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A GENERALIZATION OF THE CONSISTENCY PREDICATE

Abstract. We modify the usual arithmetical consistency predicate. We study the behavior of our predicate in fragments of arithmetic. The definition of our predicate depends on a formula J defining an initial segment and on a set Γ of arithmetical formulas. We formulate conditions on J and on Γ under which our predicate has the properties usually required from a consistency predicate. As a result we obtain a well behaving consistency predicate. Suitably choosing J and Γ we obtain a well behaving consistency predicate whose arithmetical complexity is unusual, namely is Σ_1.

1. Motivation

Our motivation is a deeper understanding of the independence phenomenon in arithmetic. To this end we generalize the usual consistency predicate. Since consistency is dual to provability, as well we may deal with provability. On one hand we restrict some usual consistency predicate, like the Hilbert or the Gentzen or the Herbrand consistency to some definable initial segment; on the other hand we include into the theory whose consistency is considered some set of true sentences. We study those properties of the initial segment involved which guarantee that our consistency predicate behaves regularly, including the Gödel phenomenon. We axiomatize those properties. Finally we illustrate our considerations by showing a consistency predicate which behaves regularly although it is Σ_1 definable, not as usually Π_1.

Restricting the consistency predicate to some definable initial segment was already considered in the literature for instance by P. Pudlak [1985]. He proved that as far as the Hilbert consistency predicate is concerned many interesting restrictions to a cut still satisfy the Gödel independence phenomenon; however as far as the Herbrand or Gentzen predicate is concerned for many natural cuts this is not the case. A similar approach for Herbrand consistency was studied carefully in [Adamowicz and Zdanowski, 2011]. Also S. Feferman [1960] studied consistency predicates for which the second Gödel

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independence theorem failed. We go in a different direction than P. Pudlak and S. Feferman.

The main theorem is formulated and proved in section 15. The last section is devoted to an illustration of the main theorem by a special case of it. The remaining sections are introductory.

2. Fragments of Arithmetic

Peano Arithmetic is the theory of the non negative part of a discretely ordered ring together with the induction scheme:

$$\forall x(\phi(x) \Rightarrow \phi(x+1)) \Rightarrow \forall x\phi(x),$$

where ϕ runs over all formulas of the language.

This scheme is equivalent to its parameter version:

$$\forall a\Big(\phi(a,0)\&\forall x(\phi(a,x) \Rightarrow \phi(a,x+1)) \Rightarrow \forall x\phi(a,x)\Big).$$

When the range of the formula ϕ is restricted to Σ_n formulas, we get the fragment $I\Sigma_n$.

For $n \geq 1$ this is considered as an strong fragment, for $n = 0$ this is the theory $I\Sigma_0$, which is considered as Weak Arithmetic. The distinction between strong and weak refers to the provability of the totality of the exponential function:

$$\forall x \exists y\ y = 2^x,$$

where the notation $y = 2^x$ is an abbreviation of an arithmetical formula defining the graph of the function 2^x.

Let the above sentence be denoted by *exp*. We have

$$I\Sigma_1 \vdash exp,$$

and

$$I\Sigma_0 \nvdash exp\,.$$

Note that a Σ_0 formula, which is also denoted by Δ_0, is a formula whose quantifiers are all bounded.

Thus, the theory $I\Sigma_0$ is more often denoted by $I\Delta_0$.

One may also consider the theory $I\Delta_0 + exp$, or intermediate theories $I\Delta_0 + \Omega_n$.

We define the following functions:

$$\omega_0(x) \quad = x^2,$$

$$\begin{aligned}\omega_1(x) &= 2^{(log\, x)^2},\\ \omega_2(x) &= 2^{2^{(log\, log\, x)^2}},\\ \omega_{i+1}(x) &= 2^{\omega_i(log\, x)}.\end{aligned}$$

Note that $\omega_1(x) = 2^{(log\, x)^2} = (2^{log\, x})^{log\, x} = x^{log\, x}$ and has an intermediate growth between polynomials x^n, for a fixed n, and the exponential function 2^x.

