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TEMPORAL-EPISTEMIC LOGIC

Abstract: The paper aims at providing temporal epistemic logic *TEL* with sound and complete axiomatization. This logic combines temporal and epistemic operators. Time is represented as isomorphic to the set of natural numbers, whereas knowledge is modeled as an *S5*-like modality.

Keywords: Temporal logic, Epistemic logic, Modal logic

Time and knowledge are strongly related. Temporal reasoning and knowledge representation is getting more attention in recent years. To formalize considerations on knowledge changing in time we use some combined logic. The natural way to construct this kind of logic is to insert epistemic logic in a temporal framework. We add two sets of specific operators to an existing propositional system. One set of specific operators is the set of temporal operators, which we use to describe temporal interactions. Because of the basic language in which the agent can describe his knowledge is propositional epistemic logic, then the second set of specific operators is the set of epistemic operators. We construct a multi-modal system combining tense and knowledge operators. Reasoning can be seen as an activity of an agent taking place in time. It is a stepwise process. The agent starts with a set of initial facts to which it applies some rules to arrive at a new state, in which he has more knowledge. In this new state, the agent may apply rules to arrive at a next state. We assume that semantic time is linear, discrete and has a starting point (time is isomorphic to the set of natural numbers). Moreover, we assume that the agent may perform positive and negative introspection, then knowledge is modeled as an *S5*-like modality. We construct sound and complete system of temporal epistemic logic with *Since* and *Until* operators. To prove the completeness of our system we use the method described in [6].

Temporal-epistemic logic *TEL*

The knowledge of our agent can be modeled by modalities of *S5* epistemic logic, then our epistemic language is the language of *S5* logic.

Definition 1 (*Epistemic language*)

Let *Prop* be a countably set of propositional atoms. The language \mathcal{L}_{S5} is the smallest set closed under:

1. if $p \in Prop$, then $p \in \mathcal{L}_{S5}$,
2. if $\varphi, \psi \in \mathcal{L}_{S5}$, then $\neg\varphi, K\varphi, (\varphi \wedge \psi) \in \mathcal{L}_{S5}$.

K is knowledge operator and a formula $K\varphi$ is interpreted as follows: *The agent knows that φ* . We introduce the following abbreviations:

- $\varphi \vee \psi \equiv \neg(\neg\varphi \wedge \neg\psi)$,
- $\varphi \rightarrow \psi \equiv \neg\varphi \vee \psi$,
- $M\varphi \equiv \neg K\neg\varphi$,
- $\top \equiv p \vee \neg p$,
- $\perp \equiv \neg\top$,

where: \vee and \rightarrow are classical disjunction and implication, respectively. M is a possibility operator. A formula $M\varphi$ is interpreted as follows: *φ is possible with respect to a knowledge of an agent*. \top and \perp are abbreviations for *constant true* and *constant false* respectively.

We make our temporal epistemic logic by adding a temporal dimension to *S5* epistemic logic, then a temporal epistemic language we define in the following way:

Definition 2 (*Temporal epistemic language \mathfrak{L}_{TEL}*)

The temporal epistemic language \mathfrak{L}_{TEL} is the smallest set closed under:

- If $\varphi \in \mathcal{L}_{S5}$, then $\varphi \in \mathfrak{L}_{TEL}$
- If $\varphi, \psi \in \mathfrak{L}_{TEL}$, then $\neg\varphi, (\varphi \wedge \psi), U(\varphi, \psi), S(\varphi, \psi) \in \mathfrak{L}_{TEL}$

U and S are temporal operators. A formula $U(\varphi, \psi)$ we read: *ψ holds until φ does*, and a formula $S(\varphi, \psi)$ we read: *ψ holds since φ does*. Again the abbreviations for \vee, \rightarrow, \top and \perp , we introduce, as well as:

- $F\varphi \equiv U(\varphi, \top)$,
- $P\varphi \equiv S(\varphi, \top)$,
- $G\varphi \equiv \neg(U(\neg\varphi, \top))$,
- $H\varphi \equiv \neg(S(\neg\varphi, \top))$,
- $\bigcirc\varphi \equiv U(\varphi, \perp)$,
- $\Box\varphi \equiv H\varphi \wedge \varphi \wedge G\varphi$.

