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A NEO-FREGEAN THEORY OF OBJECTS AND FUNCTIONS

Abstract. Aside from the most well-known semantic postulates underlying classical logic, the main postulate of Frege's philosophy of logic is the ontological principle that there are exactly two logical types of entities, functions and objects. The aim of this paper is to reconstruct a neo-Fregean theory which implements this principle in the simplest possible way and to examine the philosophical properties of this theory. Indicated and formalized here the so called NOF-theory has the following properties: (1) The only existential assumption of the logic underlying the NOF is the thesis of the existence of at least one object and at least one unary function. (2) The only non-tautological axiom of NOF is the thesis that two arbitrarily chosen objects are different from each other. It is also one of the axioms of Tarski-Grzegorczyk's theory of concatenation (TC). (3) A nominalistic interpretation of the NOF is acceptable, where all functions are determined in the field of linguistic expressions. (4) The concepts of class, of membership and equinumerosity are definable in the NOF. (5) The monadic second-order logic (MSO) is interpretable in the NOF. (6) In the NOF-formalization of Tarski-Grzegorczyk's theory – in contrast to the normal version of this theory (TC) – the concept of sequence is definable.

1. What is the neo-Fregean theory of objects and functions?

As we know from the history and philosophy of logic, the first clear formulation of the main semantic principles defining classical logic – the principles of bivalence, compositionality of extensions, and non-emptiness of names – are all derived from Gottlob Frege. Less known is the fact that the following ontological postulate occurs among the specific principles of Frege's philosophy of logic.

(O) There are exactly two logical types of entities (i.e. values of the quantified logical variables): functions ("unsaturated" entities) and objects (arguments of functions).

The first clear articulation of the idea of logicism also comes from Frege. According to the articulation, arithmetic based on natural numbers like finite cardinals is derivable from classical logic and some meaning postulates.

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Among these postulates, the most important role is played by the so called Hume's Principle (the term was introduced by George Boolos), which states that the powers of two classes are equal if and only if the classes are equinumerous. The methodological and philosophical status of the postulate has become a main theme of reflections and discussions in neo-Fregean philosophy of mathematics in recent decades.[1] These reflections and discussions alone – regardless of the evaluation of the results – clearly show that the concept of equinumerosity is a key component of Fregean foundations for mathematics.

These observations suggest that at the heart of the neo-Fregean philosophy of mathematics is a second order theory in which: a) there are exactly two types of quantified variables, object (individual) variables and *one-place* function variables, and b) the concept of equinumerosity is expressible. Frege did not assume (as far as I know) that the category of *many-place* functions were derivable from the category of one-place functions. However, this assumption provides the simplest way to formalize the postulate (O). It is also compatible with contemporary set-theoretic logicism, i.e. the widely accepted programme of reducing mathematics to standard set theory (since many-place functions are defined in the theory as a special kind of one-place function). The purpose of this paper is to simply reconstruct the suggested neo-Fregean theory of objects and functions, meaning: NOF, and to give a description of some of their philosophical properties.

2. Formalization of NOF-theory

NOF-language is the result of the reduction of "functional" second order logic on its extra-logical constants to a set of only two names, "**1**" and "**0**". Intuitively, these names denote two arbitrarily chosen objects. In more detail, the alphabet (of the NOF-language) consists of the following symbols.

1. Logical connectives: $\neg, \wedge, \vee, \Rightarrow, \Leftrightarrow$.
2. Identity predicate: $=$.
3. Logical quantifiers: $\exists, \forall$.
4. Names: $\mathbf{0}, \mathbf{1}$.
5. Object variables: $x_1, x_2, \ldots$.
6. Function variables: $f_1, f_2, \ldots$.
7. Parentheses: $(,)$.

[1] This trend is directly derived from Crispin Wright and Bob Hale, and indirectly from Boolos. See: [Hale and Wright, 2001; Boolos, 1998].

We define the set of terms and formulas (of the NOF-language) in the usual manner.

1. The object variables and the names are terms.
2. If f is a function variable and t is a term, then '$f(t)$' is a term.
3. If s, t are terms or function variables, then '$s = t$' is a formula.
4. If α is a formula, then '$\exists x_i \alpha$', '$\forall x_i \alpha$', '$\exists f_i \alpha$', '$\forall f_i \alpha$' are formulas.
5. If α, β are formulas, then '$\neg\alpha$', '$(\alpha \wedge \beta)$', '$(\alpha \vee \beta)$', '$(\alpha \Rightarrow \beta)$', '$(\alpha \Leftrightarrow \beta)$' are formulas.
6. No other sequence of symbols is a formula.