By Ω_i we mean the sentence stating the totality of the ω_i function:

$$\Omega_i : \ \forall x \exists y \ y = \omega_i(x),$$

where the notation $y = \omega_i(x)$ is an abbreviation of an arithmetical formula defining the graph of the function $\omega_i(x)$.

We have

$$I\Delta_0 + \Omega_i \nvdash exp\,.$$

Hence the theories $I\Delta_0 + \Omega_n$ are weak fragments of artithmetic, while $I\Delta_0 + exp$ is a strong fragment.

In a model M of $I\Delta_0 + \Omega_i$ exponentiation may not be total, hence $log(M) = \{x \in M : \ M \models \exists 2^x\}$ may be a proper initial segment. Similarly the segment $log_n(M) = \{x \in M : \ M \models \exists\, exp_n(x)\}$, where log_n denotes the n times iterated logarithm (where $log_0(x) = x$) and exp_n denotes the n times iterated exponentiation, may be a proper initial segment.

For a model M of $I\Delta_0$ we have $M \models \Omega_i$ iff $log_i(M)$ is closed under multiplication. Consequently, in a model of $I\Delta_0 + \Omega_i$, $log_{i+1}(M)$ is closed under addition, $log_{i+2}(M)$ is closed under successor and $log_{i-k}(M)$ is closed under ω_k.

Note that $log_k(M)$ is a Σ_1 definable initial segment of M.

Assume $\langle a, b\rangle$ denotes the pair-number $\frac{(a+b)(a+b+1)}{2}$.

3. Π_2 axiomatizable fragments of arithmetic

The theory $I\Delta_0$ may be axiomatized by Π_1 sentences, namely

$$\forall a, b\Big(\phi(a,0)\&\forall x < b(\phi(a,x) \Rightarrow \phi(a,x+1)) \Rightarrow \forall x \le b\phi(a,x)\Big),$$

where ϕ runs over Δ_0 formulas.

The theories $I\Delta_0 + \Omega_n$ additionally require the axiom $\forall x \exists y \ y = \omega_n(x)$, which is Π_2. Similarly the axiom exp is Π_2: $\forall x \exists y \ y = 2^x$. Hence $I\Delta_0 + \Omega_n$, $I\Delta_0 + exp$ are Π_2 axiomatizable. Other examples of Π_2 axiomatizable fragments of arithmetic are theories stronger than $I\Delta_0 + exp$, e.g. $I\Delta_0 + \sup exp$, where sup *exp* is an axiom stating the totality of the super exponential function, or $I\Delta_0 + \forall x \exists y\; y = F_n(x)$, where F_n is the nth function in the Grzegorczyk hierarchy, [Grzegorczyk, 1953], or $I\Delta_0 + \forall x \exists y\; y = F_\alpha(x)$, where F_α is the α's function in the Grzegorczyk-Wainer hierarchy [Cichon and Wainer, 1983].

Assume T is a Π_2 axiomatizable theory. Let $M \models T$.

Definition 1

An element $x \in M$ is syntactically Σ_1 definable in M if there is a Δ_0 formula θ, such that

$$M \models \exists y \Big(\theta(x,y) \& \forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta(x', y')) \Big).$$

An element $x \in M$ is Σ_1 definable in M if there is a Σ_1 formula η such that $M \models \eta(x) \& \forall x' (x' \neq x \Rightarrow \neg \eta(x'))$.

Remark 2

An element $x \in M$ is syntactically Σ_1 definable in M iff it is Σ_1 definable in M.

Proof.

Assume that x is syntactically Σ_1 definable. Let $\eta(x)$ be the formula

$$\exists y \Big(\theta(x,y) \& \forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta(x', y')) \Big).$$

Then η is as required.

