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PHILOSOPHICAL IMPORTANCE OF ANDRZEJ GRZEGORCZYK'S WORK ON INTUITIONISTIC LOGIC

Abstract. We compare Grzegorczyk's semantics for Heyting's intuitionistic predicate calculus with the Intuitionistic Kripke Models. The main problem with model-theoretic semantics for intuitionistic logic is that the concept of truth which implicitly is contained in this logic is different than the classical absolute concept of truth. Intuitionistic 'truth' is temporal. We compare Kripke's and Grzegorczyk's account of intuitionistic 'truth'. The main advantage of Grzegorczyk's semantics is simply the absence of the truth relation which occurs awkwardly in Kripke's semantics. Grzegorczyk replaces the truth relation with the fundamental pragmatic relation of forced assertion between an information state and a statement. Grzegorczyk investigated the relation of assertion in several subsequent papers and defined at least five types of this relation. He also discerned the classical and constructivist assertions. We argue that Grzegorczyk's semantics for Heyting's intuitionistic predicate calculus might be regarded as a predecessor of different present semantics which have arisen in the contemporary informational turn in logic.

1. Introduction

The most important of A. Grzegorczyk's papers on intuitionistic logic come from the years 1964–1971. In [Grzegorczyk, 1964], A. Grzegorczyk published his profound paper establishing the completeness of Heyting's propositional and predicate calculus with respect to the interpretation of this calculus based on the concept of forced assertion. Although A. Grzegorczyk's [Grzegorczyk, 1964] paper adds one more interpretation of intuitionistic calculus to the interpretations existing before, like that of Gödel [Gödel, 1933], Kolmogorov [Kolmogorov, 1932], Jaśkowski [Jaśkowski, 1936], Tarski [Tarski, 1938], Kleene [Kleene and Vesley, 1965], Beth [Beth, 1959], and Kripke [Kripke, 1963; 1965], his interpretation has interesting philosophical consequences, and expresses his deep understanding of intuitionism. In his [Grzegorczyk, 1967] paper, A. Grzegorczyk divides all formal interpretations of intuitionistic calculus into two groups: recursive interpretations

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and topological ones, and places his own interpretation amid the interpretations belonging to the topological group, along with the interpretations of Jaśkowski [Jaśkowski, 1936], Tarski [Tarski, 1938], Beth [Beth, 1959] and Kripke [Kripke, 1963; 1965]. The importance of A. Grzegorczyk's work on intuitionistic logic consists not only in establishing the mathematical result, but foremost is a deep philosophical understanding of intuitionistic logic as a logic of investigation. This has been done in his [Grzegorczyk, 1964] paper, together with subsequent works: his [Grzegorczyk, 1967] and [Grzegorczyk, 1968] papers, as well as in his [Grzegorczyk, 1971] paper. In his [Grzegorczyk, 1968] paper, A. Grzegorczyk has written:

> S. A. Kripke on the Oxford 1963 Colloquium spoke about interpretation of intuitionistic logic in [3]. I did not attend this meeting; but I also published similar ideas in 1964 in [2] developing a bit more the philosophical interpretation [Grzegorczyk, 1968, 86].

In this passage, A. Grzegorczyk mentions Kripke's famous paper on intuitionistic logic and his own 1964 paper. Although both semantics for intuitionistic predicate logic belong to the topological type, there are certain important differences between them, which are not always noted in relevant literature and appreciated. It is true that both semantics prove completeness theorems for Heyting's intuitionistic predicate calculus, but they are based on different concepts and different formalisms.

2. Grzegorczyk's Semantics vs. Intuitionistic Kripke Models

Kripke defines an intuitionistic model structure as an ordered triple:

$$\langle G, K, R\rangle$$

which may be also understood as a tree model structure, where K is a set, G is an element of K, and R is a reflexive and transitive relation on K. An intuitionistic model on this structure is defined as the binary function:

$$\phi(P, H)$$

where P ranges over arbitrary proposition letters, and H ranges over elements of K. The range of this function is the set of truth-values $\{T, F\}$, and it satisfies the following hereditary condition:

$$\text{If } \phi(A, H) = T \text{ and } HRH', \text{ then } \phi(A, H') = T.$$

This condition tells us that if we already have a proof of an arbitrary formula A at the time point H, then we still have the proof of A in any later time point H'. Further conditions which satisfy the modeling function are defined by induction on the number of connectives in A. We shall only turn attention to the most important conditions which are radically different from the respective conditions in the model-theoretic semantics of classical logic.

