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A Formal Proof of Euler's Polyhedron Formula

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Abstract. Euler's polyhedron formula asserts for a polyhedron p that

$$V - E + F = 2,$$

where V, E, and F are, respectively, the numbers of vertices, edges, and faces of p. This paper concerns a formal proof in the MIZAR system of Euler's polyhedron formula carried out [1] by the author. We discuss the informal proof (Poincaré's) on which the formal proof is based, the formalism in which the proof was carried out, notable features of the formalization, and related projects.

1 Euler's Polyhedron Formula

Euler first discussed his formula in a 1750 letter to Christian Goldbach:

> *Recently it occurred to me to determine the general properties of solids bounded by plane faces, because there is no doubt that general theorems should be found for them, just as for plane rectilinear figures, whose properties are: (1) that in every plane figure the number of sides is equal to the number of angles, and (2) that the sum of all the angles is equal to twice as many right angles as there are sides, less four. Whereas for plane figures only sides and angles need to be considered, for the case of solids more parts must be taken into account. [16]*

Euler does not use the term *polyhedra* but rather "solids bounded by plane faces". He goes on to enumerate some interesting propositions about polyhedra such as:

> *6. In every solid enclosed by plane faces the aggregate of the number of faces and the number of solid angles exceeds by two the number of edges, or $F + V = E + 2$.*[1]

and

[1] Euler's text has been modified to bring it into line with the notation used in this paper: he did not use the conventional English abbreviations "V", "E", and "F".

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11. The sum of all plane angles is equal to four times as many right angles as there are solid angles, less eight, that is $4V - 8$ right angles.[2]

Euler expresses surprise that he has not been able to find a precedent for these relations:

I find it surprising that these general results in solid geometry have not been previously noted by anyone, so far as I am aware;[3] *and furthermore, that the important ones, Theorems 6 and 11, are so difficult that I have not yet been able to prove them in a satisfactory way.*

It was not long before Euler presented his results publicly [8]. Like the letter to Goldbach, Euler's paper was programmatic: he was trying to encourage the study of three-dimensional solids as an extension of planar geometry. The "most difficult" propositions he mentioned to Goldbach were discussed in detail, though he acknowledges that his presentation does not constitute a proof. Indeed, in the preface to his paper Euler qualifies his work thus:

I for one have to admit that I have not yet been able to devise a strict proof of this theorem. As however the truth of it has been established in so many cases, there can be no doubt that it holds good for any solid. Thus the proposition seems to be satisfactorily demonstrated.

Euler was not satisfied with the unfinished state of his theorem and continued working with polyhedra. Eventually he did find a satisfactory proof [7].

Perhaps because of its simplicity and elegance, many other mathematicians studied the polyhedron formula and tried to give new proofs. Cauchy, for example, connected the study of polyhedra to planar graphs: project a polyhedron onto a plane, triangulate it, and take away one triangle at a time in a way that preserves χ until only a triangle remains; we obtain the desired result $\chi = 2$ by noting that the projection with which we started "removes" a face from the polyhedron (which effectively sends one of the polyhedron's faces onto an unbounded planar region). Unlike Euler, whose conception of polyhedra was that of solid (which one can slice, as with a knife), Cauchy apparently viewed polyhedra as wireframes.

Poincaré provided a new conception of polyhedra based on incidence matrices with which he gave his own proof [22, 21] of Euler's formula.[4] Poincaré's abstract, combinatorial conception of polyhedra makes no mention of points in $\mathbf{R}^3$, nor does it come from projecting polyhedra onto a plane. Poincaré's approach even allows

[2] Euler proved that proposition 6 is equivalent to proposition 11. This is an interesting equivalence because one statement has a combinatorial flavor, while the other has an analytic flavor. Proposition 11 can be seen in the famous Gauss-Bonnet formula [27].

[3] Unknown to Euler, Descartes had actually given a proof of Proposition 11 [15]. This result of Descartes's, seems to have been missing at Euler's time; it was rediscovered in the 19th century, long after Euler's death [24].

