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CHARACTERIZATION OF CLASSES OF FRAMES IN MODAL LANGUAGE

Abstract. In the paper some facts about the definability of classes of Kripke frames for tense logic are discussed. Special attention is given to the classes of frames definable by Grzegorzczuk's Axiom:

$$\Box(\Box(\phi \rightarrow \Box\phi) \rightarrow \phi) \rightarrow \phi$$

as interpreted in temporal logic.

1. Temporal logic

Let us consider Kripke¹ (relational) semantics $\langle T, < \rangle$, where T is a non-empty set (of time points) and $<$ is a binary relation on T , $< \subseteq T \times T$ (the precedence relation – *earlier/later*). The flow of time \mathfrak{T} is represented as $\langle T, < \rangle$. If a pair of elements (t, t_1) belongs to $<$ we say that t is *earlier than* t_1 , or: t_1 is *later than* t .

$\{t : n < t\}$ is the future of n . The past of n is defined likewise: $\{t : n > t\}$.

Models of temporal logic TL are triples $\langle T, <, V \rangle$ consisting of a frame $\langle T, < \rangle$, the flow of time \mathfrak{T} , together with a valuation V , where V is a function assigning each propositional letter a subset of T : $V \rightarrow 2^T$. Intuitively, V to a propositional letter assigns a set of moments of time in that the letter is satisfied (true).

Besides classical propositional connectives ($\neg, \vee, \wedge, \rightarrow, \leftrightarrow$) we will have temporal operators that are defined on $\langle W, < \rangle$.

¹ The question who invented relational semantics is a subject of discussion. Disputed is the role of Alfred Tarski [Goldblatt, 2005, p. 17]. There are some reasons to point to Leibniz as its inventor. For Leibniz the actual world is the one of the best of all possible worlds. He maintains that [Goldblatt, 2005, p. 18]:

Not only will they hold as long as the world exists, but also they would have held if God had created the world according to a different plan.

An answer to the question who invented relational models is given by Saul Kripke [Goldblatt, 2005, p. 22].

To each operator defined with the help of $<$ there is a symmetric-to-it operator defined with the help of the converse (inverse, transpose) relation of $<$, i.e. with the help of $>$. Temporal logic (*TL*) uses this possibility, distinguishing past tense and future tense operators.

Vocabulary

1. p_0, p_1, \dots – propositional letters, *Prop*;
2. $\neg, \vee, \wedge, \rightarrow, \leftrightarrow$ – propositional connectives: negation, disjunction, conjunction, implication, equivalence, respectively;
3. G, F, H, P – tense operators:
 - G – it will always be the case that ϕ , or: henceforth, ϕ ;
 - F – it will be the case that ϕ , or: ϕ is true at some time in the future;
 - H – it has always been the case that ϕ , or: hitherto, ϕ ;
 - P – it has been the case that ϕ , or: ϕ was true at some time in the past.

Tense logic is the study of tense operators, and of the logical relations between sentences having tense. The study of this logic has been initiated and developed by Arthur Norman Prior, e.g. [1957; 1962; 1967; 1968].

Definition 1 (well formed formula)

$$\phi ::= p_i, i \in \mathbb{N} \mid \neg\phi \mid \phi \vee \psi \mid \phi \wedge \psi \mid \phi \rightarrow \psi \mid \phi \leftrightarrow \psi$$

We will also frequently refer to the mirror image of a formula; this is simply the formula one obtains by simultaneously replacing all H s and P s with G s and F s, respectively, and vice versa. The mirror image of a formula ϕ will be denoted: $MI(\phi)$.

Definition 2 (the satisfiability of a formula at a point of time)

Let \mathfrak{M} be a model $\langle T, <, V \rangle$ and let $t \in T$:

1. $\mathfrak{M}, t \models \phi$ iff $t \in V(\phi)$, if $\phi \in Prop$;
2. $\mathfrak{M}, t \models \neg\phi$ iff $\mathfrak{M}, t \not\models \phi$;
3. $\mathfrak{M}, t \models \phi \vee \psi$ iff $\mathfrak{M}, t \models \phi$ or $\mathfrak{M}, t \models \psi$;
4. $\mathfrak{M}, t \models \phi \wedge \psi$ iff $\mathfrak{M}, t \models \phi$ and $\mathfrak{M}, t \models \psi$;
5. $\mathfrak{M}, t \models \phi \rightarrow \psi$ iff $\mathfrak{M}, t \not\models \phi$ or $\mathfrak{M}, t \models \psi$;
6. $\mathfrak{M}, t \models \phi \leftrightarrow \psi$ iff $\mathfrak{M}, t \models \phi$ if and only if $\mathfrak{M}, t \models \psi$;
7. $\mathfrak{M}, t \models H\phi$ iff $\forall t_1, t_1 < t : \mathfrak{M}, t_1 \models \phi$;
8. $\mathfrak{M}, t \models P\phi$ iff $\exists t_1, t_1 < t : \mathfrak{M}, t_1 \models \phi$;

