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Stanisław Krajewski

EMERGENCE IN MATHEMATICS?

Abstract. Emergence is difficult to define. In mathematics, a subjective, indeed psychological, definition of emergence seems reasonable. The following conditions seem necessary: the appearance of surprising properties that are inescapably surprising, even for the expert. It is another matter whether this is a sufficient condition for the presence of emergence. To the canonical examples of emergence – life, mind, (self)consciousness – some mathematical examples can be added: fractals (already proposed by other authors) and the emergence of undecidability (and incompleteness) of natural numbers when they are considered as a structure with both addition and multiplication.

1. Emergence, an Introduction

The notion of emergence has been proposed as a means of describing situations in the material world in which growing complexity either causes the appearance of essentially new features or provides an occasion for essentially new features to appear. To be sure, it is not arbitrary new features that are meant here, since *some* new features must necessarily appear when any change takes place; rather, what is meant are essential, fundamentally new features: new higher-level qualities that are "ungraspable," that is, cannot be grasped (or understood) from a lower level. Unfortunately, it is not obvious what these "levels" are. It is also far from clear what "ungraspability" should mean: it can be explained as irreducibility, non-educibility, indeterminateness, or unexpected character of the phenomenon. Fortunately, some examples are beyond doubt. Thus, the emergence of life, of the mind, and of consciousness illustrate higher-level, irreducible, absolutely new features – or rather dimensions – that cannot be grasped from a lower level, that is, from the level of inanimate matter or unconscious life. The notion of emergence was introduced in the 1920s. In recent decades it has been revived. It is claimed that emergence would be found in various connected structures that could be regarded as separate entities: a book by Steven Johnson is titled *Emergence: The Connected Lives of Ants, Brains, Cities,*

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and Software. Other scholars refer to the internet and other self-organizing structures, and more metaphysically minded authors use the concept of emergence to talk about spirituality – even, as does Ken Wilbur, spirituality on a cosmic scale. The construction of hierarchies of emergence can result in a variety of religious claims. Unsurprisingly, such claims are divergent. Thus, Philip Clayton in the book *Mind and Emergence: From Quantum to Consciousness* writes that Terrence Deacon draws Buddhist conclusions, Harold Morowitz finds Jewish-Spinozian ones, while Nancy Murphy, Niels Gregorsen and Clayton himself interpret such ultimate emergence in a Christian framework.

Emergence can also be described by saying that "the whole is more than the sum of its parts," but this is hardly helpful. Every whole built from parts – not just collected as is the case with sets in the distributive sense – is more than the totality of its parts. What sets emergence apart is that it occurs when novelty is essential – another concept difficult to define – and is not reducible to the connections linking the parts. However these terms just used, "essential novelty" and "(ir)reducibility," seem just as difficult to explain as the term "emergence" itself. My aim in this paper is to analyze a seemingly simpler, but nonetheless elusive subject, the appearance of emergence, if there is any, in mathematics.[1]

2. The Psychological Character of Emergence in Mathematics

If a mathematical structure is enriched, it becomes more complicated and one can detect in it new features and new phenomena. Sometimes these can represent something genuinely new. If, for instance, integers are extended to rational numbers the ordering of numbers becomes dense, and if we further extend the structure to real numbers a new feature characterizes the ordering – it becomes continuous. It is rather doubtful, however, whether such phenomena, ubiquitous in mathematics, can be described as "emergence." Indeed, I feel it would trivialize the concept of emergence if it were used in such cases. Similarly doubtful is the application of the concept of emergence when, in the spirit of Nicholas Bourbaki, structures are gradually expanded – from sets to ordered sets to added algebraic structures to topo-

[1] This paper is based on a Polish paper „Emergencja w matematyce?", in *Struktura i emergencja* (red. Michał Heller i Janusz Mączka), Biblos, Kraków 2006, 110–118, published also as Chapter 1 of my book *Czy matematyka jest nauką humanistyczną?*, Copernicus Center Press, Kraków 2011, 11–20.

logical ones to their combinations, etc. The problem is, then, what novel phenomena can rightly be described as "emergent." Are there any identifiable features of mathematical situations that are necessary for emergence to occur?