[4] Poincaré was interested more broadly in the new subject of topology, of which he was one of the earliest explorers; his new proof of Euler's polyhedron formula was but one element in his wider topological program.

for polyhedra of arbitrary dimension; the general result[5] states that

$$\sum_{k=0}^{d-1}(-1)^k N_k = 1 + (-1)^{d+1},$$

where the integer d is the dimension of p and N_k is the number of k-polytopes of p. The classical three-dimensional version stated by Euler is obtained by setting $d := 3$. The familiar property of a polygon that the number of vertices is equal to the number of edges is obtained by putting $d := 2$. (And a 1-dimensional polyhedron is just a line segment with its two endpoints, which also falls out of the general Euler relation by putting $d := 1$.)

So far no definition of polyhedron has been given, nor have we placed any restriction on the domain of validity of Euler's relation. It is a commonplace that one has to be careful with how one defines one's terms, and the term "polyhedron" is no exception. Grünbaum writes:

> *The "Original Sin" in the theory of polyhedra goes back to Euclid, and through Kepler, Poinsot, Cauchy, and many others, in that at each stage, the writers failed to define what are the 'polyhedra'.* [13]

In addition to defining polyhedra, it is a further task to specify the domain of validity for Euler's relation to hold; it turns out that around the time of Cauchy's proof in the early 19th century, it started to become clear to mathematicians that Euler's polyhedron formula does not hold for all polyhedra. In 1811, for example, L'Hulier described "exceptions" to Euler's polyhedron formula, classifying them into three kinds. Research on polyhedra in the 19th century gradually revealed that for Euler's relation to hold one should focus on *simple connectedness*, which roughly asserts that any two vertices can be connected by a path of edges and that the faces can be continuously collapsed to a point.

(Lakatos's history [18] of Euler's polyhedron formula is an entertaining discussion of some of the historical twists and philosophical problems surrounding the result.[6])

Poincaré's definition, on which the formalization to be described is based, is probably the simplest to describe. Following Poincaré, a polyhedron is characterized by a list of *incidence matrices*, which can be understood as functions f from a cartesian product $A \times B$ of sets A and B to $\{0, 1\}$, where $f(a, b) = 1$ is understood as "a is incident with b" and $f(a, b) = 0$ is understood as "a is not incident with b". Thus to specify a polyhedron of dimension $d + 1$, one just gives d incidence matrices. Let us call such a structure an *abstract* or *combinatorial polyhedron.*

5 Poincaré was not the first to generalize Euler's polyhedron formula to higher dimensions; that was done by L'Hullier.

6 Indeed, a motivation for carrying out the formalization described here was to study Lakatos's philosophy of mathematics.

2 Poincaré's Proof of Euler's Polyhedron Formula

As part of his algebraic topological program, Poincaré gave a new proof of Euler's polyhedron formula. In this section we give a sketch of Poincaré's proof; for a more detailed discussion, consult Lakatos [18] (chapter 2) or Coxeter [5] (chapter 9).

First, we should say how Poincaré defines polyhedra. In his framework, a three-dimensional polyhedron is determined by five pieces of data:

- A set of vertices (the 0-polytopes),
- A set of edges (the 1-polytopes),
- A set of faces (the 2-polytopes),
- An incidence matrix that says which vertices belong to which edges, and
- An incidence matrix that says which edges belong to which faces.[7]

Conventionally there is also a 3-polytope, namely the whole polyhedron p, and we specify a new incidence matrix declaring that all faces are incident with p. Symmetrically, we conventionally define a single -1-polytope and declare that it is incident with each vertex.

More generally, a d-dimensional polyhedron is characterized by a pair $(\mathcal{F}, \mathcal{I})$ ($\mathcal{F}$ for "faces", $\mathcal{I}$ for "incidences") of finite sequences, where

- $d = \operatorname{len} \mathcal{F}$,
- $\operatorname{len} \mathcal{F} > 0$,
- $\operatorname{len} \mathcal{I} = \operatorname{len} \mathcal{F} - 1$,
- For $0 \leq n < \operatorname{len} \mathcal{F}$, we have that $\mathcal{F}_n$ is a non-empty finite set (the set of k-polytopes of p), and
- For $0 \leq n < \operatorname{len} \mathcal{I}$, we have that $\mathcal{I}_n$ is an incidence matrix for $\mathcal{F}_n$ and $\mathcal{F}_{n+1}$.

