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Zygmunt Vetulani

WORDNET AND GÖDEL'S COMPLETENESS THEOREM

Abstract. We claim that Princeton WordNet-like lexical data bases (wordnets) may be considered as a natural conceptualization of the world in the form of a language-derived ontology determined by the linguistic concept of synonymy. We discuss some constraints on synonymy relations which must be satisfied in order to make sure that wordnet will behave as ontology and will reflect linguistic relations. We show a close relationship between the concept of wordnet and Gödel's Completeness Theorem whose proof is based on the fact that every consistent formal theory has a model. In particular, we show that, under some assumptions, wordnets may be generated by Henkin's algorithm of constructing such a model.[1]

1. Introduction

Using a wordnet (which encodes relations between words) to natural language processing depends on how much these relations correspond to relations between the entities in the real world. There is a general agreement to consider, after the Princeton WordNet creators, that the main wordnet organizing relations are hyponymy/hyperonymy and synonymy [Miller *et al.*, 1990]. This opinion has important consequences due to the mathematical properties of these relations: transitivity, asymmetry and irreflexivity for hyponymy and hyperonymy, as well as reflexivity, symmetry, and transitivity for synonymy.

2. Words and word-meaning pairs

Let us notice that satisfaction of these properties depends on some important prerequisites. First, due to the common (for natural languages) phe-

[1] In this paper we will use the term "wordnet" to designate lexical data bases inspired by Princeton WordNet.

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nomenon of polysemy, we assume that the above mathematical properties have to be applied not to words but to disambiguated words called word-meaning pairs in the rest of this paper. In other terms, a word-meaning pair is a word with one precise meaning. Meaning identification is indispensable to understanding text or discourse. Meaning distinction is a common practice in most dictionaries. Some of them also include information about pragmatic aspects such as frequency, register etc. As the ability to recognize the correctness/incorrectness of a sentence (abstraction making of its truth value) is a part of language competence, we will propose to use the following criterion to recognize that the given word W is used with different meanings in the given sentences A and B. This is so when the following hold:

- both sentences A and B are correct,
- there exists such a meaning of W that A is correct with this meaning while B is not, or conversely.

For instance the word "window" in sentences like "He was seen this morning watching through the window" and "You need to open the window in the top left corner of the screen".

3. Synonymy

For the word-sense pairs we define synonymy with the help of the notion of substitutivity. Already Leibnitz used this concept in his definition of synonymy of words (citation after [Vossen, 2002]):

> two expressions are synonymous if the substitution of one for the other never changes the truth value of a sentence in which the substitution is made.

According to [Vossen, 2002], Miller and Fellbaum observed that with such strong definiens, very few words will have synonyms – which is against common intuition about synonymy; they propose to modify Leibnitz's definition to some "linguistic context C" as follows [Miller *et al.*, 1990]:

> two expressions are synonymous in a linguistic context C if the substitution of one for the other in C does not alter the truth value.

Vossen [2002] remarks that [Miller *et al.*, 1990] suggest that it is enough to find one such context to apply the substitutivity criterion for synonymy. As an example of a pair of synonyms in the sense of Miller's definition, Vossen discusses "appearance" and "arrival" (see citation in Section 4, below). These words are presented as synonymous, as in some appropriate contexts C they satisfy the above substitutivity criterion of synonymy. On the other hand, comparing them respectively to "disappearance" and "departure", one observes that some kind of opposition holds between "arrival" and "departure" but doesn't hold between "arrival" and "disappearence" despite that "disappearence" and "departures" are considered both as being antonyms for, respectively "appearence" and "arrival". This means that the appropriate extension of the context can lead to the conclusion that these two words (with unchanged meaning) are no more substitutable in the extended context. (This example is also explored in the discussion of antonymy in Section 5). Applying different contexts to compare different word pairs may therefore result in a situation where synonymy will no more be transitive, which would invalidate the idea of synset as an equivalence class. It seems that a (theoretical) solution to this problem (in application to word-meaning pairs rather than to words) will consist in further generalization of the approach proposed by Miller and Fellbaum, namely in applying the substitutivity requirement with respect to some (possibly large) class of sentences (context class) selected in a way to guarantee synonymy to be reflexive, symmetric, and transitive (i.e. to be an equivalence relation).[2] For a given class Z of sentences we will define "synonymy respective to Z" by restricting substitutivity to Z. It is clear that if Z stands for the totality of sentences about the world, then synonymy with respect to Z will be identical with the synonymy in the sense of Leibnitz (for word sense pairs).[3]

Having the relation of synonymy already defined in such a way that it has the properties of an equivalence relation, we define synsets as equivalence classes with respect to this relation.