The interpretation of introduced temporal operators is as follows:

- $F\varphi$ – sometimes in the future φ ,
- $P\varphi$ – sometimes in the past φ ,
- $G\varphi$ – always in the future φ ,
- $H\varphi$ – always in the past φ ,
- $\bigcirc\varphi$ – at the next moment of time φ ,
- $\square\varphi$ – always φ .

Semantics of *TEL*

Definition 3 (*TEL-model*)

$$TEL\text{-model } \mathfrak{M} = \langle W, \mathbb{N}, \{A_t : t \in \mathbb{N}\}, R, V \rangle$$

where:

- W is nonempty set,
- \mathbb{N} is the set of natural numbers,
- $\{A_t : t \in \mathbb{N}\}$ is a collection of accessibility relations,
- $R(\subseteq \mathbb{N} \times \mathbb{N})$ is “earlier-later” relation,
- $V : Prop \rightarrow 2^{\mathbb{N} \times W}$.

W is the set of worlds. We assume, that semantic time in our system is isomorphic to the set of natural numbers, then \mathbb{N} is the set of moments of time. Each moment of time is assigned to a model of epistemic *S5* logic, hence to each moment of time we assign to accessibility relation (one relation for each moment of time). Thus for each $t \in \mathbb{N}$, $A_t \subseteq W \times W$.

V is function mapping to each propositional letter p subset $V(p)$ of cartesian product $\mathbb{N} \times W$. $V(p)$ is the set consist of pairs (t, w) such that p is true in the world w at the moment t .

Definition 4 (*The satisfiability of a formula*)

The satisfiability of a formula $\varphi \in \mathcal{L}_{TEL}$ in a model \mathfrak{M} , at a moment of time $t \in \mathbb{N}$, in a world $w \in W$, denoted by $\mathfrak{M} \models \varphi[t, w]$, is defined inductively as follows:

1. $\mathfrak{M} \models \varphi[t, w] \quad \equiv (t, w) \in V(\varphi)$, if $\varphi \in Prop$,
2. $\mathfrak{M} \models \neg\varphi[t, w] \quad \equiv$ it is not the case that $\mathfrak{M} \models \varphi[t, w]$,
3. $\mathfrak{M} \models (\varphi \wedge \psi)[t, w] \quad \equiv \mathfrak{M} \models \varphi[t, w]$ and $\mathfrak{M} \models \psi[t, w]$,
4. $\mathfrak{M} \models U(\varphi, \psi)[t, w] \quad \equiv \exists t' tRt'$ such that $\mathfrak{M} \models \varphi[t', w]$ and
 $\forall u \in T$ such that $(tRu$ and $uRt')$
holds $\mathfrak{M} \models \psi[u, w]$,

5. $\mathfrak{M} \models S(\varphi, \psi)[t, w] \equiv \exists t' t'Rt$ such that $\mathfrak{M} \models \varphi[t', w]$ and
 $\forall u \in T$ such that $(t'Ru$ and $uRt)$ holds
 $\mathfrak{M} \models \psi[u, w]$,
6. $\mathfrak{M} \models K\varphi[t, w] \equiv \forall w' \in W$ such that $wA_t w'$ holds $\mathfrak{M} \models \varphi[t, w']$.

Definition 5 (*The truth of formula in a model*)

Let $\mathfrak{M} = \langle \mathbb{N}, W, \{A_t : t \in T\}, R, V \rangle$.

If for all $t \in \mathbb{N}$ and for all $w \in W$, $\mathfrak{M} \models \varphi[t, w]$, then the formula φ is true in a model \mathfrak{M} , and we write $\mathfrak{M} \models \varphi$.