We will sometimes use the (metalogical) letters x, y, z as object variables, f, g, h – as function variables, α, β, γ – as variables ranging over formulas, s, t – as variables ranging over terms and function variables.

Axioms of the NOF-theory consist of logical axioms (1–5) and a specific axiom ($\mathbf{NOF^{01}}$).

1. Every instance of the tautology of classical propositional calculus.
2. Every instance of the axioms of classical logic, common to first and second order logic (i.e. schemes of two versions of axioms, objectual and functional, *dictum de omni* and existential introduction).
3. Every instance of the comprehension schema for functions (FCP):[2]

$$\forall x \exists! y \alpha(x, y) \Rightarrow \exists f \forall x \forall y (f(x) = y \Leftrightarrow \alpha(x, y)),$$

provided that f does not occur free in $\alpha(x, y)$.

4. Every instance of the axioms for identity:

$$\forall x x = x.$$

$\forall s \forall t (s = t \Rightarrow (\alpha(s) \Rightarrow \alpha(t)))$, provided that t is free for s in $\alpha(s)$.

5. The axiom of extensionality (for functions):

$$\forall f \forall g (\forall x (f(x) = g(x)) \Rightarrow f = g).$$

$\mathbf{NOF^{01}}$. $\neg \mathbf{1} = \mathbf{0}$.

NOF-theory (in short: NOF) is determined by the axioms and standard rules of inferences: *modus ponens*, two versions of (objectual and functional) rules of generalization and two versions of rules for existential introduction. A thesis of the NOF-theory is a formula derivable from the axioms with the use of the rules. If α is a thesis of NOF, we will sometimes write: $\emptyset \vdash_{NOF} \alpha$.

[2] We assume, as usual, that every expression of the form '$\exists! z \beta(z)$' is an abbreviation for the formula '$\exists z \forall y (\beta(z) \Leftrightarrow z = y)$'.

3. Reconstruction of the concepts of class and equinumerosity

We define (following Frege and John von Neumann) the concepts of class and membership relation:
Df1. $CL(f) =_{df} \forall x(f(x) = \mathbf{1} \vee f(x) = \mathbf{0})$.
Df2. $x \in f =_{df} CL(f) \wedge f(x) = \mathbf{1}$.

Then classes are (total) characteristic functions in NOF. Instead of function variables running over classes, we will sometimes use meta-variables X, Y, Z etc.[3]

We derive the principles of extensionality and comprehension for classes from definitions Df1, Df2 and axiom $\mathrm{NOF}^{\mathbf{01}}$ (and also from logical axioms).

Fact 1

$\emptyset \vdash_{NOF} \forall X \forall(\forall x(x \in X \Leftrightarrow x \in Y) \Rightarrow X = Y)$.

Fact 2

$\emptyset \vdash_{NOF} \exists f \forall x(x \in f \Leftrightarrow \alpha(x))$, provided that f is not free in α.

Sketch of the proof.

We acknowledge this fact by transforming the scheme obtained from the substitution of the formula:

$$\alpha(x) \wedge y = \mathbf{1} \vee \neg\alpha(x) \wedge y = \mathbf{0},$$

where y is not free in $\alpha(x)$, for $\alpha(x, y)$ in Axiom 3 (the comprehension scheme for functions). Since the antecedent of the obtained scheme is true, we can detach the consequent. Now we can substitute the constants $\mathbf{1}$ and $\mathbf{0}$ for y in this consequent and then use Df1, Df2 and $\mathrm{NOF}^{\mathbf{01}}$. By simple logical transforming of the result, we obtain the formula in question. ∎

We may, as usual, define – with the use of the obtained comprehension scheme – Boolean operations for classes.

Fact 3

Boolean algebra of classes is a fragment of NOF.