In the more general setting we again stipulate that there is one d-dimensional polytope, namely p, that is incident with all $(d-1)$-polytopes; also, we stipulate that there is one -1-dimensional polytope that is incident with all 0-polytopes.

Theorem 1. *For every simply connected polyhedron p, we have*

$$\sum_{k=0}^{d-1} N_k = 1 + (-1)^{d+1},$$

where d is the dimension of p and N_k is the number of polytopes of p of dimension k.

For a polyhedron p and an integer k, let the k-chains of p be the powerset of the set of k-polytopes of p. The k-chains of p naturally form a vector space over the two-element field F_2, where vector addition is represented by disjoint union (symmetric difference); call this space C_k. The relation between C_k and polyhedra can be seen in the fact that the dimension of C_k is precisely N_k, the number of k-polytopes of

[7] In fact, Poincaré used a single incidence matrix to represent a polyhedron. The matrix is a block matrix, two of whose blocks are just the zero matrix, expressing the fact that vertices are not (strictly speaking) incident with faces but only with edges.

p. (Reason: the singleton subsets of $\mathcal{F}_k$ are a basis for C_k.) The boundary $\partial_k c$ of a k-chain c is the $(k-1)$-chain

$$\{x \in \mathcal{F}_{k-1} : x \text{ is incident with an odd number of } k\text{-polytopes of } c\}.$$

In other words, a $(k-1)$-polytope x belongs to the boundary of a k-chain c iff

$$\sum_{y \in c} \mathcal{I}_{k-1}(x, y) = 1,$$

where the sum is taken modulo 2. The boundary operation ∂_k is a linear transformation from C_k to C_{k-1}. It turns out that the k-chains c whose boundary is empty (all $(k-1)$-polytopes are incident with c an even number of times) form a subspace, Z_k, of C_k. Such k-chains are called k-circuits (sometimes also called k-cycles). Another important subspace of the k-chain space C_k consists of those k-chains that are the boundary of a $(k+1)$-chain; for lack of a better name, let B_k (for "bounding") denote this subspace.

The property of *simple connectedness* is the property that $B_k = Z_k$, that the k-circuits are the bounding k-chains. The inclusion $B_k \subseteq Z_k$ says that $\partial_{k+1}\partial_k \equiv 0$. The reverse inclusion intuitively says that the only way something can be a cycle is if it "traverses" a "face". This fails in cases where, for example, a face has a hole in it (one can go around the boundary of the inner hole, but there's no face that one is traversing).

Proof of Theorem 1. If p is simply connected, then

$$Z_k = B_k,$$

so that

$$\dim Z_k = \dim B_k.$$

Since $N_k = \dim C_k$, we have by the rank+nullity theorem that

$$N_k = \dim C_k = \dim B_{k-1} + \dim Z_k = \dim B_{k-1} + \dim B_k.$$

Thus

$$\sum_{k=0}^{d-1} (-1)^k N_k = \sum_{k=0}^{d-1} (-1)^k (\dim B_{k-1} + \dim B_k) = \dim B_{-1} + (-1)^{d-1} \dim B_{d-1}.$$

The last equation follows because of the hypothesis of simple connectedness. Now $\dim B_{-1} = 1$, since B_{-1} is a two-element vector space (it contains the empty chain as well as the singleton chain containing the unique -1-polytope). And $\dim B_{d-1} = 1$ for the same reason: it contains the empty chain as well as the "full" chain containing all the $(d-1)$-polytopes, so that it has at least two elements; if c is a $(d-1)$-chain different from the "full" $(d-1)$-chain and the empty chain, then it is not in the range of ∂_d, since by stipulation *all* $(d-1)$-polytopes are incident to the unique d-polytope p. The proof is complete.

3 Overview of the Formalization

In this section we describe a formalization of Poincaré's proof of Euler's polyhedron formula that was carried out in the MIZAR system. MIZAR is based on classical first-order logic with equality and Tarski-Grothendieck set theory, a strong theory of sets that is equivalent to the Zermelo-Fraenkel theory together with an axiom asserting the existence of an inaccessible cardinal.