(Atom) If A has no connectives, then it is a proposition letter P, and $\phi(P, H) = T$ or F.

(Neg) $\phi(\neg A, H) = T$ iff for all $H' \in K$ such that $HRH', \phi(A, H') = F$; otherwise $\phi(\neg A, H) = F$.

(Imp) $\phi(A \supset B, H) = T$ iff for all $H' \in K$ such that $HRH', \phi(A, H') = F$ or $\phi(B, H') = T$; otherwise $\phi(A \supset B, H) = F$.

Kripke explains these conditions in the following way.

> To assert A intuitionistically in the situation H, we need to know at H not only that A has not been verified at H, but that it cannot possibly be verified at any later time, no matter how much information is gained; to assert $A \supset B$ in a situation H, we need to know that in any later situation H' where we get a proof of A, we also get a proof of B [Kripke, 1965, 99].

This small piece of Kripke's semantical model theory for Heyting's propositional calculus is evidence that the concepts of intuitionistic 'truth' and 'falsity' are different from the respective classical concepts. Truth is not an absolute concept, but a temporal one: A proposition becomes true only when it is proved. By relativization of truth-conditions to points in time, Kripke's semantics reflect this understanding of 'truth' to some degree, although the very concept of being intuitionistically true at a certain point of time remains philosophically unclear in Kripke's semantics. An analogous observation concerns also the concept of being intuitionistically false at a certain point of time. In Kripke's semantics, both concepts are formally represented by the function which maps pairs (proposition; time point) into the set of truth-values $\{T, F\}$. The falsity of A cannot be explained by the truth of the negation of A as is the case in the model-theoretic semantics of classical logic, where the equivalence holds: A is false if and only if $\sim A$ is true. In the intuitionistic Kripke models, we can obtain:

$$\phi(A, H) \neq T \text{ and } \phi(\neg A, H) \neq T.$$

This is the case if at point H we do not have enough information to prove proposition A, but we also do not know at H that it is impossible to prove A and assert $\neg A$. Such impossibility means proving that proposition A leads to a contradiction. Note that intuitionistic logic belongs to the family of constructive logics, therefore it adopts a constructive conception of proof. At first sight, one may think that if the intuitionistic truth of A is reducible to the provability of A, then the intuitionistic falsity of A is simply the lack of such a proof of A at a certain time point. This would be something less than having the proof that A leads to a contradiction. There are those who argue for the intuitionistic non-constructive falsity as the lack of proof of A at a time point.[1] This conception of falsity says nothing about accomplishing any construction, and has nothing to do with the intuitionistic refutation. This non-constructive conception of intuitionistic falsity expresses only the possibility of refutation, and in spite of arguments proffered in its favor, it does not respect the original intuitionistic idea of falsity of A as provability of the contradictoriness of A. Kripke [Kripke, 1965, 98], mentions situations where we lack enough information to prove a proposition A, but he explains that in these situations A *has not been verified*, not that A has been proved false.

Grzegorczyk's semantics of intuitionistic logic is philosophically much more illuminating than Kripke's semantics. A. Grzegorczyk [Grzegorczyk, 1964] models intutionistic logic as a certain logic of scientific research:

> Scientific research (e.g. an experimental investigation) consists of the successive enrichment of the set of data by new established facts obtained by means of our method of inquiry. When making inquiries we question Nature and offer her a set of possible answers. Nature chooses one of them [Grzegorczyk, 1964, 596].

Accordingly, the scientific research is modelled formally as a triple:

$$R = \langle J, o, Pr \rangle.$$

J stands for the set of all possible experimental data (the information set, finite or infinite); o stands for the initial information (possibly empty), and Pr stands for the function of all possible prolongations or extensions of the information. The experimental data a is understood as ordered finite collections of atomic sentences having the forms of atomic formulas (without variables) of the classical language of predicate calculus:

$$a = (A_1, \dots A_n); A_i := \text{atomic sentence}.$$

[1] See [Shramko, 2012].

