mathcal{AP}}$, where $i \in I$.
- $\mathcal{F} = \{V_i : i \in I\}$.
- $m_i = \langle T_i, R_i, V_i \rangle$ where $i \in I$.
- $\mathfrak{M} = \{\langle T_i, R_i, V_i \rangle : V_i \in \mathcal{F}, i \in I\}$, so $\mathfrak{M} = \{m_i : i \in I\}$.

$I$ is a set of indexes of states of knowledge. $T_i$ is the set of moments of time in the state of knowledge indexed by $i$. $R_i$ is a binary relation on the set of moments of time in a state of knowledge indexed by $i$. The relation $R_i$ is understood as a *before-after* relation on the set of moments of time of the state of knowledge indexed by $i$. $\mathcal{T}$ is the time in the state of knowledge indexed by $i$. $T$ is the set of all moments of time of all states of knowledge. $R$ is a binary relation on $T$. Let us remark, that $R \subseteq T \times T$. $\mathcal{T}$ is the sum of times of all states of knowledge. $V_i$ is a function mapping to moments $t \in T_i$ subsets $V_i(t)$ of the set of propositional letters. $\mathcal{F}$ is a set of valuations. $m_i$ is the state of knowledge indexed by $i$. $\mathfrak{M}$ is a model based on time $\mathcal{T}$ and a class of function $\mathcal{F}$.

In model $\mathfrak{M}$ we define relation $\leq (\subseteq \mathfrak{M} \times \mathfrak{M})$

**Definition 2.2**

For any $i, j \in I$ :

$$m_i \leq m_j \equiv (T_i \subseteq T_j, R_i \subseteq R_j, \underset{t \in T_i}{\forall} V_i(t) \subseteq V_j(t)).$$

The fact that the states $m_i, m_j$ are in the relation $\leq$ ( $m_i \leq m_j$) is understood as follows: the state $m_j$ has no lesser level of knowledge than the state of $m_i$.

The relation $\leq$ is determined by the inclusions: a set of moments of time, relations of temporal successions and sets of events known at particular moments in time. So, the relation $\leq$ has all the properties of inclusions. In particular $\leq$ is reflexive and transitive.

**Theorem 1**

For any $m_i (\in \mathfrak{M})$:

$$m_i \leq m_i.$$

**Theorem 2**

For any $m_i, m_j, m_k (\in \mathfrak{M})$:

$$\text{if } (m_i \leq m_j \text{ and } m_j \leq m_k), \text{ then } m_i \leq m_k.$$

The relation $\leq$ partially orders the set of states of knowledge. Let us consider some possible cases when $m_i \leq m_j$.

The first possible case is:

$$T_i = T_j \text{ and } R_i = R_j \text{ and } \underset{t \in T_i}{\forall} (V_i(t) \subseteq V_j(t)).$$

This occurs when the sets of moments of time of states of knowledge $m_i$ and $m_j$ are the same ($T_i = T_j$ ). The same in both states of knowledge is the relation ($R_i = R_j$). State $m_j$ is formed by changing the value of the $V_i$. In other words, in this case, the state of knowledge $m_j$ is obtained by increasing the number of known facts at particular moments in time.

Another possible situation is the following:

$$T_i \subseteq T_j, R_i \subseteq R_j \text{ and } \underset{t \in T_i}{\forall} (V_i(t) \subseteq V_j(t)).$$

In this case, the state $m_j$ is formed by adding new moments of time to the time structure of the state $m_i$. At no moment of time $t\,(\in T_i)$ does not change the set $V_i\,(t)$. Change of a level of knowledge is based on the occurrences in the state $m_j$ of new moments of time (in the future or in the past). Because in the state of knowledge $m_j$ we have new moments of time, all components of $m_i$ change. We have a change in the set of moments of time. We have a change in the relation *before-after*, because some moments of time of the state $m_i$ will be in relation *before-after* with new moments of time. And finally we have a change in the $V_i$ function because its domain is changed (subsets of a set of propositional letters will be mapped to new moments of time).

Yet another possibility is:

$$T_i = T_j, R_i \subseteq R_j \text{ and } \underset{t \in T_i}{\forall} (V_i(t) \subseteq V_j(t)).$$

It might also be that the reason for change of a level of knowledge of a state of knowledge $m_i$ is not a change of the set of moments of time of $m_i$ but a change of the relation *before-after*. The change of the relation *before-after* involves known facts in these moments of time which have new connections with other moments of time.