Assume conversely, that η is a Σ_1 definition of x. Assume η is of the form $\exists y \eta'(x,y)$, where η' is Δ_0. Let $\theta(x,y)$ be $\eta'(x,y) \& \forall y' < y \neg \eta'(x, y')$. Suppose for some $x', y' \in M$, we have $\langle x', y' \rangle < \langle x, y \rangle$ and $\theta(x', y')$. Then either $x' \neq x$ or $x' = x$ and $y' < y$. The second case contradicts the definition of θ. In the first case $M \models \exists y \eta'(x', y)$, whence $M \models \eta(x')$, contradicting $M \models \forall x' (x' \neq x \Rightarrow \neg \eta(x'))$. Hence $\forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta(x', y'))$, whence θ is as required in 1. ■

Remark 3

If x is Σ_1 definable in M, then w.l.o.g. we may assume that it is definable by the formula $\exists y \theta(x,y)$, where θ is of the form $\theta'(x,y) \& \forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta'(x', y'))$. Hence θ provably can have at most one witness.

Proof.

In 1 instead of θ we may take $\theta'(x,y) \& \forall x', y' (\langle x', y' \rangle < \langle x, y \rangle \Rightarrow \neg \theta'(x', y'))$. ■

Definition 4

Let $\mathcal{K}(M)$ denote the set of those x in M which are Σ_1 definable in M. In $\mathcal{K}(M)$ we consider addition and multiplication inherited from M.

Theorem 5

$$\mathcal{K}(M) \prec_{\Sigma_1} M.$$

Proof.

Recall the Tarski Vaught criterium:

For $M_1 \subseteq M_2$ we have $M_1 \prec_{\Sigma_1} M_2$ iff for any $a_1, \ldots, a_n \in M_1$ and $\eta \in \Sigma_1$, whenever there is an $x \in M_2$ such that $M_2 \models \eta(x, a_1, \ldots, a_n)$, then there is an $x \in M_1$ such that $M_2 \models \eta(x, a_1, \ldots, a_n)$.

We apply this criterium.

So, assume $a_1, \ldots, a_n \in \mathcal{K}(M)$, $\eta \in \Sigma_1$ and there is an $x \in M$ such that $M \models \eta(x, a_1, \ldots, a_n)$. Assume η is of the form $\exists y \eta'(y, \ldots)$, where η' is Δ_0. Hence $M \models \exists x, y \eta'(x, y, a_1, \ldots, a_n)$. Let $\eta_1, \ldots, \eta_n$ define $a_1, \ldots, a_n$, respectively and assume η_i is $\exists y \eta'_i$. Assume that $\eta', \eta'_1, \ldots, \eta'_n$ provably can have at most one witness.

Thus,

$$M \models \exists x, y, y_1, \ldots, y_n(\eta'(x, y, a_1, \ldots, a_n) \& \eta'_1(y_1, a_1) \& \ldots \& \eta'_n(y_n, a_n)).$$

We have

$$M \models \exists u \exists \tilde{a}_1, \ldots, \tilde{a}_n, \tilde{x}, \tilde{y}, \tilde{y}_1, \ldots, \tilde{y}_n \leq u \\ (\eta'(\tilde{x}, \tilde{y}, \tilde{a}_1, \ldots, \tilde{a}_n) \& \eta'_1(\tilde{y}_1, \tilde{a}_1) \& \ldots \& \eta'_n(\tilde{y}_n, \tilde{a}_n)).$$

Hence, for some $\tilde{x} \in M$,

$$M \models \exists u \exists \tilde{a}_1, \ldots, \tilde{a}_n, \tilde{y}, \tilde{y}_1, \ldots, \tilde{y}_n \leq u \\ (\eta'(\tilde{x}, \tilde{y}, \tilde{a}_1, \ldots, \tilde{a}_n) \& \eta'_1(\tilde{y}_1, \tilde{a}_1) \& \ldots \& \eta'_n(\tilde{y}_n, \tilde{a}_n)).$$

By the fact that η'_i and η have at most one witness in M, we have $\tilde{a}_1 = a_1, \ldots, \tilde{a}_n = a_n$ and $\tilde{x}$ is Σ_1 definable in M by the formula

$$\exists u \exists \tilde{a}_1, \ldots, \tilde{a}_n, \tilde{y}, \tilde{y}_1, \ldots, \tilde{y}_n \leq u \\ (\eta'(\tilde{x}, \tilde{y}, \tilde{a}_1, \ldots, \tilde{a}_n) \& \eta'_1(\tilde{y}_1, \tilde{a}_1) \& \ldots \& \eta'_n(\tilde{y}_n, \tilde{a}_n)).$$