Sentences containing logical constants do not represent experimental data. The relation of extension of the information in research R : *b is an extension of a in research R* is defined inductively for a finite number of new atomic sentences. This concept is used in the main definition of Grzegorczyk's semantics; the definition of the concept of forced assertion: *the information state a in research R forces us to assert the statement expressed by the formula* χ[2]. The relation of forced assertion will be denoted here as $>$.

Definition 1 [Grzegorczyk, 1964, 597]

(Atom) $a > \chi$ iff $\chi \in a$, if χ is an atomic formula without variables.

(Neg) $a > \neg\chi$ iff $\forall b[(b$is an extension of a in $R) \rightarrow\sim (b > \chi)]$.

(Imp) $a > \chi \supset \psi$ iff $\forall b[(b$ is an extension of a in$R) \rightarrow (b > \chi \rightarrow b > \psi)]$.[3]

I omit in Definition 1 the conditions for disjunction and conjunction, as well as for existential and universal quantification. Definition 1 (with other resources) enables us to formulate and to prove the completeness theorem for Heyting's propositional calculus.

Theorem 2 [Grzegorczyk, 1964]

A formula χ (without quantifiers) is logically true in formal intuitionistic logic if and only if each information state a of every research R forces us to assert the statement expressed by the formula χ.

It is easy to notice that the Law of the Excluded Middle does not satisfy the condition of Theorem 2. It may be the case that in a certain initial information state a we do not have $Q(z)$. Therefore, the state a does not force us to assert $Q(z) \vee \neg Q(z)$. On Grzegorczyk's interpretation of intuitionistic logic, the Law of the Excluded Middle belongs to our ontological assumptions about the world, and as such it lies beyond scientific methods.

A. Grzegorczyk's completeness theorem for the intuitionistic logic of quantifiers (Theorem 2 in [Grzegorczyk, 1964]) requires a certain modification of Definition 1 proffered above. The modification concerns the clause for atomic formulas, as well as for compound ones. For example, the clause for atomic formulas takes the following form:

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2 This concept has its origin in P. Cohen's concept of forcing.

3 The metalanguage in Grzegorczyk's semantics is classical.

(Atom) $a > A_i$ iff irrespectively of how we continue our research R from the state a, we obtain information b such that b contains the statement A_i.

The clause formulated in (Atom) may be understood as *potential forcing* by the information state a of research R. The modification consists in adding the initial phrase: *irrespective of how we continue our research R from the state a, we obtain information b, such that* .... To formalize this phrase, A. Grzegorczyk introduces the notion of branch of the research R.

R is a *branch* of $R = \langle J, o, P \rangle \equiv^{Df} X \subset J \wedge o \in X \wedge \forall a(a \in X \rightarrow$ there exists one and only one b such that $b \in P(a)$ and $b \in X)$.

The idea of potential forcing, and the definition of forcing appropriate for statements expressed in the language of predicate calculus, may be now formalized in the following way:

$$X[(X \text{ is a branch of} R \wedge a \in X) \rightarrow \exists b(b \in X \wedge \ldots)].$$

Theorem 3 [Grzegorczyk, 1964]

A formula χ is provable in the formal intuitionistic logic of quantifiers if and only if the statement expressed by the formula χ is forced by each information state of every research.

The main advantage of Grzegorczyk's semantics is the absence of the truth relation and the falsity relation which are present in Kripke's model-theoretic semantics for Heyting's intuitionistic predicate calculus. Grzegorczyk's account suits well the anti-realism of philosophical intuitionism. This philosophical standpoint was noted by M. Dummett:

> Thinking of a statement as true or false independently of our knowledge involves a supposition of some external mathematical reality, whereas thinking of it as being rendered true, if at all, only by a mathematical construction does not. [Dummett, 1977, 12]

Kripke is aware of this philosophical attitude characteristic of intuitionism, and provides informal comments, as well as formal translations of his model-theoretic account into the intuitionistic discourse. Nevertheless, his fundamental semantic definition formulates truth-conditions with the help of the function ϕ which maps the pairs, each of which consists of a formula and a time point, into the truth-values T and F. But as we have already mentioned, intuitionism accepts the idea of temporal truth, according to which a proposition becomes true only when it is proved.