9. $\mathfrak{M}, t \models G\phi$ iff $\forall t_1, t < t_1 : \mathfrak{M}, t_1 \models \phi$;
 10. $\mathfrak{M}, t \models F\phi$ iff $\exists t_1, t < t_1 : \mathfrak{M}, t_1 \models \phi$;

Kamp [1968] introduced two operators U (until) and S (since), and he showed that over the class of complete linear temporal orders, the formalism is expressively complete [Gabbay, 1981a; Gabbay and Hodkinson, 1990].

11. $\mathfrak{M}, t \models U(\phi, \psi)$ iff $\exists t_1, t < t_1 : \mathfrak{M}, t_1 \models \phi$ and $\forall t_2, t < t_2 < t_1 : \mathfrak{M}, t_2 \models \psi$,
 12. $\mathfrak{M}, t \models S(\phi, \psi)$ iff $\exists t_1, t_1 < t : \mathfrak{M}, t_1 \models \phi$ and $\forall t_2, t_1 < t_2 < t : \mathfrak{M}, t_2 \models \psi$,

The mirror image of ϕ , $MI(\phi)$, is obtained by simultaneously replacing S by U and U by S , everywhere in ϕ , and other temporal operators according to the former rule of forming of $MI(\phi)$.

Definition 3 (validity of a formula in a model)

$$\langle T, <, V \rangle \models \phi \text{ iff } \forall t \in T : \mathfrak{M}, t \models \phi.$$

Definition 4 (validity of a formula in a frame)

$$\langle T, < \rangle \models \phi \text{ iff } \forall V : \rightarrow 2^T : \langle T, <, V \rangle \models \phi.$$

Definition 5 (validity in a class of frames)

Let \mathfrak{F} be a class of frames. $\mathfrak{F} \models \phi$ if and only if for any \mathfrak{T} : if $\mathfrak{T} \in \mathfrak{F}$, then $\mathfrak{T} \models \phi$.

Less would do since actually all propositional truth functions can be defined for instance in terms of \neg and \rightarrow ; moreover P can be defined as $\neg H\neg$ and F can be defined as $\neg G\neg$. H and G can be defined with the help of U and S : $H\phi \leftrightarrow S(\perp, \phi)$, $G\phi \leftrightarrow U(\perp, \phi)$.

2. Definability of classes of frames

Definition 6

A formula ϕ characterizes a class of frames \mathfrak{F} if and only if

$$\mathfrak{T} \models \phi \text{ iff } \mathfrak{T} \in \mathfrak{F}.$$

Let \mathfrak{F}_ϕ denote the class of frames characterized by a formula ϕ .

If a class of frames is characterized by a formula ϕ , or – in other words – is definable by ϕ , we say that the formula expresses this class of frames. Conditions \mathfrak{C} imposed on the relation $<$, $< \in \mathfrak{C}$, are expressible by a formula ϕ if and only if the class of frames $\langle T, < \rangle$, such that $<$ fulfills the conditions \mathfrak{C} , is characterized (definable) by the formula ϕ .

We ask what classes of frames are characterized by a formula (or set of formulas) of TL .

Lemma 7

If $\langle T, < \rangle \models \phi$, then $\langle T_1, <_1 \rangle \models \phi$, where T_1 has exactly one element, and $<$ is empty or universal.

Proof.

By definition ϕ is satisfied for any valuation such that $V(p_i) = V(p_j)$, for any $i, j \in \mathbb{N}$. We do not assume that if $t_1 < t_2$, then $t_1 \neq t_2$, and there is not such an assumption in definitions of temporal operators. Hence $\langle T_1, <_1 \rangle \models \phi$, where T_1 has exactly one element. In one element set $\{t\}$ there are definable only two binary relations: empty and universal, i.e. \emptyset and $\{t, t\}$, respectively. ■

Let us remark that the empty relation is irreflexive.

Theorem 8

Irreflexivity is not expressible in TL .²

Proof.