Hence $\tilde{x} \in \mathcal{K}(M)$ and $M \models \eta(\tilde{x}, a_1, \ldots, a_n)$. Hence for some $x \in \mathcal{K}(M)$ and $M \models \eta(x, a_1, \ldots, a_n)$ and the Tarski Vaught criterium is fulfilled. ■

Corollary 6

Every element of $\mathcal{K}(M)$ is Σ_1 definable in $\mathcal{K}(M)$.

Proof.

Let $x \in \mathcal{K}(M)$ and let $\eta \in \Sigma_1$ define syntactically x. Then $M \models \eta(x)$. By Σ_1 elementariness, $\mathcal{K}(M) \models \eta(x)$. ■

Corollary 7

$\mathcal{K}(M) \models T$.

Proof.

Let $\forall x \exists y \eta(x, y)$ be an axiom of T, where $\eta \in \Delta_0$. Let $x \in \mathcal{K}(M)$. Then, $M \models \exists y \eta(x, y)$. By Σ_1 elementariness, $\mathcal{K}(M) \models \exists y \eta(x, y)$. ■

Corollary 8

A Π_2 axiomatizable theory T has a model which is pointwise Σ_1 definable, i.e. whose every element is Σ_1 definable in it.

4. Coding of truth

Definition 9

Let $x, t \in M$, $t \in \{0,1\}^x$. We say that t codes the Σ_1 truth of M if $x > \mathbb{N}$ and for every sentence $\phi \in \Sigma_1$ we have

$$t(\phi) = 1 \text{ iff } M \models \phi.$$

5. Existence of models whose Σ_1 truth is not coded

Theorem 10

If $M \models T$ is pointwise Σ_1 definable, then $\Sigma_1(M)$ is not coded in M.

Proof.

Suppose the converse. Let $x \in M$ be a code for $\Sigma_1(M)$. Let η be the Σ_1 definition of x. Then we have for ϕ running over Σ_1 sentences:

$$\phi \text{ iff } \forall x(\eta(x) \Rightarrow \phi \in x).$$

This gives a Π_1 definition of the Σ_1 truth, a contradiction with the Tarski theorem. ■

Theorem 11 [Wilkie and Paris, 1978]

Every model for $I\Delta_0 + B\Sigma_1$ has a Σ_1 elementary submodel satisfying $I\Delta_0 + B\Sigma_1$ whose Σ_1 truth is not coded.

6. Maximal theories

Definition 12

A set of Σ_1 sentences $T^\#$ is maximal w.r.t. T if it is maximal consistent with T.

Remark 13

A Π_2 axiomatizable theory T has a model which is pointwise Σ_1 definable and satisfies a maximal theory $T^\#$.

Proof.

Let $M \models T + T^\#$ and take $\mathcal{K}(M)$. ■

7. Initial segments

I is an initial segment of M if $I \subseteq M$ and for every $x \in I$, $y \leq x$ we have $y \in I$. I is definable if there is a formula η such that $I = \{x \in M : M \models \eta(x)\}$.

If I is definable we identify I with its definition.

Note that $log_k(M)$ is a Σ_1 definable initial segment of M.

If $a \in M$ is definable, then $\{x \in M : x \leq a\}$ is a definable initial segment. If $M \models PA$ then every definable initial segment of M is of this form.

I is a cut if I is definable and is, provably in T, an initial segment and I is provably closed under successor.

Note that log_k is not a cut.

In the case where $T = PA$, there are no proper definable cuts.

In a model of $I\Delta_0 + \Omega_n$ or of $I\Delta_0 + exp$, $\mathbb{N}$ may be a definable proper initial segment.

