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AN INFINITE SEQUENCE OF PROPOSITIONAL CALCULI

It is described an infinite sequence of propositional calculi where the next calculus has more compound form of deduction theorem.

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1. Introduction

J. Łukasiewicz was the first logician who introduced a 3-valued propositional calculus [1] in order to consider a three values: falsehood, truth, and admissibility. This calculus has a sole binary bundle $\supset$. A. Church proposed to prove [2] that the deduction theorem of Łukasiewicz's calculus said: if $G, A \vdash B$ then $G \vdash A \supset (A \supset B)$. This fact is generalized in this paper.

2. Syntax

It is defined a next infinite sequence of propositional calculi Cn, $n \geq 1$. The propositional calculus Cn has next symbols: $\supset$ (binary bundle), (,) (two round brackets), $f_1, f_2, \ldots, f_n$ (constants) and an infinite list of variables p, q, r, s, p_i, q_i, r_i, s_i ($i \geq 1$).

Formula (in the sense of a well-formed formula) of the propositional calculus Cn is defined inductively.

1. Every constant is a formula.

2. Every variable is a formula.

3. If A and B are formulas then $(A \supset B)$ is a formula too.

Let $A \overset{n}{\supset} B$ be a formula $A \supset (A \supset \ldots (A \supset B) \ldots)$ that contains n occurrences of the formula A and one occurrence of the formula B, $n \geq 1$; for example, $p \overset{3}{\supset} q$ is $p \supset (p \supset (p \supset q))$.

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Now for every n $(n \geq 1)$ we introduce next formulas

$$A \overset{n}{\supset} \left(B \overset{n}{\supset} A\right), \tag{1n}$$

$$\left(A \overset{n}{\supset} \left(B \overset{n}{\supset} C\right)\right) \overset{n}{\supset} \left(\left(A \overset{n}{\supset} B\right) \overset{n}{\supset} \left(A \overset{n}{\supset} C\right)\right), \tag{2n}$$

$$\left(\left(A \overset{n}{\supset} f_n\right) \overset{n}{\supset} f_n\right) \overset{n}{\supset} A, \tag{3n}$$

where A, B and C are arbitrary formulas of the propositional calculus Cn.

Axiom schemes of the propositional calculus Cn $(n \geq 1)$ are the formulas (1n), (2n), (3n).

A formula D is called an axiom according to the given axiom scheme if the D is received from this axiom scheme by substitution for A, B, C concrete formulas; for example, $p \supset (q \supset p)$ is an axiom of the calculus $C1$ according to the axiom scheme (11).

The calculus Cn has one rule of inference: B is derived from $A \overset{n}{\supset} B$ and A (this rule is called modus ponens).

A sequence

$$F_1, F_2, \ldots, F_m \tag{4}$$

of formulas of the calculus Cn is called a proof of this calculus if every F_i is an axiom according to some axiom scheme or it is derived from F_j and F_k by modus ponens where $j < i$ and $k < i$ $(m \geq 1)$. The formula F_m from the sequence (4) is called a theorem of calculus Cn and it is written $\vdash_{Cn} F_m$.

Theorem 1

$\vdash_{Cn} A \overset{n}{\supset} A$ where the A is any formula of the calculus Cn, $n \geq 1$.

Proof. Let A be any formula of the calculus Cn, $n \geq 1$. An adequate sequence consists of 5 formulas F_i.

The F_1 is $A \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)$; it is the axiom scheme (1n).

The F_2 is $A \overset{n}{\supset} \left(\left(A \overset{n}{\supset} A\right) \overset{n}{\supset} A\right)$; it is the axiom scheme (1n).

The F_3 is $\left(A \overset{n}{\supset} \left(\left(A \overset{n}{\supset} A\right) \overset{n}{\supset} A\right)\right) \overset{n}{\supset} \left(\left(A \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)\right) \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)\right)$; it is the axiom scheme (2n).

The F_4 is $\left(A \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)\right) \overset{n}{\supset} \left(A \overset{n}{\supset} A\right)$; it is derived from the F_3 and the F_2 by modus ponens.

The F_5 is $A \overset{n}{\supset} A$; it is derived from the F_4 and the F_1 by modus ponens.

Theorem 1 is proved.