**Abbreviation**

By $m_i^*$ (where $i \in I$ ) we means any $m_j$ ($\in \mathfrak{M}$) such that

$$m_i \leq m_j.$$

We now give a definition of the satisfiability of a sentence in the model[3].

**Definition 2.3**

Satisfiability of the sentence $\varphi$ in the model $\mathfrak{M}$, in the state $m_i$, in the moment $t$ we define in the following way:

a) $\mathfrak{M}, m_i, t \models \varphi \quad \equiv \quad \varphi \in V_i(t)$, if $\varphi \in \mathcal{AP}$,

b) $\mathfrak{M}, m_i, t \models \neg\varphi \quad \equiv \quad \underset{m_i^* \in \mathfrak{M}}{\forall} \mathfrak{M}, m_i^*, t \not\models \varphi$

c) $\mathfrak{M}, m_i, t \models \varphi \vee \psi \quad \equiv \quad \mathfrak{M}, m_i, t \models \varphi$ or $\mathfrak{M}, m_i, t \models \psi$,

d) $\mathfrak{M}, m_i, t \models \varphi \wedge \psi \quad \equiv \quad \mathfrak{M}, m_i, t \models \varphi$ and $\mathfrak{M}, m_i, t \models \psi$,

e) $\mathfrak{M}, m_i, t \models \varphi \rightarrow \psi \quad \equiv \quad \underset{m_i^* \in \mathfrak{M}}{\forall} (\mathfrak{M}, m_i^*, t \not\models \varphi \text{ or } \mathfrak{M}, m_i^*, t \models \psi)$,

f) $\mathfrak{M}, m_i, t \models F\varphi \quad \equiv \quad \underset{t_1 \in T_i}{\exists} (tR_i t_1 \text{ and } \mathfrak{M}, m_i, t_1 \models \varphi)$,

g) $\mathfrak{M}, m_i, t \models G\varphi \quad \equiv \quad \underset{m_i^* \in \mathfrak{M}}{\forall} \ \underset{t_1 \in T_i^*}{\forall} (\text{if } tR_i^* t_1, \text{ then } \mathfrak{M}, m_i^*, t_1 \models \varphi)$,

h) $\mathfrak{M}, m_i, t \models P\varphi \quad \equiv \quad \underset{t_1 \in T_i}{\exists} (t_1 R_i t \text{ and } \mathfrak{M}, m_i, t_1 \models \varphi)$,

i) $\mathfrak{M}, m_i, t \models H\varphi \quad \equiv \quad \underset{m_i^* \in \mathfrak{M}}{\forall} \ \underset{t_1 \in T_i^*}{\forall} (\text{if } t_1 R_i^* t \text{ then } \mathfrak{M}, m_i^*, t_1 \models \varphi)$.

We now give some basic definitions.

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[3] This definition uses quantifier symbols: $\forall$ – *for any*, $\exists$ – *exists*. The symbols $\forall$ and $\exists$ are not symbols of the language of $IK_t'$ system. We use them as metalanguage symbols. Moreover, we use the symbol $\nvDash$ . By $\mathfrak{M}, m_i^* t, \nvDash \varphi$ we mean *it is not true, that* $\mathfrak{M}, m_i^*, t \vDash \varphi$.

**Definition 2.4**

$\mathfrak{M} \models \varphi$ ($\varphi$ is true in a model $\mathfrak{M}$) iff for any state of knowledge $m_i (\in \mathfrak{M})$ and for any $t (\in T_i)$ holds $\mathfrak{M}, m_i, t \models \varphi$.

**Definition 2.5**

$\mathcal{T} \models \varphi$ ($\varphi$ is true in time $\mathcal{T}$) iff $\varphi$ is true in a model $\mathfrak{M}$ for any nonempty class of function $\mathcal{F}$ $(= \{V_i : i \in I\})$.

**Definition 2.6**

$\models \varphi$ ($\varphi$ is true) iff for any $\mathcal{T}$, $\mathcal{T} \models \varphi$.

Between tense logic systems based on classical logic and tense logic systems based on intuitionistic logic, there are many differences. One of them is that in the intuitionistic tense logics falsehood $\neg\varphi$ does not imply the truth of $\varphi$.