In order to move toward a solution let us consider an example that at first glance seems promising. Passing from finite to infinite structures is very natural in mathematics, or rather in modern mathematics, as until the 19th century this step was considered unacceptable. Georg Cantor, however, has introduced us to the realm of actually infinite sets. We consider longer and longer sequences and naturally move to considering transfinite sequences, which can be manipulated in much the same way as the finite ones. Infinite sets, too, are treated in the same way as finite sets, or to be more specific, operations such as taking the union of a set of sets or the powerset of a given set, which raise no reservations in the case of finite sets, are also executed on actually infinite sets. In fact, the possibility of the unrestricted extension of those operations to the realm of infinite sets can be seen as the essence of the Cantorian revolution in mathematics. It is certain that new features, new regularities appear in the realm of infinite sets. The main novelty can be seen in the presence of, well, infinity! Are we dealing with "emergence" here? The answer is not easy. While we have no criterion for recognizing emergence, it seems that the mechanism producing the new situation is relatively clear: the potential infinity, or the potential to extend the finite beyond any limit, leads to a jump into the actual infinity. The situation is by no means simple, but the novelty is created by our decision to perform a jump. Therefore, we cannot say that the result is unexpected. And it seems to me that in order to talk about emergence it is necessary to face something truly unexpected. Some surprise is needed, the appearance on the higher level of something that is not simply higher, more complex, but is also *astonishing* – an unforeseen feature or regularity, impossible to anticipate at the lower level.

A necessary feature has just been formulated that must characterize a situation in which emergence occurs, namely, the presence of something unexpected, a *surprise*. This criterion is not very clear, and what may seem worse, it is subjective. Is it useful? Is there anything surprising on the level of infinite sets with respect to the level of finite sets? Of course, new features appear. One of them is noteworthy, the possibility that a part be equal to the whole, or rather as big as the whole, in the sense that there is a one-to-one correlation of the part with the whole. This is surprising at first. A moment later, however, the surprise disappears. The numeric equivalence of integers with the even numbers is so simple that no surprise remains. More generally, the one-to-one correspondence of a set and its proper part can be treated as a characterization of infinity ("Dedekind infinite sets"). Other equivalences proved by Cantor, like the denumerability of the set of rational numbers, are only slightly more difficult. Their proofs are transparent, as is the proof of the uncountability of the set of real numbers (if the existence of this set as a completed entity is assumed). Once understood they no longer cause surprise.

It can be realized now that another necessary criterion has just been formulated: to witness emergence one must not only feel a surprise, but also sense a surprise that is impossible to overcome, an *inescapable surprise*. Now this criterion is not just subjective, it seems purely psychological. Is this acceptable? My thesis is that the psychological nature of emergence is unavoidable at least in the case of mathematics. I leave out the question whether the psychological definition is appropriate for science as well, although my guess is that some psychological dimension is inevitable.

In the case of infinity, is the feeling of unexpected developments present or not? It is possible to show more and more difficult theorems about infinite sets: statements that need a lot of effort to understand. A real expert in the field can, however, understand them so well that the difference between them and the basic properties is only one of quantity. To witness emergence we need, I believe, to face a qualitative difference, which even for an expert indicates a different order of events. To answer whether the expert perceives a qualitative difference there, we must refer to a psychological approach. Certainly, it would be good to have an objective definition of qualitative as distinct from merely quantitative difference in the degree of surprise provoked by theorems. Yet to formulate such a definition seems to be just as difficult as is the definition of emergence.

Therefore, having accepted the psychological nature of emergence, we are looking for an example of the situation in which a new property appears, one that is unexpected and one where the feeling of surprise is lasting (independent of the level of expertise and familiarity with the subject) so that we can be sure that we face a genuinely new quality. If the structure gets richer, can some new unanticipated properties appear? Properties that are unexpected in an inescapable manner? To repeat, I do not mean here just any new property, but an essentially new property or regularity. The requirement that such a property must be inescapably unexpected defines a necessary condition for emergence. If sufficiently many examples can be found then, hopefully, one would be able to say whether this is a sufficient condition as well.

3. Mathematical examples

Mathematics of the last several decades provides some highly interesting examples. Fractals are generally known by now, even among the mathematically illiterate. Iterating simple functions produces unanticipated, immensely complicated structures that are self-similar: zooming in we encounter the same structure again and again. The complication is infinite. What is even more relevant in our context is that some such iterations provide a huge variety of patterns, or "landscapes." I believe that even the best experts are repeatedly and inescapably surprised when they study the successive regions of the Mandelbrot set. Our psychological criterion for the presence of emergence is satisfied.