Among the many candidate systems (*e.g.*, ISABELLE, HOL LIGHT, COQ) with which the formalization could have been carried out, MIZAR was selected because of its familiar logical foundations (first-order set theory), its everyday knowledge representation language (dependent types, structures, flexible notation for functions and predicates), its standard proof language (a kind of natural deduction), and its large library of formalized mathematical knowledge on which one can build.[8] But it must be admitted that the choice of MIZAR over the other candidates was somewhat arbitrary. Nonetheless, it seems plausible that, if one were to compare the formalization in MIZAR under discussion with a formalization of the same proof in some other system, one would find considerable overlap.[9]

3.1 Main Formalizations

One often finds when formalizing that, in addition to the logical and mathematical details in a formal proof that must be supplied, one must also formalize various kinds of "background" knowledge. And one often finds that the simplest mathematical facts are (apparently) missing from the library of formalized mathematics[10]. Like Euler writing to Goldbach, we can be surprised that "these general results have not been previously noted by anyone".[11] The formalization of Poincaré's proof of Euler's polyhedron formula in MIZAR was no exception to this phenomenon. But this is understandable; just as libraries of implemented algorithms for various programming languages do not eliminate the need for programmers to adjust them to their specific problems, so too do general mathematical facts in a formal library require further specification before they can be applied.

The contribution naturally divided into three MIZAR "articles" (collections of definitions, theorems). They were:

[8] At the time the formalization began, no formal proof of Euler's formula was known. But independently, another formal proof has been carried out in the COQ system[6].

[9] It would be interesting to discover cases where one *learns* something different about a proof (and not about the different systems or the different logics on which they are built) when formalizing it in one system as compared with what one learns from another formalization of the same proof.

[10] There are two kinds of missing knowledge: well-known (perhaps named) mathematical results can be contrasted with details that, in an less formal context, are left tacit.

[11] And, conversely, often one discovers that mathematical knowledge that we previously thought to be unformalized does in fact exist in the library. At one point the author thought that he had a *proof* that the MIZAR library did not contain a formalization of the fact that $\{0, 1\}$ can be made into a two-element field. This turned out to be mistaken.

– `RANKNULL`: The rank+nullity theorem;
– `BSPACE`: The vector space of subsets of a set based on disjoint union; and
– `POLYFORM`: Euler's polyhedron formula.

We now briefly discuss some notable features of these formalizations.

The rank+nullity theorem The rank+nullity theorem states that if T is a linear transformation from a finite-dimensional vector space V to a finite-dimensional space W, then

$$\dim V = \dim \operatorname{im} T + \dim \ker T.$$

We were able to straightforwardly formalize a standard proof [19] of the result, but some formal groundwork had to be laid for that to be possible.

Much basic linear algebra has already been formalized in MIZAR; there are a number of theorems and definitions concerning subspaces [30], linear combinations [29], dimensions of vector spaces [33] and linear spans of sets of vectors [28]. But some of the linear algebraic facts involved in a proof of the rank+nullity theorem were unavailable and had to be formalized. To carry out the formalization, we defined:

1. the image and kernel of a linear transformation, and the fact that these form subspaces of the domain and range of a linear transformation;
2. the restriction of a linear combination to a set of vectors; and
3. the image and inverse image of a linear combination under a linear transformation.

The first item is straightforward, but the second and third items may require some explanation. In MIZAR, a linear combination is represented as a function from a vector space to the field of scalars whose carrier (the set of vectors not mapped to zero) is finite.[12] The restriction of a linear combination l on a vector space V to a subset X of V is thus naturally represented by the function

$$\lambda v \in V. \begin{cases} l(v) & \text{if } v \in X \\ 0_V & \text{otherwise} \end{cases}.$$

Suppose that T is a linear transformation from a vector space V to a vector space W, both over a field F, and that l is a linear combination of vectors in V. Thus l represents the linear combination

$$a_1 v_1 + \cdots + a_n v_n,$$

where n is a natural number, $a_k \in F$ and $v_k \in V$ and $a_k \neq 0_F$ $(1 \leq k \leq n)$. Since T is a linear transformation, we ought to have

$$T(a_1 v_1 + \cdots + a_n v_n) = a_1 T(v_1) + \cdots + a_n T(v_n).$$

[12] This is a case where a representation of a mathematical object contains more information than meets the eye. When represented this way, linear combinations tacitly build in the commutativity of vector addition. $u + v$ is represented by a function f that sends u and v to 1 and every other vector to 0. The same function f also represents $v + u$.