This idea contradicts the central idea of realism which underlies classical model-theoretic semantics and classical logic, where the Law of the Excluded Middle is accepted as a tautology. From this classical point of view, the idea of temporal truth is contentious and counterintuitive. Accordingly, there seems to be a little embarrassment in making use of such a conception of truth in any model-theory. Kripke in his [Kripke, 1963; 1965] paper does not appeal to this idea of truth, but encounters many obstacles in expressing intuitionistic semantics in a model-theoretic framework with the classical concepts of truth and falsity. In consequence, he makes much effort to make sense of his meta-theoretical evaluations in terms of classical truth and falsity applied to intuitionistic language with its intended interpretation.[4]

The solution chosen by A. Grzegorczyk for building a semantics of the intuitionistic language is better, since it avoids the clash of the intuitionistic with classical model-theoretic ideas. From a philosophical point of view, most important in Grzegorczyk's semantics is the fundamental pragmatic relation of forced assertion between an information state and a statement. This relation is defined by A. Grzegorczyk implicitly along with the definition of the connectives and quantifiers of the intuitionistic predicate language. Although formally, this relation is modelled as binary, in fact it is a ternary pragmatic relation: *the information state a forces the subject s to assert the statement expressed by the formula* χ. It is an implicit assumption of Grzegorczyk's semantics that the relation remains unchanged when applied to different subjects, and for this reason the relativization to the subject s may be omitted.

3. The Concept of Assertion in Grzegorczyk's Semantics

Many other philosophical advantages of Grzegorczyk's semantics are discussed by A. Grzegorczyk in his later papers [Grzegorczyk, 1967; 1968; 1971]. A. Grzegorczyk [Grzegorczyk, 1968] proffers a slightly different account of his [Grzegorczyk, 1964] result. The differences are of a philosophical nature. The main relation of forcing assertion by an information state is now called the relation of *strong assertion* of a sentence at a time point in a given inquiry:

[4] Cf. for example Kripke's comments in [Kripke, 1965, 95].

$$As_E(\chi, t).$$

$$\chi \in Atom \to [As_E(\chi, t) \equiv \chi \in A_E(t)].$$

$$As_E(\neg\chi, t) \equiv \forall s \in T[ts \to \sim As_E(\chi, s)].$$

$$As_E(\chi \supset \psi, t) \equiv \forall s \in T[ts \to (\sim As_E(\chi, s) \vee As_E(\psi, s))].$$

$A_E(t)$ stands for the set of atomic empirical sentences we assert in performing experiments prescribed to the time point $t \in T$ by the program of the inquiry E. A. Grzegorczyk claims that intuitionistic calculus forms the logic of the *strong assertion*. The strong assertion is compared with the *admissibility* relation: χ is admitted as supposition in the time point t in E, which may be regarded as a *weak* assertion:

$$Ad_E(\chi, t).$$

The inquiry E is understood as a triple:

$$\langle A, R, L \rangle.$$

A is the set of time points in which can be admitted χ; R is the set of time points in which can be admitted $\neg\chi$, for χ atomic; L is the conjunction of all theories accepted as background for E. The inquiry E defined as above requires discerning the admissibility conditions of atomic and negated atomic sentences, and next the proper admissibility conditions of compound and negated compound sentences. The conditions for atomic and negated atomic sentences are simple and take the following form, respectively (the index E has been omitted):

$$Ad(\chi, t) \equiv t \in A(\chi);$$

$$Ad(\neg\chi, t) \equiv t \in R(\chi).$$

The conditions for compound sentences are more complicated, and we only note that the following equivalence holds for the admissibility relation:

$$Ad(\neg\neg\chi, t) \equiv Ad(\chi, t).$$

The admissibility relation is a weak counterpart of the strong assertion which is characteristic of intuitionistic logic.[5] A. Grzegorczyk [Grzegorczyk, 1967] considers also another concept of assertion understood intuitively as *to be allowed*, meant as the relation:

$$Al_R(\chi, a).$$

[5] The admissibility relation is a formalization of the same kind of relation which occurs in Popper's logic of scientific discovery.