Let us consider some particular case. A sentence $\varphi$ is not known in the state $m_i$ at the time $t (\in T_i)$, however $\varphi$ is known at this time in the state $m_j$ which has no lesser level of knowledge than the level of knowledge of the state $m_i$. If from the fact that $\varphi$ is not known at the time $t$ in the state of knowledge $m_i$ we conclude that in this state of knowledge at time $t$ $\neg\varphi$ is known, then – according to the definition of satisfiability – $\varphi$ could not be known at $t$ in any state of knowledge $m_i^*$. In particular, the sentence $\varphi$ could not be known at the time $t$ in the state $m_j$. This leads to a contradiction, because we get the sentence $\varphi$ is known and unknown in $t$ in the state $m_j$. If $\varphi$ is known at some moment of time in state $m_i$, then in each state of knowledge $m_i^*$ at $t$ the sentence $\varphi$ is known. However, if at some moment of time the sentence $\varphi$ is not known, then it does not mean that at this moment, in every state of knowledge $m_i^*$ the sentence $\varphi$ is known.

Now we prove the lemma, which expresses the monotonicity of knowledge in the system $IK_t'$. What is known in the state $m_i$ is also known in every state of knowledge with a level of knowledge which is not lesser than the level of knowledge of the state $m_i$.

**Lemma 1**

For any $\varphi$ and for any $m_i, m_j \in \mathfrak{M}$:

$$\text{if } (m_i \leq m_j \text{ and } \mathfrak{M}, m_i, t \models \varphi), \text{ then } \mathfrak{M}, m_j, t \models \varphi.$$

**Proof.**

We prove this by induction with respect to the length of $\varphi$. Let us assume that $m_i \leq m_j$.

($\varphi \in \mathcal{AP}$)

First, let us consider a case when $\varphi$ is a propositional letter.
From the definition 2.2 if $m_i \leq m_j$, then for any $t \in T_i$ holds

$$V_i(t) \subseteq V_j(t). \tag{1}$$

If $\mathfrak{M}, m_i, t \models \varphi$, then from the definition 2.3

$$\varphi \in V_i(t). \tag{2}$$

From (1) and (2) we have

$$\varphi \in V_j(t). \tag{3}$$

Because $\varphi$ is a propositional letter, then from (3) and the definition 2.3 we have $\mathfrak{M}, m_j, t \models \varphi$.

**Inductive assumption:**

Let $\varphi, \psi$ satisfy the following conditions:

a) if $\mathfrak{M}, m_i t \models \varphi$, then $\mathfrak{M}, m_j, t \models \varphi$,

and

b) if $\mathfrak{M}, m_i, t \models \psi$, then $\mathfrak{M}, m_j, t \models \psi$.

Now, let us consider sentences built from $\varphi, \psi$, connectives and tense operators.

($\neg\varphi$)

Let us assume that $\mathfrak{M}, m_i, t \models \neg\varphi$.
From the definition 2.3.b we have:

$$\text{for any } m_k, \text{ such that } m_i \leq m_k \text{ it is true, that } \mathfrak{M}, m_k, t \not\models \varphi. \tag{1}$$

Let us consider any state of knowledge $m_l$ with a level of knowledge not lesser than the level of knowledge of the state $m_j$,

$$m_j \leq m_l. \tag{2}$$

From (2), the assumption $m_i \leq m_j$ and a transitivity of relation $\leq$ we have $m_i \leq m_l$. Thus, from (1) we have $\mathfrak{M}, m_l, t \not\models \varphi$. Because $m_l$ is any state of knowledge with a level of knowledge not lesser than the level of knowledge of $m_j$, we have:

$$\text{for any } m_l \text{ such that } m_j \leq m_l \text{ is true, that } \mathfrak{M}, m_l, t \not\models \varphi. \tag{3}$$

From (3) and the definition 2.3.b we have

$$\mathfrak{M}, m_j, t \models \neg\varphi.$$

$(\varphi \wedge \psi)$

Let us assume $\mathfrak{M}, m_i, t \models \varphi \wedge \psi$.
Thus, from the definition 2.3.d:

$$\mathfrak{M}, m_i, t \models \varphi, \tag{1}$$

and

$$\mathfrak{M}, m_i t \models \psi. \tag{2}$$

From 1) and point a) of inductive assumption we have:

$$\mathfrak{M}, m_j, t \models \varphi. \tag{3}$$

Analogous, from 2) and point b) of inductive assumption we have:

$$\mathfrak{M}, m_j, t \models \psi. \tag{4}$$

From (3), (4) and the definition 2.3.d we have $\mathfrak{M}, m_j, t \models \varphi \wedge \psi$.