4. Wordnet as ontology

Relations holding between word-meaning pairs (disambiguated words) may in a natural way be mapped to synsets if only synonymy is congruent with these relations. By definition, the given equivalence relation is told to be congruent for the relation R if the fact that R holds/does not hold for some elements E means that, respectively, R holds/does not hold for the elements

[2] We consider this solution to be theoretical, as we are aware of the possibly large size of such a theoretical context, so that in practice a limited context will have to be used.

[3] The problems discussed above were (probably) the reasons for Vossen and others [Vossen, 2002] to make another decision for EuroWordNet. Instead of referring to the concept of substitutivity they decided to define synonymy using linguistic tests to compare the extension of words (or word sense pairs).

which are equivalent to E.[4] In a case where synonymy is congruent with respect to hyperonymy and hyponymy we say that the quotient structure (composed of synsets and hyperonymy/hyponymy mapped to synsets) is a *basic wordnet* (or briefly *wordnet*).

The basic wordnet can be considered as an ontology in which concepts (represented by synsets) are directly related to the language. The interest of considering a wordnet as an ontology for NLP applications is that it directly reflects the conceptualization of the world in the same way as does the natural language (more precisely the natural language that the corresponding wordnet is derived from), i.e. it is culturally dependent.[5] This is why wordnets are interesting as ontology candidates for natural language processing applications.

The fact that synonymy may be defined with reference to an external parameter (context class) which may be modified according to the users' needs makes it possible to extend the basic wordnet by introducing other relations mapped from the linguistic relations between word-meaning pairs, such as antonymy, metonymy, and other. In practice, introduction of new relations to the existing wordnets may be difficult because it is necessary to make sure that the synonymy is congruent with the linguistic relations we wish to be mapped to synsets. This may require redefinition of the synonymy (by modification of the context class used in the definition of synonymy) and change the wordnet granulation (refinement).

The case of antonymy is a well described example of a mapping-to-wordnet problem for a linguistic relation. Both G. A. Miller and P. Vossen, designers and developers of wordnets, articulated their doubts about the possibility to express antonymy at the wordnet level. Vossen [2002] wrote the following in the final EuroWordNet report.

> Antonymy relates lexical opposites, such as "to ascend" and "to descend", "good" and "bad" or "justice" and "injustice". It is clear that antonymy is a symmetric relation, but little more can be said, since it seems to encode a large range of phenomena of opposition, e.g. "rich" and "poor" are scalar opposites with many values in between the extremes, "dead" and "alive" can be seen as complementary opposites (...). It is also unclear whether antonymy stands between either word forms or word meanings. For instance, "appearance" and "arrival" are, in the appropriate senses, synonyms; but linguistic

[4] By "mapping to synsets" we mean the act of defining the relation R' operating on the equivalence classes to hold if and only if the relation R holds for some elements of these equivalence classes; the equivalence classes form the so called quotient structure.

[5] The EuroWordNet and other similar projects (as e.g. BalkaNet) are nice and quite successful attempts to integrate various "national", culturally-dependent wordnets.

> intuition says that the appropriate antonyms are different for each word ("disappearance" and "departure"). With respect to this, EWN (EuroWordNet) will assume the solution adopted by Miller's WordNet, that is, antonymy is considered to be a relation between word forms, but not between word meanings – namely synsets. Therefore, in the example above, the antonymy relation will hold between "appearance" and "disappearance", "arrival" and "departure" as word forms. In those cases that antonymy also holds for the other variants of the synset we use a separate NEAR_ANTONYM relation. (...)