In the case of operators which are defined without any assumption about the relation $<$, by the lemma 7, any formula that is satisfied in any frame $\langle T, < \rangle$ is also satisfied in a frame $\langle T_1, <_1 \rangle$, where T is a one-element set and $<_1$ is either empty or universal. If $<_1$ is universal, then $<$ is reflexive. Empty relations are irreflexive, but not all irreflexive relations are empty.

Let us consider a language whose some operators are defined by assuming that if $t < t_1$, then $t \neq t_1$. In such a case if a formula with such an operator is satisfied in a model $\langle T, <, V \rangle$, then the formula is satisfied in a model $\langle T, \leq, V \rangle$. Hence irreflexivity is not expressible. ■

Irreflexivity is not the only property which is not expressible in TL .

² Irreflexivity is expressible by Gabbay's [1981b] Irreflexivity Rule, IRR:

$$\frac{q \wedge H(\neg q) \rightarrow \phi}{\phi},$$

provided that the propositional letter q does not appear in the formula ϕ .

We ask what class of frames is characterized by $MI(\phi)$ if ϕ characterizes a class \mathfrak{F} .

Definition 9 (converse of a binary relation)

The converse of binary relation $<$ is the relation $>$ such that:

$$t < t_1 \text{ iff } t_1 > t.$$

Let class of frames $\check{\mathfrak{F}}$ be such that:

$$\text{if } \langle T, < \rangle \in \mathfrak{F}, \text{ then } \langle T, > \rangle \in \check{\mathfrak{F}}.$$

Let us note that $\check{\check{\mathfrak{F}}} = \mathfrak{F}$.

Lemma 10

$$\langle T, <, V \rangle, t \models \phi \text{ iff } \langle T, >, V \rangle, t \models MI(\phi).$$

Proof.

The lemma is provable by the structural induction.

Let us consider only one case, namely of $S(\psi, \chi)$. By assumption we have that for any V, t :

$$\langle T, <, V \rangle, t \models \psi \text{ iff } \langle T, >, V \rangle, t \models MI(\psi),$$

and

$$\langle T, <, V \rangle, t \models \chi \text{ iff } \langle T, >, V \rangle, t \models MI(\chi).$$

By definition of MI we have $MI(S(\psi, \chi)) = U(MI(\psi), MI(\chi))$.

$\langle T, <, V \rangle, t \models \psi \models S(\psi, \chi)$ by definition of S is equivalent to:

$$\exists t_1, t_1 < t : \mathfrak{M}, t_1 \models \psi \text{ and } \forall t_2, t_1 < t_2 < t : \mathfrak{M}, t_2 \models \chi.$$

It is equivalent to:

$$\exists t_1, t_1 > t : \mathfrak{M}, t_1 \models MI(\psi) \text{ and } \forall t_2, t_1 > t_2 > t : \mathfrak{M}, t_2 \models MI(\chi).$$

It is equivalent to:

$$\langle T, V > \rangle, t \models U(MI(\psi), MI(\chi)),$$

and, finally:

$$\langle T, <, V \rangle, t \models S(\psi, \chi) \text{ iff } \langle T, >, V \rangle, t \models MI(S(\psi, \chi)).$$

■

By lemma 10 we have:

Theorem 11

A formula ϕ characterizes a class of frames \mathfrak{F} if and only if the formula $MI(\phi)$ characterizes $\check{\mathfrak{F}}$.

The formula:

$$Gp \rightarrow Fp$$

characterizes endless time (forward seriality), $T^{\infty+}$: $\forall t \exists t_1 : t < t_1$.

Time without a beginning (backwards seriality), $T^{\infty-}$: $\forall t \exists t_1 : t_1 < t$ is characterized by the formula:

$$Hp \rightarrow Pp.$$

F-LIN, linearity in the future (forward linearity): $\forall t, t_1, t_2$: if $t < t_1, t < t_2$, then $t_1 < t_2$ or $t_1 = t_2$ or $t_2 < t_1$ is characterized by:

$$Fp \rightarrow G(Pp \vee p \vee Fp).$$

P-LIN, linearity in the past (backward linearity): $\forall t, t_1, t_2$: if $t_1 < t, t_2 < t$, then $t_1 < t_2$ or $t_1 = t_2$ or $t_2 < t_1$, is characterized by:

$$Pp \rightarrow H(Pp \vee p \vee Fp).$$

Some relations are such that they are equal to its converse. If a relation is reflexive, irreflexive, symmetric, antisymmetric, asymmetric, transitive, total, trichotomous, a partial order, total order, strict weak order, total preorder (weak order), or an equivalence relation, its inverse is, too.