$(\varphi \vee \psi)$

Proof is analogous to the case of conjunction.

$(\varphi \to \psi)$

Let us assume $\mathfrak{M}, m_i, t \models \varphi \to \psi$.
From the definition 2.3.e we have:

$$\underset{m_i^* \in \mathfrak{M}}{\forall} (\mathfrak{M}, m_i^*, t \not\models \varphi \text{ or } \mathfrak{M}, m_i^*, t \models \psi), \tag{1}$$

Let us consider any state of knowledge $m_l$ with a level of knowledge which is not lesser than the level of knowledge of the state $m_j$:

$$m_j \leq m_l. \tag{2}$$

From (2), the assumption $m_i \leq m_j$ and the transitivity of the relation $\leq$ we have $m_i \leq m_l$,. Thus from (1) we have $\mathfrak{M}, m_l, t \not\models \varphi$ or $\mathfrak{M}, m_l, t \models \psi$. Because $m_l$ is any state of knowledge with the level of knowledge which is not lesser than the level of knowledge of the state $m_j$, we have:

$$\text{for any } m_l \text{ such that } m_j \leq m_l \text{ holds: } \mathfrak{M}, m_l, t \not\models \varphi \text{ or } \mathfrak{M}, m_l, t \models \psi). \tag{3}$$

From(3) and the definition 2.3.e) we have $\mathfrak{M}, m_j, t \models \varphi \to \psi$.

$(G\varphi)$

Let us assume $\mathfrak{M}, m_i, t \models G\varphi$. From the definition 2.3.g:

$$\underset{m_i^* \in \mathfrak{M}}{\forall} \underset{t_1 \in T_i^*}{\forall} (\text{ if } tR_i^* t_1, \text{ then } \mathfrak{M}, m_i^*, t_1 \models \varphi), \tag{1}$$

Let us consider any state of knowledge $m_l$ with a level of knowledge which is not lesser than the level of knowledge of the state $m_j$,:

$$m_j \leq m_l. \tag{2}$$

From (2), assumption $m_i \leq m_j$ and the transitivity of the relation $\leq$ we have $m_i \leq m_l$. Thus, from (1) we have:

$$\text{for any } t_1 (\in T_l) \text{ such that } tR_l t_1 \text{ holds } \mathfrak{M}, m_l, t \models \varphi. \tag{3}$$

Because the state of knowledge $m_l$ is a state of knowledge with a level of knowledge which is not lesser than the level of knowledge of the state of knowledge $m_j$, we have

$$\underset{m_l}{\forall} \underset{t_1 \in T_l}{\forall} (\text{ if } m_j \leq m_l \text{ and } tR_l t_1, \text{ then } \mathfrak{M}, m_l, t_1 \models \varphi). \tag{4}$$

From (4) and definition 2.3.g we have $\mathfrak{M}, m_j, t \models G\varphi$

$(H\varphi)$

Proof is analogous to the case of the $G$ operator.

$(F\varphi)$

Let us assume $\mathfrak{M}, m_i, t \models F\varphi$. From definition 2.3.f there is a moment $t_1 (\in T_i)$, $tR_i t_1$, such that:

$$\mathfrak{M}, m_i, t_1 \models \varphi. \tag{1}$$

From (1) and point a) of the inductive assumption

$$\mathfrak{M}, m_j, t_1 \models \varphi. \tag{2}$$

From the assumption $m_i \leq m_j$, the definition 2.2 and the definition of inclusion we have:

$$t \in T_j, t_1 \in T_j, tR_j t_1. \tag{3}$$

Thus from (2),(3) and the definition 2.3.f we obtain $\mathfrak{M}, m_j, t \models F\varphi$.

$(P\varphi)$

Proves analogous to the case of the $F$ operator. □

We have shown thus monotonicity of knowledge is described using the language of $\mathfrak{L}_{IK'_t}$. Everything that is known in the state of $m_i$, is also known in every state of knowledge with a level of knowledge not lesser than the level of knowledge of the state $m_i$.