It seems however that with an appropriate understanding of antonymy and synonymy there is no need to go so far and to resign from having antonymy defined on synsets. Let us assume that A is a set of pairwise orthogonal binary attributes. By antonymy (restricted to nouns) with respect to A we mean such a relation which holds between two word-meaning pairs if and only if there is exactly one attribute from A for which these word-meaning pairs take opposite values. The sufficient condition for synonymy to be congruent with respect to antonymy is that "antonyms of any two synonymous word-meaning pairs are synonymous to each other". To make this condition true we must further restrict the synonymy relation by considering appropriate sentences related to the attributes A as a part of the context set used to define synonymy. It follows that imposing these new restrictions to the definition of synonymy may cause further fragmentation of synsets.

5. Gödel's completeness theorem and wordnets

Considering wordnets as natural[6] ontologies in which concepts are represented by language entities appears to be compatible with the correspondence between semantic consequence (entailment) and syntactic provability in first-order logic established by Kurt Gödel [1929; 1930]. This correspondence directly follows from the so called completeness theorem which is a simple conclusion from the statement that "each consistent first order theory has a model"[7] (i.e. there exists a world in which the theory is true). The Henkins [1949] proof of this theorem shows how to construct, for a consistent theory in first-order logic, an algebraic structure which is a model for this theory. It appears that, for a given wordnet, a consistent theory T

[6] "Natural" means here "corresponding to the conceptualization shared by language users".

[7] We use the term "model" in the sense it has within the mathematical *model theory.*

may be found for which the Henkins construction will result with a model isomorphic with this wordnet. The Henkins idea of constructing the model consists in adding new constants, and then in constructing (in the extended language) a complete set of sentences[8] consistent with the given theory. In particular among the new individual constants the model construction algorithm selects individual names for entities whose existence is postulated by existential sentences belonging to the constructed complete set of sentences. The idea of the construction of a model of the theory T follows the following lines. First we extend the language L of the theory T to the language L' introducing a new countable infinite set C of individual constants. Then we build an infinite sequence $\langle S_n \rangle_{n=0,1,\ldots}$ of set of sentences of the language L'. Let $P = \langle P_n \rangle_{n=0,1,\ldots}$ be the sequence enumerating all prenex-normal-form formulas of the language L'. We construct successive consistent extensions S_n of T by first putting T as S_0 and then considering one by one the formulas from the sequence of sentences P. At each step we check whether the considered formula belongs to $Cn(S_n)$ or not.[9] If the negation of the formula P_n considered in step n belongs to $Cn(S_n)$ (i.e. is a logical consequence of S_n), then we add it to S_n and obtain S_{n+1}, otherwise we consider two subcases. If the formula P_n is not an existential sentence, then we simply add it to S_n to obtain S_{n+1}. Otherwise, together with this existential sentence we add to S_n the sentence in which the existential quantifier is omitted and the variable bounded by the quantifier in the considered sentence is replaced by a new (i.e. still not used in this extension procedure) constant from C (this sentence is a constructive witness of the existence of an entity with the required property). We denote by S the union $\bigcup_{n \in N} S_n$ of all successive consistent extensions of T. The resulting set of formulas S is consistent and complete, i.e. for each sentence ϕ of the extended language L' either this sentence ϕ or its logical negation $not(\phi)$ belongs to S. Then we define in the set C of individual constants the following relation $\sim$ which turns out to be an equivalence relation. Specifically, we define

$$s_1 \sim s_2 \leftrightarrow_{df} \text{`}s_1 = s_2\text{'} \in Cn(S)$$

This relation can be extended to the whole Herbrand universe U of the language L' (here the Herbrand universe consists of all terms with no

[8] By a *complete set of sentences* we mean any set S of sentences that for any given sentence φ, either ϕ belongs to S or its negation $not(\phi)$ belongs to S.