Thus, it is natural to define the image of l under T to be the MIZAR-linear combination

$$\lambda w \in W. \begin{cases} l(T^{-1}(\{w\})) & \text{if } w \in \operatorname{im} T \\ 0_F & \text{otherwise} \end{cases}.$$

The problem with this definition is that it works only if T is injective. We are supposed to define the image of any linear transformation T on any linear combination l, so we need to allow for the possibility that some of the $T(v_i)$'s are equal. A definition that gets around this problem is

$$T(l) := \lambda w \in W. \sum l(T^{-1}(w)).$$

This definition allows us to add together the coefficients, given by l, of those vectors in V that are identified by T. It is interesting to note how the formal definition of the image of a linear combination under a linear transformation differs from the informal (or semi-formal) notation above. This case provides an interesting example of a formal analysis of informal notation.

The inverse image operation also deserves to be mentioned. Suppose that X is a subset of a vector space V, that T is a linear transformation from V to W, and that l is a linear combination of $T(X)$ (that is, that l is a function from W to F with finite support whose value is 0_F outside of $T(X)$). This is a precise way of saying that l looks like

$$b_1 T(v_1) + \cdots + b_n T(v_n),$$

for some natural number n and $v_k \in X$. We want to say that the inverse image of l is the linear combination

$$b_1 v_1 + \cdots + b_n v_n.$$

This is correct, but only on the assumption that the vectors $T(v_1), \ldots, T(v_n)$ are distinct. One way to ensure this is by requiring that $T|X$ is one-to-one, and that is in fact what we did when defining the inverse image operation in MIZAR and suited the formalization task at hand. As it stands, the inverse image operation in MIZAR is a partial operation. The restriction of injectivity of the restriction is, however, unnecessary and it would be valuable to extend the formalization to account for the general case.

The vector space of subsets of a set based on disjoint union Another result needed for a formalization of Poincaré's proof of Euler's polyhedron formula is the fact that the power set of a set forms a vector space over the two-element field F_2. Vector addition is disjoint union (symmetric difference), and scalar multiplication is defined by

$$0 \cdot x := \emptyset, 1 \cdot x := x.$$

This fact seems to be standard, but we were unable to find any conventional name for this space. For lack of a better notation, let $B(X)$ (for "Boole") be the vector space of subsets of X based on disjoint union.

Approximately half of the article `BSPACE` is devoted to proving that $B(X)$ is indeed a vector space. The other half is devoted to some facts about the linear algebraic features of the singleton subsets of X, namely that

– they are a linearly independent set of vectors, and
– if X is finite, then they span $B(X)$.[13]

Polyhedra Perhaps surprisingly, the formalization of Poincaré's proof was rather straightforward. The highlight of the article is the generalized relation, as well as special cases for one-, two-, and three-dimensional polyhedra. The statement of the main theorem, in the MIZAR syntax, is

```
p is simply-connected implies p is eulerian;
```

where of course `p` has type `polyhedron`. The term "Eulerian" is a neologism that means that a polyhedron satisfies Euler's relation; it appears in Lakatos [18]. The definitions of the two properties are

```
p is simply-connected
 means
for k being Integer
 holds k-circuits(p) = k-bounding-chains(p);
```

and

```
p is eulerian
 means
Sum (alternating-proper-f-vector(p))
  = 1 + (-1)|^(dim(p)+1);
```

(The f-vector of a polyhedron p is the sequence

$$s := N_{-1}, N_0, N_1, \ldots, N_d,$$

where $d = \dim p$ and N_k is the number of polytopes of dimension k. It could also be reasonably defined as a bi-infinite sequence indexed by the integers containing the terms displayed above with all other terms being 0. The terminology is standard [4], but to ease the formalization two related neologisms were coined: *proper f-vector* and *alternating proper f-vector*. By definition deleting the first and last terms of s gives the proper f-vector of p; alternating the signs of the sequence yields the alternating proper f-vector of p.) We also proved a lemma on telescoping sums that apparently did not exist in the MIZAR library:

```
for a,b,s being FinSequence of INT
 st len s > 0 &
    len a = len s & len b = len s &
    (for n being Nat st 1 <= n & n <= len s
      holds s.n = a.n + b.n) &
    (for k being Nat st 1 <= k & k < len s
      holds b.k = -(a.(k+1)))
 holds Sum s = (a.1) + (b.(len s))
```

[13] The condition of finiteness is necessary because linear combinations must be finite; if X is infinite no finite linear combination of singletons can equal X.