The formula $Gp \rightarrow p$ characterizes a reflexive time. The converse of reflexive relation is a reflexive relation. The formula $Hp \rightarrow p$ is inferable from $Gp \rightarrow p$, and vice versa, i.e. in \mathbf{K}_t , the system of minimal tense logic, $G\phi \rightarrow \phi$ is mutually provable from $H\phi \rightarrow \phi$:

1. $\neg\phi \rightarrow GP\neg\phi$ axiom of \mathbf{K}_t
2. $GP\neg\phi \rightarrow P\neg\phi$ assumption
3. $\neg\phi \rightarrow P\neg\phi$ Syll. (1,2)
4. $\neg P\neg\phi \rightarrow \neg\neg\phi$ Trans., 3
5. $H\phi \rightarrow \phi$ by definition of H and double negation.

The proof in the other way is similar.

It has been shown that in \mathbf{K}_t the formulas characterizing transitive time:

$$FFp \rightarrow Fp$$

and

$$PPp \rightarrow Pp$$

are mutually inferable (e.g. [McArthur, 1976, p. 26]). The same is true about formulas characterizing dense time (e.g. [McArthur, 1976, pp. 31–32]):

$$Fp \rightarrow FFp$$

and

$$Pp \rightarrow PPP.$$

The empty relation and its converse are equal. Empty relation is characterized by $G(p \wedge \neg p)$. Also this relation is characterized by $H(p \wedge \neg p)$, the *IM* of the formula $G(p \wedge \neg p)$. Are these formulas mutually inferable in \mathbf{K}_t ?

Theorem 12

$$\mathbf{K}_t \cup \{G(\phi \wedge \neg\phi)\} \not\vdash H(p \wedge \neg p).$$

Proof.

Let now:

- $\mathfrak{M}, t \models H\phi$ iff $\forall t_1, t_1 \leq t : \mathfrak{M}, t_1 \models \phi$;
- $\mathfrak{M}, t \models P\phi$ iff $\exists t_1, t_1 \leq t : \mathfrak{M}, t_1 \models \phi$;

Under this interpretation if the relation $<$ is empty, all the theorems of $\mathbf{K}_t \cup \{G(\phi \wedge \neg\phi)\}$ are valid, but $H(p \wedge \neg p)$ is not satisfiable in any model. ■

A binary relation $<$ is symmetric if and only if:

$$\text{if } t < t_1, \text{ then } t_1 < t.$$

The symmetry of $<$ is expressible in *TL* (Brouwer axiom):

$$p \rightarrow GFp.$$

The symmetrical relation is equal to its converse. The *IM*($p \rightarrow GFp$) derivable in $\mathbf{K}_t \cup \phi \rightarrow GF\phi$ [McArthur, 1976, pp. 34–35].

Some classes of relations are not characterized by any formula, e.g. – as it is stated in theorem 8 – the class of irreflexive times.³ In particular this concerns so called negatively definable classes.⁴

The notion of definability is such that if ϕ characterizes a class of frames, then ϕ is valid in any frame of this class. To distinguish this sort of definability, we call it positive characterization.

³ For general results about definability see [van Benthem, 2001].

⁴ Venema [1993] discusses a ‘negative’ way of defining frame classes in (multi)modal logic. In a metatheorem on completeness he defined the conditions under which a derivation system is strongly sound and complete with respect to the class of frames determined by its axioms and rules.

Definition 13 (negatively definable class)

A formula ϕ negatively characterizes a class of frames \mathfrak{F} if and only if for any $\langle T, < \rangle \in \mathfrak{F}$ for every t there is a valuation V such that $\langle T, < \rangle, t \models \neg\phi$.

Let \mathfrak{F}_ϕ denote the class of frames positively characterized by ϕ and $\mathfrak{F}_{-\phi}$ denote the class of frames negatively definable by ϕ . It is an interesting question about the relations between both two classes.

First of all let us remark that the classes \mathfrak{F}_ϕ and $\mathfrak{F}_{-\phi}$ are disjoint.

Theorem 14

For any $\phi : \mathfrak{F}_\phi \cap \mathfrak{F}_{-\phi} = \emptyset$.

We ask if both these classes are complementary, i.e. if for any $\phi : \mathfrak{F}_\phi \cup \mathfrak{F}_{-\phi} = \mathfrak{F}$, where \mathfrak{F} is the class of all frames. The answer is negative: frames characterized by $Gp \rightarrow Fp$ (endless times) are not complementary with a class of frames definable negatively, i.e. the class of all frames such that the relation $<$ is empty. There are frames such that only some, but not all, points do not have a successor.