The intuitionistic negation is defined in terms of this assertion as:

$$Al(\neg\chi, a) \equiv\sim Al(\chi, a).$$

This is the main concept of Grzegorczyk's semantics for modal logics based on strict implication. He defines the set of theorems of the logic of strict implication as identical with the set of those formulas which are allowable by each information state of every research R. In this sense the following paradoxes of the material implication are not allowable:

$$p \to (q \to p);$$

$$p \to (\sim p \to q).$$

It turns out also that the famous Grzegorczyk formula, G, is allowable in the above sense. Formula G does not belong either to the system $S4$, or $S5$.[6]

The pragmatic act of assertion is characteristic of human cognition, while the intuitionistic and constructive logics may be regarded as logics of cognition. Accordingly, not truth, but assertion is the crucial concept underlying the intutionistic and constructive logics which one may observe in Grzegorczyk's semantic reconstruction of these logics. Intuitionistic and constructive logic remain in a close relationship with the contemporary epistemic logics, which are logics of knowledge, belief, and information. The main difference between the latter and their predecessors consists in this, that constructivism and intuitionism provide us with the logic of the process of investigation, while epistemic logics formulate the principles of an agent's knowledge or agent's beliefs. J. van Benthem [van Benthem, 1993] depicts the difference with the help of the distinction between *implicit* and *explicit* knowledge. In the topological semantics of intuitionistic logic, we have to do with the information loading of some logical constants such as negation and implication, as we have seen above. This property of the logical constants is called implicit knowledge. On the other hand, in epistemic logic, which retains the classical account of logical constants, but adds the explicit modal operator, K, we have to do with explicit knowledge. We could say that the implicit knowledge of intuitionistic logic is properly expressed by the assertibility conditions; the explicit knowledge of epistemic logic, by truth conditions describing sufficient and necessary conditions of any formula, K_χ, true in a model M and world s. J. van Benthem [van Benthem, 2009] considers embedding intuitionistic logic

[6] More about Grzegorczyk's modal logic may be found in [Maksimova, 2007].

in explicit modal-temporal theories of information processes, which enables us to define such concepts as "always in the future" and "necessarily now". He argues that epistemic logic being a result of the embedding gives an account of the rational agents in actions of observation, inference, and communication. J. van Benthem is right as to his conclusion; nevertheless the embedding of intuitionistic logic in modal logic, or in modal epistemic logic, is connected with the loss of specific semantic features of intuitionism which have been clearly proved by A. Grzegorczyk.

A deep analysis of the relation of assertion is to be found in one Grzegorczyk [Grzegorczyk, 1971] paper. A. Grzegorczyk includes in the methods of assertion the checking, deducing, and construction of an asserted sentence with semantic terms. By checking he understands applying an algorithm. The method of deducing assumes a recursive set of axioms and inference rules. A. Grzegorczyk argues that the three methods are connected with the three types of assertion:

- Classical assertion;
- Relativistic assertion;
- Constructivistic assertion.

The classical assertion is absolute, that is, non-relativistic. It does not come into degrees, and is independent of the time of assertion and the method of assertion. By this conception, the asserted sentence is identified with the true sentence being effectively justified. This conception is closest to classical logic, although, it does not imply that it is asserted either χ or non-χ, no matter whether our meta-theory is classical or not. This conception of asserting may be combined with the method of algorithmic checking if atomic sentences are formulated in the language of arithmetic. For example, if we have an atomic sentence of the form:

$$n + m = k,$$

where n, m, k are names of natural numbers, then the atomic sentence is asserted if it is justified by an algorithm. Next, the method of classical assertion tells us what it is to assert a compound sentence.