The lemma is a formalization of the claim that if s, a, and b are are sequences of integers, all of the same length n, and if $s = a + b$ but $b_k = -a_{k+1}$, then $\sum s = a_1 + b_n$. In Poincaré's proof, thanks to the assumption of simple connectedness, the sum on the left-hand side of the Euler relation turns out to be telescoping in this way.

4 Discussion

4.1 Filled gaps

One of the aims of formal mathematics is to give gap-free proofs of mathematical theorems. One could take a skeptical view and doubt the validity of virtually every proof in mathematics; for the skeptic, all proofs are informal and are (potentially) rife with logical gaps. There is a kernel of truth in the skeptical view, but the paucity of *interesting* gaps—oversights, ambiguities, or errors that, once exposed, would alter the views of the working mathematician—makes the view less plausible [9, 25]. One might say that a formalized proof of a theorem gives us better grounds to believe the theorem than were available before the proof was formalized, but at present it seems to be an open philosophical challenge to say why this should be so, while acknowledging the rarity of interesting gaps.

Were any interesting gaps uncovered in the formalization of Poincaré's proof of Euler's polyhedron formula? If a gap can be both interesting and small, then the answer might be "yes", but is more likely "no". In Coxeter's *Regular Polytopes*, we apparently see a proof that "the boundary of any $(k+1)$-chain is a k-circuit" [5]. But this simply cannot be proved because there are counterexamples.

But it is not clear whether Coxeter is making an invalid inference here. An alternative explanation is that, rather than *proving* that $\partial\partial \equiv 0$ for all polyhedra, Coxeter was instead *motivating* the assumption of this property. Lakatos seems to have observed as much; in his discussion of Poincaré's proof, we find this exchange:

> GAMMA: *I think that the boundary of a decent k-chain should be closed. For instance I could not possibly accept as a polyhedron a cube with the top missing; and I could not possibly accept as a polygon a square with an edge missing. Can you prove, that the boundary of any k-chain is closed?*
> EPSILON: *Can I prove that the boundary of the boundary of any k-chain is zero?*
> GAMMA: *That is it.*
> EPSILON: *No, I cannot. This is indubitably true. It is an axiom. There is no need to prove it.*

Lakatos is right that this principle (that $B_k \subseteq Z_k$) must be an "axiom" in some form. In the formalization under discussion, it is contained in the definition of simple connectedness.

4.2 A proof-theoretic corollary

The result of the formalization is that Euler's polyhedron formula (understood à la Poincaré) is a first-order logical consequence of the axioms of Tarski-Grothendieck set theory (TG). But it should be clear that the full strength of TG set is not *required* for Poincaré's proof; it would be quite surprising if Poincaré's proof of Euler's polyhedron formula required the existence of infinitely many inaccessible cardinals. After all, following Poincaré, polyhedra are conceived as certain combinatorial structures that, presumably, could be completely captured in an arithmetical theory. And thanks to the fact that our work on a formal version of Euler's polyhedron formula is quite detailed, one has a clear basis with which to start proving Euler's polyhedron formula in a weaker theory than TG.

The characteristic axiom of TG asserts: for every set N there exists a set M such that

- $N \in M$,
- M is closed under taking subsets,
- M is closed under the powerset operation, and
- if $X \subseteq M$ and $X \not\sim M$, then $X \in M$.

Such a set M might be called a universe containing N; accordingly, let us call this principle the *universe axiom*. Some important consequences of the universe axiom (none of which are axioms of TG) are:

- The existence of an infinite set,
- The axiom of choice, and
- Powerset.