[9] $Cn(S_n)$ is the set of all logical consequences of S_n, i.e. the set of all sentences provable from S_n.

free variables). This relation is a congruence with respect to all relations $\Re$ defined in a natural way through S for all R as follows:

$$\Re(t_1, t_2, \ldots, t_k) \leftrightarrow_{df} \text{`}\Re(t_1, t_2, \ldots, t_k)\text{'} \in Cn(S)$$

The set of equivalence classes over U with induced relations forms the quotient structure. The rest of the proof of Gödel's theorem consists in showing that this quotient structure is a model for S, and therefore for T.[10]

Our further considerations depend on the assumption following Montague's famous conjecture that natural language may (with some limits) be considered as a formal language and, therefore, that Tarski's truth concept is applicable. In particular we will also consider word-meaning pairs instead of words. The validity of considerations will be limited to the fragment of natural language which, in accordance to the ideas of Richard Montague, may be considered as equivalent to the language of first-order predicate logic.[11] With this assumption, the set of true sentences about the real world may be considered as a theory of the natural model (the natural model being the conceptualization of the real world). Clearly, the theory of the natural model is consistent.

The relation of synonymy, introduced above in our paper with reference to the idea of substitutivity, may be formalized in the following way. By an *NL context* (or *context*, for short) we mean a text with a variable X occurring one or more times. By $K(s)$ we will denote the context K with all occurrences of X substituted by (the appropriate form of) s. Let us consider the finite class of contexts $K = \{context_i : i = 1, 2, \ldots, k\}$ with the property of distinguishing all word meanings for any given word. Let us consider the relation $\approx$ in the set of word-meaning pairs of the natural language defined as follows:

$$s_1 \approx s_2 \leftrightarrow_{df} \text{the sentence `}\forall_{i=1,2,\ldots,k}\ (context_i(s_1) \leftrightarrow context_i(s_2))\text{'} \\ \textit{is true in the natural model}$$

The relation $\approx$ is an equivalence relation in the set of word-meaning pairs and – according to our terminology – the corresponding equivalence classes are synsets.

Let us observe that the correct use of synonyms by natural language speakers do not entail contradictions (provided that the speakers correctly distinguish the word-meanings). This means that we may assume that

[10] For more details the reader may consult the Grzegorczyk [1974] or Shoenfield [1967] textbook on the foundations of logic.

[11] c.f. the paper by R Montague [1970] on English as formal language.

the word sense pairs satisfy the additional *equality axiom for synonyms* in the form of the following formula:

$$`\forall_{i=1,2,\ldots,k}(context_i(s_1) \leftrightarrow context_i(s_2))' \rightarrow s_1 = s_2$$

In other terms, the equality axiom for synonymy is consistent with the theory of the natural model (for the correctly chosen set of contexts K). This last remark allows us to claim that Henkin's model constructed to prove Gödel's conjecture, when applied to natural language, is built out of natural language synsets (the relations $\sim$ and $\approx$ are in fact identical).

6. Conclusions

The above considerations allow us to notice a close connection between two, at first sight very different from each other, scientific ideas. The first one is that wordnets conceived as networks of connections between natural language words appear to be natural ontologies whose concepts are directly linked to language entities (which means that these ontology concepts may be represented in computers in a way that eases their application in natural language processing). We have discussed some essential theoretical problems related to the theoretical foundations of the concept of wordnet, mainly those connected with the nature of the relation of synonymy and we have presented the algebraic structure of the wordnet(s), as well as some fine problems connected with the operation of mapping the linguistic relations to the universe of synsets. The second of the two is the idea of the constructive proof of Gödel's completeness theorem which contributes to a better understanding of the relationship between syntactic consequence (entailment) and semantic consequence. The key element of this proof is a procedure to construct the model for a consistent theory. This model is built out of terms of a (formal) language. We have shown (under some assumptions) that a model constructed according to this procedure for a consistent set of true sentences about the world is equal to a wordnet. This means that the model postulated by Gödel's theorem corresponds to the natural conceptualization of knowledge about the world represented in natural language. Our final message is that one may safely claim that Kurt Gödel's work was an early portent for the idea of a wordnet as a natural ontology whose concepts are directly linked to words.

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Zygmunt Vetulani
Adam Mickiewicz University in Poznań
`vetulani@amu.edu.pl`