Example 14

Let $\theta(x)$ be the formula expressing "x is the least proof of the inconsistency of $I\Delta_0 + \Omega_n$". Let $M \models I\Delta_0 + \Omega_n + \exists x\theta(x)$ and let a satisfy θ in M. Assume that elements of the form $\omega_1^n(a)$, for $n \in \mathbb{N}$, are cofinal in M. Then $\mathbb{N}$ is definable in M by the formula $\eta(u)$:

$$\exists x \exists y(\theta(x) \& y = \omega_1^u(x)).$$

It is more difficult to show a model of $I\Delta_0 + \Omega_n$, where $\mathbb{N}$ is Π_1 definable.

8. Defining $\mathbb{N}$

We assume that J_T is a Σ_1 or Π_1 formula. We shall identify J_T with the set definable by the formula J_T.

We assume the following:

Key properties

1) J_T is an initial segment provably in T,
2) $\mathbb{N} \subseteq J_T$ provably in T
3) J_T is $\mathbb{N}$ in some models of T.

9. What J_T can be

Assume $T \supseteq I\Delta_0 + exp$.

Consider the following formula $\mathbb{N}_{T,\Sigma_1}(x)$ expressing the meaning that there is a set (i.e. a characteristic function of a set) of size x consisting of Σ_1 sentences containing all true Σ_1 sentences and x-consistent with T:

We may call $\mathbb{N}_{T,\Sigma_1}(x)$, the amount of the codability of the Σ_1 truth.

$$\exists t \in \{0,1\}^x \Big(\forall \varphi < x(Sat_{\Sigma_1}(\varphi) \Rightarrow t(\varphi) = 1)$$

$$\& \text{ the theory } \{\varphi < x : \ t(\varphi) = 1\} \text{ is } x\text{-consistent with } T\Big)$$

We can refine $\mathbb{N}_{T,\Sigma_1}(x)$ so that it will be Π_1 definable. Assume T contains a Σ_1 sentence η which is a Σ_1 definition and is false in $\mathbb{N}$. Assume η is of the form $\exists y \eta'(y)$. Consider the formula $\mathbb{N}_{T,\Sigma_1}(x)$:

$$\forall y \forall w \Big(\eta'(w) \Rightarrow \exists t \in \{0,1\}^x \Big(t \leq w \& \forall \varphi < x(Sat_{\Sigma_1}(\varphi^y) \Rightarrow t(\varphi) = 1)$$

$$\& \text{ the theory } \{\varphi < x : \ t(\varphi) = 1\} \text{ is } x\text{-consistent with } T\Big)\Big)$$

This formula is Π_1.

Dual

Consider the following formula $\mathbb{N}_{T,\Pi_1}(x)$ expressing the meaning that there is a set (i.e. a characteristic function of a set) of size x consisting of Π_1 sentences containing all true Π_1 sentences and x-consistent with T:

We may call $\mathbb{N}_{T,\Pi_1}(x)$, the amount of the codability of the Π_1 truth.

$$\exists t \in \{0,1\}^x \Big(\forall \varphi < x(Sat_{\Pi_1}(\varphi) \Rightarrow t(\varphi) = 1)$$

$$\& \text{ the theory } \{\varphi < x : \ t(\varphi) = 1\} \text{ is } x\text{-consistent with } T\Big)$$

This can be made Σ_1.

10. For what T, do $\mathbb{N}_{T,\Pi_1}$, $\mathbb{N}_{T,\Sigma_1}$ have key properties?

Let T denote a Π_2 axiomatizable consistent recursive theory containing $I\Delta_0 + exp$.

E.g. $I\Delta_0 + exp$, $I\Delta_0 + \Omega_1$. We may deal with the language containing a constant $\underline{a}$ and we include into our theories the sentence $\zeta(\underline{a})$, where ζ is a fixed Δ_0 formula. In this case T usually is not a true theory, i.e. every model of T is non standard.

- T has pointwise Σ_1 definable models. Every model of T has a Σ_1 elementary submodel pointwise Σ_1 definable.
- T has models in which witnesses for true Σ_1 sentences are cofinal.
- T has models in which the set $\Sigma_1(M)$ of true Σ_1 sentences is not coded.