### 2.3. Axiomatization of $IK'_t$

The $IK'_t$ is axiomatizable. The set of axioms of $IK'_t$ consists of axioms $A1-A10$, which are substitutions of intuitionistic propositional logic axioms and specific axioms $H1-G7$. Rules of inferences in $IK'_t$ are: Modus Ponens $MP$ and temporal generalization rules $RH$, $RG$.

**2.3.1. Axioms**

For any $\varphi, \psi, \gamma \in FOR(\mathfrak{L}_{IK'_t})$ :

A1) $\varphi \to (\psi \to \varphi)$,

A2) $(\varphi \to \psi) \to \{[\varphi \to (\psi \to \gamma)] \to (\varphi \to \gamma)\}$,

A3) $[(\varphi \to \gamma) \wedge (\psi \to \gamma)] \to [(\varphi \vee \psi) \to \gamma]$,

A4) $(\varphi \wedge \psi) \to \varphi$,

A5) $(\varphi \wedge \psi) \to \psi$,

A6) $\varphi \to [\psi \to (\varphi \wedge \psi)]$,

A7) $\varphi \to (\varphi \vee \psi)$,

A8) $\psi \to (\varphi \vee \psi)$,

A9) $(\varphi \wedge \neg\varphi) \to \psi$,

A10) $(\varphi \to \neg\varphi) \to \neg\varphi$,

H1) $H(\varphi \to \psi) \to (H\varphi \to H\psi)$,

H2) $H(\varphi \to \psi) \to (P\varphi \to P\psi)$,

H3) $\varphi \to HF\varphi$,

H4) $PG\varphi \to \varphi$,

H5) $P(\varphi \vee \psi) \to (P\varphi \vee P\psi)$,

H6) $(P\varphi \to H\psi) \to H(\varphi \to \psi)$,

H7) $P\varphi \to \neg H\neg\varphi$,

G1) $G(\varphi \to \psi) \to (G\varphi \to G\psi)$,

G2) $G(\varphi \to \psi) \to (F\varphi \to F\psi)$,

G3) $\varphi \to GP\varphi$,

G4) $FH\varphi \to \varphi$,

G5) $F(\varphi \vee \psi) \to (F\varphi \vee F\psi)$,

G6) $(F\varphi \to G\psi) \to G(\varphi \to \psi)$,

G7) $F\varphi \to \neg G\neg\varphi$.

Rules:

Modus Ponens $MP$: $\dfrac{\varphi \to \psi, \varphi}{\psi}$.

Temporal generalization rules:

$$RH: \quad \frac{\vdash_{IK'_t} \varphi}{\vdash_{IK'_t} H\varphi} \qquad\qquad RG: \quad \frac{\vdash_{IK'_t} \varphi}{\vdash_{IK'_t} G\varphi}.$$

In the $K_t$ system (minimal system of tense logic based on classical propositional logic) specific axioms are $H1$, $G1$, $H3$, $G3$. Other axioms of the $IK'_t$ are theorems of $K_t$ system. Because when we prove these theorems we use some theses of classical propositional logic which are not provable in intuitionistic logic, but are true in any model, then we add these formulas to the set of axioms of $IK'_t$ system. Thus the axioms $H2$, $H4$, $H5$, $H6$, $H7$ and $G2$, $G4$, $G5$, $G6$, $G7$ are also axioms of $IK'_t$.

The system $IK'_t$ is complete.

**Theorem 3** ([5])

Let $\Sigma$ be a set of sentences of the language $\mathfrak{L}_{IK'_t}$. For any $\varphi \in \Sigma$ :

$$\Sigma \vdash_{IK'_t} \varphi \text{ iff } \Sigma \models_{IK'_t} \varphi.$$

## 3. Conclusion

Intuitionistic logic and knowledge are closely related. An epistemic approach is the epicenter of the intuitionistic Brouwerian explanation of the truth as provable by ideal mathematics, or more generally, the ideal cognitive agent. Intuitionistic Kripke models well model the evolutionary process of acquiring knowledge (information) by agents. It could be asked whether such an approach, the notion of *being true* and *to be known* are to be understood by the ideal cognitive agent as two different terms, or should the terms be equated with each other? It seems that if we use intuitionistic logic for modeling mathematical knowledge only, these two notions are not significantly different. However, when we apply this logic to modeling empirical reasoning, the distinction between these concepts is necessary.

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