Definition 6 (*The truth of a formula*)

If φ is true in all models, then φ is true and we write $\models \varphi$.

Axiomatization of *TEL*

TEL system is axiomatizable. To create axiom system for *TEL* we combine axioms for modal logic *S5* with axioms for temporal logic of time isomorphic to the set of natural numbers. Axioms for the logic *S5* is well known. Axioms for temporal logic for natural numbers were given by Burgess in the paper [2].

Definition 7 (*Axioms for TEL*)

Axioms system for temporal-epistemic logic *TEL* is as follows:

1. All instances of propositional tautologies
2. $G(\varphi \rightarrow \psi) \rightarrow (U(\varphi, \xi) \rightarrow U(\psi, \xi))$,
3. $G(\varphi \rightarrow \psi) \rightarrow (U(\xi, \varphi) \rightarrow U(\xi, \psi))$,
4. $(\varphi \wedge U(\psi, \xi)) \rightarrow U(\psi \wedge S(\varphi, \xi), \xi)$,
5. $(U(\varphi, \psi) \wedge \neg U(\varphi, \xi)) \rightarrow U(\psi \wedge \neg \xi, \psi)$,
6. $U(\varphi, \psi) \rightarrow U(\varphi, \psi \wedge U(\varphi, \psi))$,
7. $U(\psi \wedge U(\varphi, \psi), \varphi) \rightarrow U(\varphi, \psi)$,
8. $(U(\varphi, \psi) \wedge U(\xi, \phi)) \rightarrow ((U(\varphi \wedge \xi, \psi \wedge \phi) \vee U(\varphi \wedge \phi, \psi \wedge \phi) \vee U(\psi \wedge \xi, \psi \wedge \phi))$,
9. the mirror images of axioms 2-8¹,
10. $U(\top, \perp) \wedge S(\top, \perp)$,
11. $H\perp \vee PH\perp$,

¹ The *mirror image* of φ is obtained by simultaneously replacing S by U and U by S , everywhere in φ .

12. $F\top$,
13. $F\varphi \rightarrow U(\varphi, \neg\varphi)$.

Rules:

1. $\frac{\varphi, \varphi \rightarrow \psi}{\varphi}$ (*Modus Ponens*)
2. $\frac{\varphi}{G\varphi} \quad \frac{\varphi}{H\varphi}$ (*Temporal generalization*)
3. $\frac{\vdash_{S5} \varphi}{\vdash_{TEL} \varphi}$, for every formula $\varphi \in \mathfrak{L}_{S5}$ (*Preserve*)

Let us remark, that the axioms of (1) and (2) in some sense correspond to axioms for modal logic K . Axiom (3) relates the mirror image connectives. Axioms (6) and (7) express transitivity. Axiom (8) express linearity. Axioms (11), (12) and (13) relates respectively: having a first point, right-seriality, Prior-style Dedekind completeness.

We construct our temporal-epistemic logic according to the method of addition of a temporal dimension to the logic system described in the paper [6]. In our case we add a temporal dimension to the modal epistemic logic $S5$. We temporalize $S5$ epistemic logic.

TEL is sound and complete. The soundness of TEL we obtain by theorem 2.2 of [6] using soundness of $S5$ and soundness of the axiom system for temporal logic of time isomorphic to the set of natural numbers.

Theorem 1 (*Soundness TEL*)

The axiom system of TEL is sound.

The logic $S5$ is complete. The Burgess's system given in [2] is complete over a class of time isomorphic to the set of natural numbers. Then by theorem 2.3 of [6] we obtain completeness of TEL . Therefore, we have the following theorem:

Theorem 2 (*Completeness TEL*)

The axiom system of TEL is complete.

And finally, by theorem 3.1 of [6], decidability of $S5$ logic [8] and decidability of the temporal logic of time isomorphic to the set of natural numbers [9] we have:

Theorem 3 (*Decidability of TEL*)

The TEL logic is decidable.

R E F E R E N C E S

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