Another example is provided by the mathematics of deterministic chaos, initiated over a hundred years ago and developed in a deeper way only relatively recently. Even in completely deterministic processes tiny changes in the initial conditions can produce huge differences in results. This can explain why surprise is an inescapable property of models of weather. Some authors call this impossibility to predict the result emergence. More systematically, similar processes are studied in the theories of complexity, in which various phenomena of growing complexity are investigated. Some of them are generated by iterated applications of very simple rules. For example, John Conway's game Life is played on an infinite plane grid of regular squares where an initial arbitrary finite pattern of black squares among the remaining white ones is consecutively modified according to fixed rules that establish the color of a given square based on the colors of its immediate neighbors at the previous step. The resulting patterns undergo an evolution, and the process can be somewhat similar to the movement of schematic organisms that grow, shift, multiply, get "old," etc. A closer investigation has been made possible due to computers. A similar realm, that of finite automata, has been classified by Stephen Wolfram. In his impressive and highly unconventional book, *A New Kind of Science*, published in 2002, he attempts to demonstrate that all sorts of physical phenomena can be represented as iterations of simple algorithms. Wolfram considers rows of black and white cells that change according to fixed rules; in each step the color of a cell depends only on the colors of itself and its immediate neighbors in the previous step. Even in this simple space some rules determine extremely complicated behavior that is neither periodic nor completely random. Wolfram is ready to say that the whole universe is an automaton, or a gigantic computer. This view can be seen as the ultimate expression of the Pythagorean approach to nature. In the last analysis, our universe and everything in it would be an automaton. This includes us, our brains and ourselves – or "our selves."

While the above vision looks extreme, the fact is that some finitely describable structures can contain more complex, and actually arbitrarily complex, phenomena. The existence of such universal structures has been known in the foundations of mathematics since 1936, when Alan Turing defined what we today call a universal Turing machine. With a (coded) natural number as input, it can imitate another Turing machine, and each Turing machine can be simulated that way, given an appropriate parameter. This looks like emergence but actually the matter is no longer surprising as soon as we realize that the parameter encodes the program of the given machine that is to be simulated. Therefore, according to our criteria, emergence is not taking place here. Still, the example indicates that perhaps one can find emergence in mathematical structures analyzed with the help of methods employed in modern mathematical logic.

4. Logical foundations of mathematics

The so-called nonstandard models come to mind when mathematical logic is evoked in our context. Theories in first order logic have unintended models; that is, there exist mathematical structures that satisfy all axioms of a given theory but are not isomorphic with the "standard" model that served as the source of the axioms. In the case of set theory we are faced with the Skolem paradox: whereas set theory (formulated in first order logic) is supposed to describe all sets, of all possible cardinalities, it admits countable models that can be constructed from natural numbers. Each of them does contain higher infinities in the sense of the model, since there is no function in the sense of the model inside the model that would establish a one-to-one correlation of the sets of different cardinalities. The sets are countable only from the outside. This situation is well understood by logicians, for whom the initial paradox disappears.

Set theory was conceived as maximal, referring to all sets. At the other end there are theories of natural numbers $0, 1, 2, 3, \ldots$. All of them, if in first order logic, admit uncountable models. This is as surprising as Skolem's paradox. (It can be mentioned here that the first nonstandard model for arithmetic was also constructed by Skolem.) And actually there are many complicated nonstandard models of arithmetic, countable and uncountable. Each of them contains infinite numbers, that is, numbers bigger than each standard number $0, 1, 2, 3, \ldots$. But, again, this infinity is perceived only from outside; inside the model all elements have the same status as the standard numbers. It follows, and this is well understood by logicians, that it is impossible to express the notion of a standard number inside the model.

Nonstandard models have some properties that can be linked to emergence: they are unexpected, at least initially. However, the surprise disappears for anyone who is initiated into the theory of models of first order logic. One quickly gets used to the fact that first order logic emerges as too weak to describe the intended model properly. In addition, nonstandard models of arithmetic introduce a more sophisticated view of the aforementioned passage from the finite to the infinite. Mathematicians are free to produce all sorts of abstract models; they can be made of anything and are considered acceptable as long as they satisfy all the axioms. One of the most fruitful methods is due to Henkin: models can be constructed from abstract linguistic entities, and if we begin with an arbitrary consistent set of sentences we can add all the necessary individual constants, identify some of them, and obtain a model of these sentences. The existence of an immense variety of models ceases to be startling. More to the point, the surprise caused by non-standard models does not seem inescapable. It is, therefore, doubtful that we are really facing emergence here.