If it is assumed that an occurrence of a formula F_i into the sequence (4) is yet founded by its membership to a formula list G then such sequence is called the definition of hypothesis proof and it is written $G \vdash_{Cn} F_m$.

Theorem 2 (Deduction theorem)

If $G, A \vdash_{Cn} B$ then $G \vdash_{Cn} A \overset{n}{\supset} B$, $n \geq 1$.

Proof. Proof scheme of this theorem coincides with the proof scheme of deduction theorem involved in [2] as the calculus Cn has analogous formulas which are used in Church's proof.

Theorem 2 is proved.

3. Semantics

Let $m \dot{-} k$ be $max(0, m-k)$ where m and k are arbitrary natural numbers. Elements of $\{0, 1, \ldots, n\}$ will be assigned to formulas of the calculi Cn if its variables are assigned values from this set, the constant f_i is assigned to value i and take value of $k \supset m$ equal to $m \dot{-} k$.

At first it is noticed that value of the formula $p \overset{n}{\supset} q$ when $p = k$ and $q = m$ is equal to $m \dot{-} nk$ (it is proven by mathematical induction).

Examples.

1. Find value of the formula $p \overset{n}{\supset} \left(q \overset{n}{\supset} p\right)$ when $p = k$ and $q = m$. We have

$$k \overset{n}{\supset} \left(m \overset{n}{\supset} k\right) = \left(m \overset{n}{\supset} k\right) \dot{-} nk = \left(k \dot{-} nm\right) \dot{-} nk = 0$$

for every natural numbers k, m.

2. Find value of the formula $\left(\left(p \overset{n}{\supset} f_n\right) \overset{n}{\supset} f_n\right) \overset{n}{\supset} p$ when $p = k$. We have

$$\left(k \overset{n}{\supset} n\right) \overset{n}{\supset} n \overset{n}{\supset} k = k \dot{-} n\,(n \dot{-} nk) = 0$$

for every natural numbers k.

3. It is easy to verify that value of the formula

$$\left(p \overset{n}{\supset} \left(q \overset{n}{\supset} r\right)\right) \overset{n}{\supset} \left(\left(p \overset{n}{\supset} q\right) \overset{n}{\supset} \left(p \overset{n}{\supset} r\right)\right)$$

is equal to zero when $p = k$, $q = m$, $r = t$ where k, m, t are arbitrary natural numbers.

A formula of the calculus Cn is called exceptional if its value is equal to zero when its variables are assigned arbitrary values.

Theorem 3

If $\vdash_{Cn} A$ then the formula A is exceptional, $n \geq 1$.

Proof. Every axiom obtaining on any axiom scheme $(1n)$, $(2n)$ and $(3n)$ is exceptional according to the examples 1, 2, 3. Further modus ponens holds the exception property. Theorem 3 is proved.

Theorem 4

$\vdash_{C1} A$ if and only if the formula A is exceptional.

Proof. Formulas of the calculus $C1$ will be converted in formulas of Church's calculus P_1 [2] if the constant f_1 is changed to the constant f of the calculus P_1 and round brackets are changed to square ones. Then tautologies of the calculus P_1 will be converted in exceptional formulas of the calculus $C1$ if in the calculus P_1 truth is represented by zero and falsehood is represented by one (the constant f of the calculus P_1 is represented by falsehood). Church's Theorems 150 and 152 say that a formula of the calculus P_1 is proven if and only if it is tautology. So for as the calculi $C1$ and P_1 have the same axioms and rules of inference then this completes the proof.

Theorem 5

For every calculus Cn $(n \geq 2)$ there is its exceptional formula which is not a theorem.

Proof. Every theorem of the calculus Cn has the form $A \overset{n}{\supset} B$ according to Theorem 3 and the forms of axiom schemes. So the formula $p \overset{n-1}{\supset} p$ of the calculus Cn where $n \geq 2$ is not its theorem but this formula is exceptional. Theorem 5 is proved.

References

[1] Łukasiewicz, Jan, O logice trójwartościowej, [in:] Ruch Filozoficzny. Volume 5. 1920. pp. 169–171.

[2] Church, Alonzo, Introduction to mathematical logic. Volume 1. Princeton University Press. 1956.