11. The key properties of $\mathbb{N}_{T,\Pi_1}$, $\mathbb{N}_{T,\Sigma_1}$

Lemma 15

For every $n \in \mathbb{N}$ and every model M of T, $M \models \mathbb{N}_{T,\Pi_1}(n)$, $M \models \mathbb{N}_{T,\Sigma_1}(n)$.

Lemma 16

For every theory $T^{\#} \subseteq \Sigma_1$ which is maximal consistent w.r.t. T and every model M of $T + T^{\#}$ having the property that $T^{\#}$ is not coded in M, $\mathbb{N}_{T,\Sigma_1}$ defines $\mathbb{N}$ in M.

For every theory $T^{\#} \subseteq \Pi_1$ which is maximal consistent w.r.t. T and every model M of $T + T^{\#}$ having the property that $T^{\#}$ is not coded in M, $\mathbb{N}_{T,\Pi_1}$ defines $\mathbb{N}$ in M.

Proof.

Let M satisfy the requirements of the first part of the lemma.

We shall show that $\mathbb{N}_{T,\Sigma_1}$ defines $\mathbb{N}$ in M.

For, assume $x \in \mathbb{N}$. Let $t \in \{0,1\}^x$ be such that

$$t(\varphi) = 1 \text{ iff } M \models Sat_{\Sigma_1}(\varphi).$$

Then t is as required in $\mathbb{N}_{T,\Sigma_1}$.

Assume now $\mathbb{N}_{T,\Sigma_1}(x)$ and suppose $x > \mathbb{N}$. Take the $t \in M$ existing by $\mathbb{N}_{T,\Sigma_1}$. Then the theory

$$\{\varphi : \ M \models t(\varphi) = 1\}$$

is consistent with T since

$$M \models \text{the theory } \{\varphi < x : \ t(\varphi) = 1\} \text{ is } x\text{-consistent with } T.$$

On the other hand this theory contains $T^{\#}$, since whenever φ is true i.e. $M \models Sat_{\Sigma_1}(\varphi)$, then $t(\varphi) = 1$.

So, by the maximality of $T^{\#}$, the theory

$$\{\varphi : \ M \models t(\varphi) = 1\}$$

equals $T^{\#}$. But so, t is its code on M, a contradiction.

For $\mathbb{N}_{T,\Pi_1}$ the proof is similar. ■

12. Provability and consistency

Let $Pr_T(\phi)$ express the meaning "there is a proof of ϕ in the theory T". The most known predicates Pr are "there is a Hilbert proof", "there is a Gentzen proof", "there is a Herbrand proof", there is a "Tableau proof". All these predicates are definable by Σ_1 formulas of the form $\exists x\, Pr_T^x(\phi)$, where $Pr_T^x(\phi)$ expresses the meaning "x is a proof of ϕ in the theory T".

Dual to $Pr_T(\phi)$ is the predicate $Cons(T + \phi)$, defined as

$$Cons(T + \phi) \text{ iff } \neg Pr_T(\neg\phi).$$

Consequently, $Cons(T + \phi)$ is Π_1 in each of the above cases.

If J is an initial segment of a model M of $I\Delta_0 + \Omega_n$, then let $Pr_T^J(\phi)$ express the meaning "there is a proof belonging to J of ϕ in the theory T". Consequently, $Cons^J(T + \phi)$ expresses the meaning "there is no proof belonging to J of $\neg\phi$ in the theory T".

If J is definable by a formula $J(x)$, then $Pr_T^J(\phi)$ can be defined as $\exists x(J(x) \& Pr_T^x(\phi)$. Similarly, $Cons^J(T + \phi)$ can be defined as $\forall x(J(x) \Rightarrow \neg Pr_T^x(\neg\phi))$.

Our focus will be on $Cons^J(\cdot)$ (consistency relativized to J), for some definable initial segment J.

Assume T is recursive, consistent and contains $I\Delta_0$.

Usually a predicate $Cons(\cdot)$ is considered as expressing consistency if

$$T \text{ is consistent iff } \mathbb{N} \models Cons(T).$$

Let $Pr_T(\cdot)$ be defined as $\neg Cons(T + \neg\cdot)$.