And yet in this area of mathematics, or rather the logical foundations of mathematics, there exists a phenomenon that is, in my view, fully worthy of the name "emergence." It can be found in the realm of natural numbers: not in nonstandard models, however, but in the familiar standard model. The concept of a natural number seems very … natural. It seems that the operation of successor, "+1", describes the concept. We begin with 0 and iterate the operation indefinitely. One can remark that the concept of unlimited iteration is itself very close to the concept of a natural number. Yet, still, the resulting set of natural numbers, N, seems transparent. We have always known that there are many difficult problems involving natural numbers, yet their totality seems transparent enough to assume that there exists, at least in principle, a procedure to decide whether a given statement is true or not. And, indeed, there is such a procedure if the language is first order and its vocabulary consists of standard logical concepts (sentential connectives and quantifiers binding variables ranging over natural numbers) and the successor operation. In fact a natural set of axioms is complete; each sentence in this language can be either logically derived or refuted from the axioms. What is more, a slight extension of the theory preserves decidability. Namely, when the operation of addition is added, one gets Presburger Arithmetic. In 1929, Mojżesz Presburger demonstrated that the first order sentences true in $(N, +)$ form a complete, decidable theory, which can be axiomatized by a series of natural axioms. Incidentally, when only multiplication is considered, the resulting theory is also decidable. (This was proved by, again, Skolem.) It would seem, then, that nothing unexpected can happen, and that the elementary theory of natural numbers, taken with addition and multiplication, is decidable, as should be also the theory extended by more complicated operations, like exponentiation. That was indeed the expectation of Hilbert and all logicians until 1930. But they were wrong.

Indeed, this is common knowledge now: that standard arithmetic involving both addition and multiplication is not decidable, and that it admits no complete axiomatization as long as the set of axioms is required to be recursive. This was demonstrated by Gödel in his epoch-making paper of 1931. Actually, Gödel proved much more. His result is not only about first order logic, but about arbitrary means of effective listing of, among other things, arithmetical sentences in the first order language referring to addition and multiplication. No axiomatization, formal system, computer, or Turing machine can produce all such true, and only true, sentences.

The advent of undecidability as a consequence of one simple step consisting of piecing together multiplication and addition, should be called emergence. This is surprising; and it was surprising to all experts when it was discovered. What is more, it remains surprising. This claim may be controversial, so it requires an explanation. The contrary view would be based on the argument that the phenomenon of undecidability of arithmetic is well known now, as are many related results. Mathematicians and logicians have got used to the fact discovered by Gödel and they know that the structure of natural numbers, considered with addition and multiplication, is so involved that one can represent in it all recursive functions, and this can be proved to be sufficient to represent also some non-recursive sets. It can be added here, that the natural question, of what happens if exponentiation is added, and then further functions, admits an impressive answer: nothing new is happening. Exponentiation can be defined in Peano Arithmetic (and even in weaker theories), and all primitive recursive functions can be defined as well, as was shown by Gödel in his paper. If so, does the fact that we have learned so much eliminate the initial surprise?

My view is that it does not. The reason is that it is hard to explain why this undecidability occurs. It seemed that the iteration of the successor operation defines the natural numbers. And it still seems so. What happens when addition and multiplication are added? More can be expressed. The natural numbers, so utterly simple, become suddenly very complicated. The complexity is objective; it has nothing to do with our way of approaching it.

The numbers emerge as being anything but simple. Whatever we propose as a definition of them is necessarily inadequate. No definition, no program is sufficient. This much is known; but it is very difficult to overcome the initial surprise and to agree that no definition, no finite description can be given. After all, it seems we do *know* what the natural numbers are. Why is there no comprehensive, adequate definition of natural numbers? After all, we do give definitions that seem to grasp our intuitive concept of natural number; we formulate Peano axioms, define second order arithmetic, etc. What are those definitions, if they are not adequate? The answer is that they are definitions good enough for us, but not comprehensive enough. Apparently, they function as definitions only in conjunction with some background knowledge that is not explicit. That is, some implicit resources are inevitable. In the case of natural numbers defined by means of the successor operation, the intuitive knowledge is applied, as mentioned before, in the idea of unlimited iteration of the operation. To understand what this means, the concept of potential infinity must be available, and, indeed, some understanding of the natural numbers.