Theorem 15

Let $\mathfrak{F}_\phi, \mathfrak{F}_{-\phi}$ be both non-empty. For any $\phi : \mathfrak{F}_\phi \cup \mathfrak{F}_{-\phi} \neq \mathfrak{F}$.

Proof.

We have to show that for any ϕ there are frames \mathfrak{T} such that there are moments such that for any valuation ϕ is satisfied and that there are moments such that there are valuations such that ϕ is not satisfied. These conditions are fulfilled by the following frame.

Let $\langle T_1, <_1 \rangle$ be a frame positively definable by ϕ and let $\langle T_2, <_2 \rangle$ be a frame negatively definable by ϕ . The frame is such that:

- $T = T_1 \times \{1\} \cup T_2 \times \{2\}$ and
- $< = <_1^* \cup <_2^*$, where
- $<_1^* \subset T_1 \times \{1\}$ and $(t, 1) <_1^* (t_1, 1)$ iff $t <_1 t_1$ and similarly
- $<_2^* \subset T_2 \times \{2\}$ and $(t, 2) <_1^* (t_1, 2)$ iff $t <_2 t_1$. ■

Is the class \mathfrak{F}_ϕ equal to $\mathfrak{F}_{-\phi}$? The formula $Gp \rightarrow Fp$ characterizes endless time. The class of frames negatively definable by this formula is characterized by the formula $G(p \wedge \neg p)$ and the class of frames negatively definable by $G(p \wedge \neg p)$ is characterized by $Gp \rightarrow Fp$.

As we see, some classes negatively definable are characterized by formulas of TL , e.g. class negatively definable by $Gp \rightarrow Fp$, and some classes are not, e.g. class negatively definable by $Gp \rightarrow p$.

3. Grzegorzcyk's Axiom and classes of frames definable by it

It is well known that there is a translation of intuitionistic logic into the modal logic **S4** via provability operator [Boolos, 1993]. The fact was suggested by Gödel [Gödel,] and proved by Tarski [McKinsey and Tarski, 1948]. There is translation T [[van Benthem, 2001, p. 385] such that for each formula ϕ of the language of intuitionistic logic **INT**:

$$\mathbf{INT} \vdash \phi \text{ iff } \mathbf{S4} \vdash T(\phi).$$

Andrzej Grzegorzcyk [1964; 1967] investigated relational and topological semantics for intuitionistic logic. Grzegorzcyk found a modal formula grz :

$$\Box(\Box(\phi \rightarrow \Box\phi) \rightarrow \phi) \rightarrow \phi^5$$

that is not provable in $S4$ but the intuitionistic logic is translatable into a normal extension of **S4** by grz , **S4.Grz** [Solovay, 1976; Goré *et al.*, 1997; Goldblatt, 1978].⁶ Moreover, it has been established that **S4.Grz** is the greatest normal extension of **S4** for which Gödel's translation of **INT** is still full and faithful [Esakia, 1976; Bezhanishvili, 2009], [van Benthem, 2001, Theorem 82, p. 385]. Strong provability operator "... is true and provable" provides a better model for provability than the operator "... is provable". The logic of the strong provability operator is known to coincide with Grzegorzcyk logic **Grz** [Boolos, 1993].

Grzegorzcyk's axiom defines the class of Kripke frames that fulfills the following conditions [van Benthem, 2001, p. 385]:

- $\forall x : x < x$ – reflexivity,
- $\forall x, y, z : \text{if } x < y, y < z, \text{ then } x < z$ – transitivity,
- from no t is there an ascending chain $t = t_1 \leq t_2 \leq \dots$ with $t_i \neq t_{i+1}, i = 1, 2, \dots$ – well-foundedness.

The last condition implies antisymmetry.

The Hilbert-style axiomatic calculus **K** is composed of the classical propositional calculus, the axiom schemata:

- $\Box(\phi \rightarrow \psi) \rightarrow (\Box\phi \rightarrow \Box\psi)$ – Distribution Axiom,

and the inference rules:

- from ϕ and $\phi \rightarrow \psi$ infer ψ – modus ponens,
- from ϕ infer $\Box\phi$ – necessitation.

S4 is defined as **K** plus the axiom schemata:

- $\Box\phi \rightarrow \Box\Box\phi$.