We define a translation function $\star$ from the set of formulae of the monadic second order logic (MSO) to the set of NOF-formulae:

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[3] Formulas '$\forall X\alpha$', '$\exists X\alpha$' represent, respectively, '$\forall f(CL(f) \Rightarrow \alpha(X/f))$', '$\exists f(CL(f) \wedge \alpha(X/f))$' (provided that f is not free in α).

$$
\begin{aligned}
(x = y)^\star &= \text{`}x = y\text{'},\\
(X_i x)^\star &\equiv \text{`}\forall x(f_i(x) = \mathbf{1} \vee f_i(x) = \mathbf{0}) \wedge f(x) = \mathbf{1}\text{'},\\
(\neg\alpha)^\star &\equiv \text{`}\neg(\alpha)^\star\text{'},\\
(\alpha \wedge \beta)^\star &\equiv \text{`}(\alpha)^\star \wedge (\beta)^\star\text{'},\\
(\alpha \vee \beta)^\star &\equiv \text{`}(\alpha)^\star \vee (\beta)^\star\text{'},\\
(\alpha \Rightarrow \beta)^\star &\equiv \text{`}(\alpha)^\star \Rightarrow (\beta)^\star\text{'},\\
(\alpha \Leftrightarrow \beta)^\star &\equiv \text{`}(\alpha)^\star \Leftrightarrow (\beta)^\star\text{'},\\
(\exists x\alpha)^\star &\equiv \text{`}\exists x(\alpha)^\star\text{'},\\
(\forall x\alpha)^\star &\equiv \text{`}\forall x(\alpha)^\star\text{'},\\
(\exists X_i\alpha)^\star &\equiv \text{`}\exists f_i(CL(f_i) \wedge (\alpha)^\star)\text{'},\\
(\forall X_i\alpha)^\star &\equiv \text{`}\forall f_i(CL(f_i) \Rightarrow (\alpha)^\star)\text{'}.
\end{aligned}
$$

We can easily state that the determined function leads all MSO-theses to NOF-theses.

Fact 4

MSO-system is a fragment of (is interpretable in) NOF-theory.

We may also define the concept of mapping of sets in NOF-theory:

Df3. $f : X \to Y =_{df} \forall x(x \in X \Rightarrow f(x) \in Y) \wedge \forall x(\neg x \in X \Rightarrow f(x) = \mathbf{0})$,

and then, as usual, the concepts of bijection and equinumerosity.

4. Is neo-Fregean logic a set theory in disguise?

Let NF be a system obtained from NOF by deletion of the axiom $\mathrm{NOF}^{\mathbf{01}}$. NF is a system without extra-logical constants that forms the logical basis for NOF.

NF does not include – unlike the full version of second-order logic and MSO – any existential commitments to classes. Two facts are its sources (quite nice from the philosophical point of view). First, NF-language does not have separate types of variables ranging over classes. Second, if classes were definable in NF, then they would be characteristic functions; however, this would require extra-logical assumption about the existence of at least two different objects. Since the said assumption does not apply to this logic, no version of set theory is interpretable in NF.

Moreover, this logic has no strong existential commitments to functions. The source of this property is in turn the fact that the comprehension scheme for functions (FCP) is the conditional form. It is easy to verify (considering even the minimal model of NF, thus any singleton) that we can define exactly one function on the basis of FCP, namely the identity function (obtained by the substitution of the formula '$x = y$' for '$\alpha(x, y)$'). Based on this, we can state the fact:

Fact 5

The set of the existential commitments of NF consists of exactly two claims:

- there is at least one object,
- there is at least one function.

This conclusion may seem quite surprising from the philosophical point of view. Previous discussions concerning the issue of the assumptions underlying the consistent and interesting (for logicists) fragments of Frege's system seem to suggest that one of the greatest difficulties of neo-Fregeanism is the question of the ontological commitment of higher order logic. This difficulty is usually associated with Quine's thesis that second-order logic is set theory in disguise.[4] Indeed, if the thesis were correct, then neo-Fregeanism would not be essentially different from the usual set-theoretical logicism.

The previous discussions assumed – as far as I know – that MSO is contained in each adequate (for logicists) fragment of Frege's logic. Under this assumption, a defense against Quine's thesis was sometimes developed, replacing the objectual interpretation of second-order quantifiers by a substitutional one. However, such a solution is not compatible with the spirit of the neo-Fregean philosophy of logic, both because of the typical assignment of this philosophy, the objectual interpretation of the logic, and because of the common tasks of neologicism (contrasting with the limited power of expression of the substitutional quantification).[5]

The concept of foundations of mathematics, in which NOF plays a central role, provides a simple method to avoid Quine's objection. The method is to exclude, from the scope of mathematical logic, systems in which the predicate variables (including monadic variables) and many-placed functional variables are quantified. As a result, we get a system of second-

[4] See for example: [Hale and Wright, 2005, pp. 197–200].