It is impossible to describe the entirety of our concept of natural numbers in a way comprehensive enough that it can be communicated to another human being or to an artificial being, say a robot, without additional assumptions about the tacit background knowledge of the recipient. Where *our* tacit knowledge comes from is an interesting issue. Usually biological evolution is offered as the source. Some intuitive common knowledge is needed to understand mathematical definitions of even the simplest concepts. As a matter of fact, they can turn out to be not so simple. The necessity of some background knowledge was obvious for traditional philosophers and also suits those modern attempts that try to uncover hidden assumptions, like phenomenology, at least since Husserl's *Lebenswelt*, and the later philosophy of Wittgenstein. The awareness that tacit background knowledge must exist is also present in analytic philosophy that, like the work of Putnam, overcomes the naïve temptation to formalize everything. One can also say that it should be obvious that in order to formalize anything, something unformalized must be left as the fundament. Gödel's theorem seems not only to confirm that intuition but also to indicate that it is necessarily so.

Interestingly enough, the thesis about the unavoidability of tacit knowledge even in apparently simple mathematics remains in place even if we take into account the possibility that the human mind is equivalent to a machine. This possibility is not excluded by Gödel's results despite claims to the contrary made by many authors who have not understood what Gödel himself noticed: that the existence of such a machine, equivalent to the mind in the realm of arithmetical sentences, does not contradict his theorem. If it exists then there must remain something beyond the transparent, understandable fragments of the program. This "something" may be located in some common knowledge that makes it possible to say that the program is correct, or it may be some innate feature or some property of hardware.

Whatever the nature of the background knowledge is, and however hard we try to understand it, it seems to me that we are unable to imagine the resulting complexity of numbers taken with both addition and multiplication if our point of departure is solely our intuitive understanding and the naïve definition of natural numbers. Thus we can say that we are facing genuine emergence.

Additionally, emergence in a loose and rather metaphorical sense can be also seen in two other aspects of the logical foundations of mathematics indicated by Gödel's proof. Thanks to his proof one can refer to "Gödelian emergence." Let us consider an axiomatic theory in a broad sense – we require only that it is rich enough to make Gödel's proof applicable. Given such a theory, if we assume the consistency of the theory then automatically the arithmetical sentence expressing the consistency of the theory can be assumed to be true. The sentence can have a simple form; namely, it can state that "there is no solution of a Diophantine equation p = 0", where p is a polynomial with integer coefficients." (The polynomial p is defined specially for the theory in question.) What is more, this is still true even if the theory is about a completely different area with no direct connection to arithmetic. As soon as we agree that the theory is consistent we can also assume that a certain specific and highly unreadable equation has no integer solutions and, still more, that this statement (stating that the equation has no solution) is not derivable in the original theory (if the standard coding procedures are used). Some kind of inexhaustibility of mathematics can be seen here; in particular, if a theory is intended to include the *whole* of mathematics, an appropriate statement about the non-existence of integer solutions of a certain equation refutes the intention. This inexhaustibility, that is, the process of going beyond any framework that we can propose that is supposed to capture mathematics, can be seen as a sort of emergence.

A related interpretation was proposed by Michael Dummett. He brings Gödel's results as an argument in favor of the intuitionist concept of number. Incompleteness, he says, is the result of the internal unclarity of the meaning of number. We know what numbers are but we cannot escape ambiguity because the principle of mathematical induction must be true for all well-defined properties of numbers, and the concept "well-defined property of number" has no fixed reference but is indefinitely extendible. According to

Dummett, Gödel has shown that the class of principles used to recognize the truth of sentences involving quantification over natural numbers cannot be precisely defined; it must be seen as an indefinitely extendible class. This conclusion is in accordance with the vision advocated by mathematical intuitionists: the class of intuitionistically acceptable proofs grows in time because we understand our mathematical constructions better and better. This extendibility of the notion of a mathematical proof and of the concept of a well-defined property can be seen as an indication that the concepts are creative. There is something new arising, something impossible to anticipate, and, therefore, we are witnessing here emergence or something akin to it.

The two prime examples of emergence in mathematics considered here, the fractals and the undecidability of the structure of the natural numbers taken with both addition and multiplication, have an interesting similarity to examples from the material world. The properties of fractals and numbers are considered as objectively existing properties of structures that are completely independent of us. In these examples, mathematics looks strikingly similar to science. It is probable that examples of emergence in mathematics can be relevant for the philosophy of natural sciences.

Stanisław Krajewski
Institute of Philosophy
University of Warsaw
`stankrajewski@uw.edu.pl`