When one inspects the deduction underlying the MIZAR proof of Euler's polyhedron formula, one can trace the argument through each of the three principles mentioned above. Since each of these three principles are consequences of the universe axiom (together, of course, with other axioms of TG), we see that the MIZAR proof of Euler's polyhedron formula uses the universe axiom. But in MIZAR this is to be expected. Indeed, the proof of *every* theorem in the MIZAR mathematical library that involves natural numbers uses the universe axiom by way of the existence of an infinite set (obtained by applying the universe axiom to $\emptyset$).

It may be somewhat surprising that the axiom of choice appears in the proof of Euler's polyhedron formula. To be clear, what is claimed is not that Euler's polyhedron formula ineliminably *depends* on the axiom of choice in the way that, say, the well-ordering principle does. Instead, what is claimed is that there is a deduction of Euler's polyhedron formula that *uses* choice. The use occurs in the proof of the rank+nullity theorem theorem. The proof proceeds by starting with a linear transformation T from a finite-dimensional vector space V to a finite-dimensional vector space W. The first step is to choose a basis A for $\ker T$; one then extends A to a basis B for all of V and, finally, one shows that $T(B - A)$ is a basis of $\operatorname{im} T$. In the actual MIZAR proof of the rank+nullity theorem, the justification for the first step (choosing a basis for $\ker T$) appeals to the theorem [28] that every vector space has a basis.[14]

[14] In the MML version 4.110.1033, released September 9, 2008, the exact MIZAR item is `VECTSP_7:def 3`. Every type in MIZAR must be provably non-empty. Interestingly, the theorem that every vector space has a basis appears not as a MIZAR theorem *per se*, but

But clearly the principle that every vector space has a basis (which, perhaps surprisingly, is equivalent over ZF [3] to the axiom of choice) is stronger than what is required for the purpose of proving the rank+nullity theorem, which after all deals with only finite-dimensional vector spaces.[15] And for finite-dimensional vector spaces, it is clear that we can produce a basis through an iterative search procedure whose formalization requires only arithmetical principles.

Some custom software (building on Josef Urban's work [31]) for computing dependency relations in MIZAR texts provides evidence that the *only* way that the universe axiom is used is by way of the three principles mentioned above (infinity, choice, powerset). This in turn is evidence that, from the provability judgment TG $\vdash$ EPF we have the improved judgment ZFC $\vdash$ EPF, where "EPF" is the Poincaré/combinatorial formalization of Euler's polyhedron formula.[16]

Applying "Kreisel's trick" to the Poincaré/combinatorial understanding of Euler's polyhedron formula, from the judgment ZFC $\vdash$ EPF we can drop choice and conclude that ZF $\vdash$ EPF. We have thus moved from the heights of TG to the more modest realm of ZF by studying the MIZAR deduction of Euler's polyhedron formula; we have established a new provability judgment without actually producing a new deduction.

One can continue the process of trying to further weaken the theory with which proof is carried out. It seems plausible that one can get away without having a *set* of natural numbers. That is, it seems plausible that one can eschew the axiom of infinity and deal with the natural numbers not as a set but as a proper class. Accepting that for the moment, we see, using the equivalence of ZF $-$ Infinity and Peano Arithmetic (PA), that Poincaré's proof of Euler's polyhedron formula can be carried out in PA.

Based on some initial studies, it appears that a formalization of Poincaré's proof can be carried out in the theory $\mathrm{I}\Delta_0(\exp)$, a first-order arithmetical theory in a language with addition, multiplication, ordering, and exponentiation with an induction scheme for Δ_0-formulas (which are permitted to contain exponentiation) [14]. It also appears that some kind of exponentiation is required. These are results in progress and have not yet been rigorously proved.

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rather as the justification for the non-emptiness of the type `Basis of V`, where V itself has the dependent type `VectSp of F`, where, finally, F has type `Field`. The proof of the non-emptiness of the `Basis` type appeals to the theorem that every linearly independent subset of a vector space can be extended to a linearly independent spanning set, *i.e.*, a basis.

15 Simpson has shown that the principle "Every vector space has a basis" is equivalent, over the second-order arithmetical theory RCA_0 (for "recursive comprehension axiom"), to the principle of arithmetical comprehension [26].