Relativistic assertion depends on many conditions: one of them may be time. On this conception, the absolute notion of assertion applied to theorems must be defined separately:

$$\chi \text{ is asserted as a theorem } \equiv^{Df} \chi \text{ is asserted in all conditions.}$$

For many collections of conditions, the set of absolutely asserted sentences is identical with intuitionistic logic.

Constructivistic assertion is a special kind of relativistic assertion relativized to method. The method is understood as an algorithm giving an expected result in a finite number of steps. In this conception, the assertion of a theorem is defined in the following way:

$$\chi \text{ is asserted as a theorem } \equiv^{Df}$$
$$\text{there is the algorithm } \alpha \text{ such that } \alpha \text{ is a justification for asserting } \chi.$$

It is instructive to compare the assertibility conditions formulated for the denial of χ in the three conceptions:

- $\sim\chi$ is asserted $\equiv$ it is *not* asserted χ.
- $\neg\chi$ is asserted in the conditions $c \equiv$ there is the open interval I for c, such that for any member $i \in I, \chi$ is *not* asserted in i.
- $\neg\chi$ is asserted as justified by the algorithm $\alpha \equiv$ for any β, if χ is asserted as justified by the algorithm β, then the algorithm $\alpha(\beta)$ leads to a *contradiction*.

It is easy to notice that only the third condition is free from a tension between truth and justification. The constructivistic assertibility conditions are in fact justification conditions formulated in terms of an algorithm and the classical concept of contradiction. Neither the classical conception of assertion, nor the relativistic conception mentions how our assertion is justified. It seems that justification, and not truth, is decisive for assertion meant as a pragmatic relation. Intuitionism and constructivism rest on the following conviction:

> It does not make sense to think of truth or falsity of a mathematical statement independently of our knowledge concerning the statement. A statement is true if we have proof of it, and false if we can show that the assumption that there is a proof for the statement leads to a contradiction. For an arbitrary statement we can therefore not assert that it is either true or false [van Dalen and Troelstra, 1988, 4].

The constructivist conception of truth does not coincide with the Tarskian conception of truth, where the truth-predicate satisfies the metalogical law of the Excluded Middle, as well as the metalogical law of Non-Contradiction. In this sense, the Tarskian truth-predicate is absolute; that is, independent of time, space, and other conditions. Even if we cannot assert that a statement is either true or false, the concept of truth in the Tarskian sense enables us to believe that the statement must be either true or false. Here is how M. Dummett wrote about the intuitionistic notion of truth:

> [...] 'is true' would have to be equated with 'has been proved' and 'is false' with 'has been refuted'. On this use, any statement A that has not yet been decided is neither true nor false; but this does not preclude its later becoming true or becoming false [Dummett, 1977, 18].

> Such a notion of truth, obvious as it is, already departs at once from that supplied by the analogue of the Tarski-type truth-definition, since the predicate *is true* thus explained is significantly tensed: a statement not now true may later become true [Dummett, 1978, 239].

M. Dummett describes the intuitionistic notion of truth as endowed with a property usually ascribed to belief, knowledge, justification, or assertion, such as the relativisation to time. Note that justification for the *realist* has distinct properties from truth, and plays a different role, a role which consists in "binding" our beliefs with respective truths.

Assertion is regarded as a speech act taking place when the speaker makes an utterance with an *assertoric force*. This conception of assertion comes from Frege [Frege, 1918]. The same idea reappears in J. Austin's works, where the general theory of speech acts was founded, dividing them in three groups: locutionary acts, illocutionary acts, and perlocutionary acts. Acts of asserting belong to illocutionary acts. Frege and J. Austin distinguished the act of assertion from other pragmatic phenomena such as presupposition and implicature. A separate problem is connected with the relationship between truth and assertion. According to A. Tarski,

asserting *that* p is materially equivalent to asserting *that* p *is true.*

A different account of that relationship may be found in [Dummett, 1959], who identifies the act of assertion with the act which aims at truth, which is interpreted that the speaker intends to convince the hearer that he/she aims at saying something true. Note that it is usually assumed that the act of assertion may be insincere in the case of lying. Assertion may be evaluated as correct in different respects. It is commonly accepted that:

asserting *that* p is correct if and only if the speaker has good evidence that it is true *that* p.