⁵ $\Box\phi$ can be read "formula ϕ is provable in Peano Arithmetic"

⁶ For an overview of results on **Grz** and its extensions see [Maksimova, 2007].

If a logic consists of \mathbf{K} , $\Box\phi \rightarrow \phi$, $\Box\phi \rightarrow \Box\Box\phi$, grz , then it is characterized by the class of reflexive, transitive and antisymmetric Kripke frames which do not contain any infinite ascending chains of distinct points. $\mathbf{S4}$ is valid in frames defined by grz . $\mathbf{S4}$ laws in $\mathbf{K} \cup grz$ were proved around 1979 by W. J. Blok and E. Pledger [van Benthem, 2001, p. 385].

\mathbf{Grz} is characterised by a class of Kripke frames which is not first-order definable, but is decidable.

The fact that \mathbf{Grz} is complete with respect to the class of upwards well-founded partially-ordered Kripke frames is provable only using some form of the Axiom of Choice [Jeřábek, 2004]. This logic is also complete with respect to Kripke frames such that T is finite and $<$ is a partial order [Segerberg, 1971], [Bezhanishvili and de Jongh, 2005, Theorem 84, p. 41].

We can ask about the class of frames negatively defined by grz or about the class of converse of the Kripke frames complete for \mathbf{Grz} . We may also consider the question of Grzegorzczuk's understanding of necessity in the context of temporal logic.

There are discussed different definitions of temporal modalities. E.g. $\Box\phi$ is defined as:

1. $\phi \wedge G\phi$ – Diodorean,
2. $H\phi \wedge \phi \wedge G\phi$ – Aristotelian
3. $P\phi$ – is based on the conviction that: *Quidquid fuit, necesse est fuisse.*⁷

\mathbf{K}_t , the minimal tense logic, is the tense logical counterpart of \mathbf{K} . In temporal logic G and H are semantical (in Kripke semantics) counterparts of \Box . In grz the \Box can be replaced by G and/or by H and grz as axiom can be added to \mathbf{K}_t .

System $\mathbf{K}_t \cup \{G(G(\phi \rightarrow G\phi) \rightarrow \phi) \rightarrow \phi\}$ is consistent – it has a model – the model of $\mathbf{S4.Grz}$. The same is true in the case of $\mathbf{K}_t \cup \{H(H(\phi \rightarrow H\phi) \rightarrow \phi) \rightarrow \phi\}$. Now, as in the case of the former system, it is complete with respect to the class that instead of well-foundedness is conversely well-founded.

⁷ Moreover, in [Anselm, Saint Archbishop of Canterbury, 1929, Book II, chapter XVIII (a)] we read: *Quidquid est, necesse est esse, et necesse est futurum fuisse. Quidquid futurum est, necesse est futurum esse.* In the English edition we read: Whatever has been, necessarily has been. Whatever is, must be. Whatever is to be, of necessity will be. This is that necessity which Aristotle treats of (“*de propositionibus singularibus et futuris*”), and which seems to destroy any alternative and to ascribe a necessity to all things [Anselm, Saint Archbishop of Canterbury, 1998, Book II, chapter XVIII (a)]. See <http://www.sacred-texts.com/chr/ans/ans118.htm>. If necessity is so conceived, the temporal possibility applies only to the future. According to Thomas Aquinas (Qu. 25, art. 4): *Praeterita autem non fuisse, contradictionem implicat* (For the past not to have been implies a contradiction). There is a Latin saying: *facta infecta fieri non possunt*; that is, what once has happened cannot become not happened.

Is consistent the system $\mathbf{K}_t \cup \{G(G(\phi \rightarrow G\phi) \rightarrow \phi) \rightarrow \phi\} \cup \{H(H(\phi \rightarrow H\phi) \rightarrow \phi) \rightarrow \phi\}$?

If we understand necessity as $P\phi$ we have:

$$P(P(\phi \rightarrow P\phi) \rightarrow \phi) \rightarrow \phi$$

we have to ask if the system $\mathbf{K}_t \cup \{P(P(\phi \rightarrow P\phi) \rightarrow \phi) \rightarrow \phi\}$ is consistent.

The formula:

$$P(P(p \rightarrow Pp) \rightarrow p) \rightarrow p$$

characterizes the class of frames such that t : if $t_1 : t_1 < t$, then $t_1 = t$. This can be understood as showing that the discussed conception of necessity is too weak from the point of view of Grzegorzczuk's axiom. Are the other conceptions of temporal necessity sufficiently strong to satisfy Grzegorzczuk's Axiom in a class of all times?

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