16 The custom code is not yet complete; certain features of the MIZAR system are not yet accounted for, such as so-called registrations and the implicit uses of Hilbert's ε-operator. Thus it is possible that some important dependency relations are not being taken into account with the present version of the software.

4.3 Streamlining the formalization

At the time of writing, no mechanism for binders (apart from the quantifiers $\forall$ and $\exists$) has been implemented in the MIZAR language. (Wiedijk has a proposal [32] for this as-yet-unimplemented feature.) For example, the definition of the so-called incidence sequence $I_{x,c}$ generated by a $(k-1)$-polytope x and a k-chain c. Using one common notation for sequences [11], $I_{x,c}$ can be defined as

$$\langle v@P_{k,n} \cdot [x \in P_{k,n}] \colon 1 \leq n \leq N_{p,k} \rangle,$$

The bracket notation "$[x \in P_{k,n}]$", from Knuth [17], denotes 1 or 0 according as the relation does or does not hold.[17] The actual MIZAR definition is somewhat more complicated:

```
incidence-sequence(x,v) -> FinSequence of F2
 means
((k-1)-polytopes(p) is empty implies it = <*>{}) &
((k-1)-polytopes(p) is non empty implies
 len it = num-polytopes(p,k) &
 for n being Nat
  st 1 <= n & n <= num-polytopes(p,k)
  holds
  it.n =
   (v@(n-th-polytope(p,k)))*incidence-value(x,n-th-polytope(p,k)));
```

A binder syntax would simplify this definition. It would also help to simplify the examples involving linear combinations that have already been discussed (in light of the fact that in MIZAR linear combinations are represented as functions). Even if these examples are unconvincing, it should be clear that, in general, notations for sequences, functions (λ-abstraction), relations, and other mathematical objects would help to streamline the MIZAR language and make it even more attractive as a formal language for mathematics than it already is.

5 Further Work

Poincaré's abstract, combinatorial conception of polyhedra facilitated formalization because the definition could be easily captured using MIZAR structures. Following Poincaré, the messy details are largely suppressed; one just formalizes the definition of simple connectedness and carries out the linear algebraic proof. Whether one regards this as a problem or a feature of Poincaré's approach is left for the reader to decide. A further challenge for formal mathematics would be to treat Euler's proof of his relation, involving "concrete" or "real" polyhedra. One could start with the relatively easy case of convex polyhedra (with which Euler was arguably working [10], even though his definition apparently permits non-convex polyhedra). It would be especially interesting to take on Euler's argument because of the subtle

[17] Perhaps even this notation could be implemented in MIZAR, but its logical properties are peculiar and would be a challenge to formally specify.

flaws that it was found to contain. The main problem was that Euler did not specify just how to carry out the slicing procedure. One can see, by inspecting simple examples, that one must be careful about the vertex about which the slicing procedure is done, because for some polyhedra and some choices of the vertex, Euler's method can lead to strange results:

> *It is not at all obvious that this slicing procedure can always be carried out, and it may give rise to 'degenerate' polyhedra for which the meaning of the formula is ambiguous.* [2]

Samelson [23] has repaired this gap in Euler's proof. Are there any others?

As mentioned earlier, for the purposes of the formalization is was not necessary to define in full generality the notion of the inverse $T^{-1}(l)$ of a linear combination l under a linear transformation T. It would be valuable for future formalizations in MIZAR of linear algebra to deal with the full generality of inverse images.

The property of a polyhedron satisfying $\partial\partial \equiv 0$ is part of the definition of simple connectedness. This property is equivalent to the inclusion $B_k \subseteq Z_k$, which says that boundaries are circuits. One might regard this not as the *definition* of simple connectedness, but rather as part of the definition of polyhedron; one would then define simple connectedness as the converse inclusion $Z_k \subseteq B_k$ (circuits are boundaries). For future formalizations using combinatorial polyhedra in MIZAR, it may be valuable (if not necessary) to carry out this rearrangement.

A further step would be to give a formal proof of Steinitz's theorem relating convex "analytic" polyhedra (whose points are in $\mathbf{R}^3$) to planar graphs [12, 20, 4].

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