The logic of assertion formulated by N. Rescher [Rescher, 1968] has the following principles. Let Axp stand for x *asserts that* p, then:

1. $\forall x \exists p Axp$;
2. $(Axp \wedge Axq) \rightarrow Ax(p \wedge q)$;
3. $\sim Ax(p \wedge \sim p)$.

The specific inference rule of this logic is called a "rule of commitment":

(C) If p implies q, then Axp implies Axq.

The principles have respectively the following intuitive meanings: someone asserts something; if it is asserted p and asserted q, then the conjunction of p and q is asserted: a contradiction is not asserted. One may say that the principles are necessary conditions of rational thinking. By contrast, the inference rule (C) is contentious. It tells us that the rational subject who asserts a certain proposition also asserts the logical consequences of the proposition. This rule is highly unrealistic, since it appeals to the conception of an ideal subject who is logically omniscient.

What is then A. Grzegorczyk's conception of assertion? First of all, A. Grzegorczyk's assertion is relativized not only to time, but also to research, or more precisely, to the programme of research (inquiry) R, accordingly

relation of assertion holds between the information state, a, obtained in the research (inquiry) R up to the time point t, and the sentence χ.

The information state makes the relation of assertion founded on facts independent of our consciousness. It is assumed that the information state makes the relation of assertion correct, since it provides good *evidence* for the truth of the sentence χ. Besides, asserting the negation of the sentence, and the implication of two sentences, is what makes the relation of assertion *strong* one. To assert the negation at the point of time t, it is necessary and sufficient never later to assert the very sentence. This condition imposes on us the requirement of careful formulation of a sentence which can never be asserted. I cannot assert now many ordinary language utterances, such as "It is not the case that I have short hair" unless I am sure that I will have long hair always in the future, and I cannot assert now "It is not the case that there is a tree nearby that building" unless I am sure that there will never be any tree nearby that building. These examples merely suggest that ordinary language negation is not intuitionistic negation, but they are not an argument in favour of the view that the concept of strong assertion and that of intiutionistic negation are useless. Intuitionistic negation serves for modelling mathematical (i.e. non-empirical) discourse negation, but not natural language negation. The strong assertion is understood as an absolute assertion in a certain sense: what has been once asserted is not retracted. In other words, the relation of strong assertion is *monotonic*, since new facts do not force us to retract what has been strongly asserted before.[7]

4. Grzegorczyk's Semantics and the Informational Turn in Logic

Note that A. Grzegorczyk's relation of assertion holds between a certain *information state* and a sentence. According to the contemporary Law of Causality of Information, there is no more information received than that which has been sent.[8] In the case of empirical research, Nature sends the information which is to be partially received, and no more can be asserted than the information which has been received.

A. Grzegorczyk's work on the semantics of intuitionistic calculus was written more than a decade before the so called *informational turn* in logic, which caused a new understanding of logic as information-based. The points of evaluation, meant as information states in A. Grzegorczyk's semantics, are allowed to be incomplete, although they are always consistent[9]. At present, we consider also inconsistent information states. From the point of view of this informational trend in logic, Grzegorczyk's semantics for intuitionistic logic may be regarded as a logic of information where the relation $a > \chi$ is understood as the relation of carrying information: *the information state a carries the information that* χ. Understood in that way, Grzegorczyk's semantics is a sort of frame semantics, where the model contains the evaluation relation: $>$. Let the frame $\mathbf{F}$ be a structure containing the set of information states S, and the binary relation of extension of the information: *b is an extension of a in the research* R, which will be denoted at present as $\supseteq$. Therefore,

$$\mathbf{F} = \langle S, \supseteq \rangle.$$

A model $\mathbf{M} = \langle F, \supseteq, > \rangle$, where the evaluation relation $>$, meant as Grzegorczyk's relation of forced assertion, satisfies the respective conditions for each connective of the intuitionistic language, as well as the following monotonicity condition, a counterpart of the hereditary condition in Kripke's semantics:

$$\text{If } a > \chi \text{ and } b \supseteq a, \text{ then } b > \chi.$$

[7] The intuitionistic negation fails to distinguish between the global future absence of verification, and local falsification. Compare [van Benthem, 2009, 253].