Some other properties are usually expected, e.g. the Hilbert Bernays derivability conditions:

- $T \vdash \phi$ implies $T \vdash Pr_T(\phi)$
- $T \vdash (Pr_T(\phi) \Rightarrow Pr_T(Pr_T(\phi)))$
- $T \vdash \Big((Pr_T(\phi) \& Pr_T(\phi \Rightarrow \psi)) \Rightarrow Pr_T(\psi)\Big)$

Note two other useful properties:

- $Cons(T) \& Pr_T(\phi)$ implies $Cons(T + \phi)$.
- If $Cons^J(\cdot)$ denotes $Cons$ relativized to a definable initial segment J, then

$$Cons^{2J}(T) \& Pr_T^J(\phi) \text{ implies } Cons^J(T + \phi),$$

where $2J = \{2x : x \in J\}$.

We shall call the above properties **basic**.

Later we shall consider some unusual consistency predicates $Cons^J(\cdot)$, for some initial segments J.

13. Usual properties of consistency

Here we formulate the most important properties of the usual consistency predicates like the Hilbert, Gentzen or Herbrand ones.

- $Cons(\cdot)$ is Π_1

Σ_1 **completeness**:

- $T \vdash (\eta \Rightarrow Pr_T(\eta))$ for $\eta \in \Sigma_1$

Gödel:

- $T \nvdash Cons(T)$;
- If T is true then $T \nvdash \neg Cons(T)$ (note that $T + \neg Cons(T) \vdash \neg Cons(T + \neg Cons(T))$
- If $\theta \Leftrightarrow Cons(T + \neg\theta)$ provably in T, then $\theta \Leftrightarrow Cons(T)$ provably in T

14. Consistency over J_T

Let $Cons(\cdot)$ denote the Hilbert or the Herbrand consistency predicate. Assume $T \supseteq I\Delta_0 + exp$. Let Γ be a class of formulas, for instance, Γ can be Σ_1 or Π_1. Assume that we are given a formula $Sat_\Gamma(\cdot)$ universal for sentences in Γ which itself is in Γ. Let $Cons(\cdot + \Gamma)$ mean the sentence stating the following: for every sentence η in Γ, if $Sat_\Gamma(\eta)$ holds, then $Cons(T + \eta)$ holds.

We consider consistency $Cons^J(T + \Gamma)$ over an initial segment $J = J_T$ depending on T. The definition of T is built into the definition of J_T.

By $Cons^{J_T}(T + \Gamma)$ we shall mean the sentence stating the following: for every $x \in J_T$ and every sentence η in Γ such that $\eta \in J_T$ if $Sat_\Gamma(\eta)$ holds, then $Cons^x(T + \eta)$ holds.

For a sentence ϕ, by $Cons^{J_T}(T + \Gamma + \phi)$ we shall mean a sentence stating the following: for every $x \in J_T$ and every sentence η in Γ such that

$\eta \in J_T$ if $Sat_\Gamma(\eta)$ holds, then $Cons^x(T+\eta)$ holds and if $\eta\&\phi \in J_T$, then $Cons^x(T+\eta+\phi)$ holds.

We assume that J_T is Γ definable and has the key properties. Below, we see that the usual properties of consistency from section 13 generalize to this case.

15. Consistency with true Γ sentences

Theorem 17

Assume T is a recursive consistent theory containing $I\Delta_0$. Let Γ be a recursive set of formulas. Let J_T be a formula in Γ having the key properties, i.e.

1. J_T is an initial segment provably in T,
2. $\mathbb{N} \subseteq J_T$ provably in T
3. J_T is $\mathbb{N}$ in some models of T.

Assume that there is a universal formula $Sat_\Gamma(\cdot)$ available in T, which is itself in Γ and is universal for sentences in Γ. We assume that Γ is closed under conjunction, double negation and has enough closure properties so that $Cons^{J_T}(T+\Gamma+\cdot)$, defined in section 14, and the Gödel sentence θ equivalent in T to $Cons^{J_T}(T+\Gamma+\neg\theta)$ are in $\neg\Gamma$.