[5] See ibid. In this article, other ways to avoid Quine's difficulty are also presented (for example the interpretation of MSO as "logic of plurality" in Boolos's style).

order logic with modest existential commitments to keep an ordinary, non-substitutional interpretation (for both types of quantification). At the same time, the system has sufficient strength of expression for introducing – through such theories as NOF – fundamental theories in mathematics.

5. Does NOF have ontological commitments?

Since the NOF is an extremely general theory of objects and functions, its extent does not exclude nominalistic interpretations in which some expressions of NOF-language are values of the object variables. A simple example of such an interpretation is the structure:

$$\mathrm{M}^{1,0} = \langle\{\mathbf{1}, \mathbf{0}\}, \mathrm{ID}\rangle,$$

where ID is the identity function (i.e. $\mathrm{ID}(\mathbf{1}) = \mathbf{1}, \mathrm{ID}(\mathbf{0}) = \mathbf{0}$). It is quite reasonable to postulate that ontological commitments of a theory are reduced to objects that are not linguistic expressions of the theory. In this sense, we can assume that NOF does not contain any ontological commitments in $\mathrm{M}^{1,0}$.

Fact 6

There are acceptable interpretations of NOF that are free of any ontological commitment.

In this context, it is quite interesting that $\mathrm{NOF}^{\mathbf{01}}$ is one of the axioms of Tarski-Grzegorczyk's theory of concatenation TC.[6] From the neo-Fregeanism perspective, as to mathematical basis, there is nothing in the way of formulating TC theory on the basis of NOF. In making such formalization, we get a “nominalistic” definition of the sequence:

$$f = (x_1, x_2, x_3 \ldots)$$

as a function defined on successive “powers” of names (the ˆ is here a symbol of the concatenation operation):

$$f(\mathbf{1}) = x_1, f(\mathbf{1}\hat{}\mathbf{1}) = x_2, f(\mathbf{1}\hat{}\mathbf{1}\hat{}\mathbf{1}) = x_3 \ldots.$$

Now the n-tuples can be represented as follows. All the arguments, different from $\mathbf{1}^k$, for $1 \leq k \leq n$, are assigned by the function f to the

[6] TC theory, derived from [Tarski, 1933], in recent years has been described and applied in examining the issue of decidability in [Grzegorczyk, 2005].

name **0** (which may be identified with the empty string).[7] With the finite strings, you can then pose the problem of neo-Fregean reconstruction of the concepts of relation and the many-placed function. This problem, like the question of the details of the project outlined above, we leave here as open.[8] Here we only note the fact.

Fact 7

Let TC^{NOF} be the result of the extension of the NOF-theory by addition of TC-axioms. In the TC^{NOF} theory, the concept of (finite or infinite) sequence is definable.

Bibliography

[Boolos, 1998] G. Boolos. *Logic, Logic and Logic*. Harvard University Press, Cambridge, Ma., 1998.

[Grzegorczyk, 2005] A. Grzegorczyk. Undecidability without arithmetization. *Studia Logica*, 79:163–230, 2005.

[Hale and Wright, 2001] B. Hale and C. Wright. *The Reason's Proper Study. Essays towards a Neo-Fregean Philosophy of Mathematics*. Clarendon Press, 2001.

[Hale and Wright, 2005] B. Hale and C. Wright. Logicism in the twenty-first century. In S. Shapiro, editor, *The Oxford Handbook of Philosophy of Mathematics and Logic*, pages 166–202. Oxford University Press, 2005.

[Tarski, 1933] A. Tarski. *Pojęcie prawdy w językach nauk dedukcyjnych*. Prace Towarzystwa Naukowego Warszawskiego. Warszawa, 1933. Zob. [Tarski, 1936 za rok 1935]. Tłumaczenie angielskie: "The Concept of Truth in Formalized Languages", w: Logic, Semantics, Metamatematics: Papers from 1923 to 1938, Oxford 1956.

[Tarski, 1936 za rok 1935] A. Tarski. Der Wahrheitsbegriff in der formalisierten Sprachen. *Studia Philosophica*, 41:261–405, 1936 (za rok 1935). Uzupełniony przekład z wydania polskiego w 1933 r., [Tarski, 1933].

[Visser, 2009] A. Visser. Growing commas. A study of sequentiality and concatenation. *Notre Dame Journal of Formal Logic*, 50(1):61–85, 2009.

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[7] The symbol "$\mathbf{1}^k$" ("power" of the letter **1**) means the product of k-times writing of the letter **1** to the empty string.

[8] This is not a very apparent problem: ordered pairs are not definable in TC (see: [Visser, 2009]).