[8] See [Pawlowski *et al.*, 2009].

[9] There are such states of information which do not force us to assert $A \vee \neg A$, while there are no such states which force us to assert $A \wedge \neg A$

J. van Benthem [van Benthem, 2009, 255] makes the important observation that intuitionistic logic registers two kinds of information:

1. Factual information about how the world is;
2. Procedural information about our current investigative process, that is, about the way we learn facts.

J. van Benthem [van Benthem, 2009] argues that branching tree-like models are closest to the intuitionistic tradition. They describe an informational process where an agent learns progressively about the state of the actual world. Surely, the evaluation conditions for the intuitionistic negation and implication in A. Grzegorczyk's semantics register procedural information, since they capture the dynamic of our investigative process.

There are many logics of information by now, each describing different aspects of this complex phenomenon.[10] A philosophically important notion in studying the phenomenon of information is the notion of information application. An interesting analysis of this notion may be found in [Sequoiah-Grayson, 2009]. The starting point of his analysis is a set of information states S with a partial order on it. The partial order is understood in a similar way as A. Grzegorczyk's relation of the extension of information. Let the partial order be denoted as $\leq$. The expression $x \leq y$ is read as: the information x is contained in the information y. The operator of the application of information holds between information states. If the operator is denoted as $\bullet$, then the expression:

$$x \bullet y \leq z$$

is understood as the result of the application of the information in x to the information in y develops into the information in z [Sequoiah-Grayson, 2009, 422]. The operator is an ontological counterpart of the binary connective, $\otimes$, occurring between formulas, which is called by S. Sequoiah-Grayson *fusion*, and the expression:

$$A \otimes B$$

is meant as the application of the information in the formula A to the information in the formula B. The evaluation conditions for this compound formula are given in terms of the model-theoretic evaluation relation, denoted by us as: $<$. Accordingly, $x < A$, is read as *x carries the information that* A,*or* x *supports* A [Sequoiah-Grayson, 2009, 410]. Thus,

$$xA \otimes B \text{ iff for some } y, z \in S \text{ such that } y \bullet z \leq x, yA \text{ and } zB.$$

[10] See [Barwise and Seligman, 1997; Floridi, 2003; Jago, 2006; Mares, 1997; Restall, 1994; Sequoiah-Grayson, 2006; Misiuna, 2012], to mention only a few.

We omit here the other connectives included to the informational language considered by Sequoiah-Grayson [Sequoiah-Grayson, 2009]. The operator of the application of information could be easily described semantically in terms of A. Grzegorczyk's semantics, augmented with the binary operator $\bullet$, in the following way:

$a > A \otimes B$ iff for some $b, c \in S$ such that $a \supseteq b \bullet c, b > A$ and $c > B$.

5. Conclusions

We have compared A. Grzegorczyk's semantics for Heyting's intuitionistic propositional and predicate calculus with the respective Kripke's semantics. Both semantics are of a topological kind, but Grzegorczyk's semantics is philosophically more acceptable than the intuitionistic Kripke models. We have argued that the main philosophical advantage of Grzegorczyk's semantics is the lack of the model-theoretic truth-relation which is present in the intuitionistic Kripke models. We have noted that A. Grzegorczyk makes a distinction between different types of assertion:

- Forced assertion [Grzegorczyk, 1964];
- Potentially forced assertion [Grzegorczyk, 1964];
- Strong assertion (compared with weak assertion; [Grzegorczyk, 1967]);
- Admissibility (weak assertion; [Grzegorczyk, 1967]);
- Relation of being allowed [Grzegorczyk, 1967].

The five types of assertion may be regarded as that of a relativistic kind. Besides relativistic assertions, A. Grzegorczyk discerns also classical and constructivistic assertions. The other peculiar property of A. Grzegorczyk's semantics, besides its philosophical insight, is its anticipatory feature. We have argued that Grzegorczyk's semantics may be regarded as a predecessor of many present semantics characteristic of the contemporary informational turn in logic.

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