Then $Cons^{J_T}(T+\Gamma+\cdot)$ has the following properties:

- $T \vdash \phi$ implies $T \vdash Pr^{J_T}_{T,\Gamma}(\phi)$

The Γ completeness:

- $T \vdash (\eta \Rightarrow Pr^{J_T}_{T+\Gamma}(\eta))$ for $\eta \in \Gamma$

The Gödel properties:

- $T \not\vdash Cons^{J_T}(T+\Gamma)$
- $T \not\vdash \neg Cons^{J_T}(T+\Gamma)$
- If $\theta \Leftrightarrow Cons^{J_T}(T+\Gamma+\neg\theta)$ provably in T, then $\theta \Leftrightarrow Cons^{J_T}(T+\Gamma)$ provably in T

Proof.

The Γ completeness is immediate. Let us focus on the Gödel properties.

Lemma 18

Let θ be the diagonal sentence such that

$$T \vdash (\theta \Leftrightarrow Cons^{J_T}(T+\Gamma+\neg\theta)).$$

Call θ the Gödel sentence.

Then

$$T \vdash (\theta \Leftrightarrow Cons^{J_T}(T+\Gamma)).$$

Proof.

Work in T. Assume θ. Then $Cons^{J_T}(T+\Gamma+\neg\theta)$, whence, in particular, $Cons^{J_T}(T+\Gamma)$. Assume $Cons^{J_T}(T+\Gamma)$. Suppose $\neg\theta$. Since $\neg\theta$ is Γ we infer $Cons^{J_T}(T+\Gamma+\neg\theta)$, whence θ. ∎

Corollary 19

$T \nvdash Cons^{J_T}(T+\Gamma)$.

Proof.

We shall prove that $T \nvdash Cons^{J_T}(T+\Gamma)$. Suppose the converse. Let θ Gödel sentence. Then, by 18, $T \vdash \theta$. Let M be a model of T. Then $M \models \theta$. Thus, $M \models Cons^{J_T}(T+\Gamma+\neg\theta)$. Since ${J_T}^M \supseteq \mathbb{N}$, the theory $T+\neg\theta$ is consistent. But on the other hand $T \vdash \theta$. Contradiction. ∎

Corollary 20

The sentence $Cons^{J_T}(T+\Gamma)$ is independent from T.

Proof.

To see that the theory $T + Cons^{J_T}(T+\Gamma)$ is consistent it suffices to observe that is is true in every model M of T in which ${J_T}^M = \mathbb{N}$. On the other hand, $T + \neg Cons^{J_T}(T+\Gamma)$ is consistent, by 19. ∎

Thus, the theorem follows. ∎

16. Consistency which is Σ_1

Here we illustrate our general considerations on the predicate $Cons^{J_T}(T+\Gamma+\cdot)$ by considering the case where T is $S+B\Sigma_1$, where S is a Π_2 axiomatizable fragment of arithmetic including $I\Delta_0 + exp + \zeta$, where ζ is a Σ_1 sentence false in $\mathbb{N}$, Γ is the class of Π_1 sentences and J_T is Π_1 definable, e.g. $J_T = \mathbb{N}_{T,\Sigma_1}$. We have the following properties:

- $T \vdash \phi$ implies $T \vdash Pr^{J_T}_{T,\Pi_1}(\phi)$
- $Cons^{J_T}(T+\Pi_1+\cdot)$ is Σ_1

Π_1 **completeness**

- $T \vdash (\eta \Rightarrow Pr^{J_T}_{T+\Pi_1}(\eta))$ for $\eta \in \Pi_1$

Gödel:

- $T \nvdash Cons^{J_T}(T+\Pi_1)$
- $T \nvdash \neg Cons^{J_T}(T+\Pi_1)$
- If $\theta \Leftrightarrow Cons^{J_T}(T+\Pi_1+\neg\theta)$ provably in T, then $\theta \Leftrightarrow Cons^{J_T}(T+\Pi_1)$ provably in T

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