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THE MAIN GENERALIZATION OF CONTINUOUS-VALUED LOGIC

The paper outlines basic results in the generalization of continuous-valued logic. The survey is based on Russian publications. We consider an order logic, which is a generalization of continuous-valued logic where the operations of maximum (disjunction) and minimum (conjunction) are substituted with the operation of selection of r-th order argument, corresponding to the values of arguments. We show that this new operation is expressed in a superposition of disjunctions and conjunctions of continuous-valued logic. Various classes of logical determinants are considered; they are thought as numerical characteristics of matrices, expressible in operations of continuous-valued logic. Namely, we investigate order determinants, which generalize order logical operation of several arguments in matrix form, and determinants with various constraints on subsets of matrix elements. Properties of all logical determinants are discussed, compared with properties of algebraic determinants; techniques of computation of logical determinants are supplied. We also investigate a predicate algebra of choice, which generalizes continuous-valued logic in case of simulation of discontinuous functions; a hybrid logic of continuous and discrete variables; a logic-arithmetic algebra, which includes, in addition to continuous-logic operations, four arithmetical operations; a complex algebra of logic, where supportive set C is a field of complex numbers. A description of each algebra includes basic laws, which are compared with the laws of conventional continuous-valued logic. Several generalizations of continuous-logic operations to operations over matrices, random and interval variables are discussed. Some applications of continuous-valued logics are shown.

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1. Introduction

Continuous-valued logic (CL) is a very rich mathematical structure, particulars of which include expressiveness, constructability and visibility. These features open a wide road to the huge varieties of kinetic potential applications in mathematics, engineering, economy, biology, sociology, and history. Much more areas of possible application are still uncovered; therefore we have to search for novel generalizations of CL.

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There is a large variety of generalizations of CL. Some of them, namely logic determinants, are oriented toward application of CL to systems of higher dimensions. Others aim to expand domains of logical functions by enriching continuous set with discrete set (hybrid logic) or by transition from real numbers to complex numbers (complex logic). There are also generalizations of CL derived from the modification of a structure of variables, they are related to matrix, random or interval variables. At last, we can generalize CL by including some non-logical operations, e.g. arithmetical operations. All these generalizations allow us to apply mathematical apparatus of CL to investigation of complex natural and engineering systems. Thus, in particular, hybrid logic may be applied to design and analysis of analog-digital devices, interval logic is appropriate for systems with uncertain parameters, logic-arithmetic algebra is perfect for systems with discontinuity. All these classes will be discussed in this paper.

2. Order logic and order logical determinants

Let us start with a function

$$f^r(a_1,\ldots,a_n) = a^r, \quad a^1 \le \ldots \le a^n, \quad r = \overline{1,n}, \tag{1}$$

which selects the r-th serial element a^r of a set $A = \{a_1,\ldots,a_n\}$, $a_i \in C$. The r-operation f^r generalizes operations of disjunction $\vee = \max$ and conjunction $\wedge = \min$ of CL; it is equivalent to disjunction when $r = n$, and it is a conjunction-like function when $r = 1$. The algebra $\{A; f^r,\ r = \overline{1,n}\}$ is called an algebra of serial logic. Every possible operation in the serial logic is build of the serial operations like f^r, $r = \overline{1,n}$, and their various superpositions. To define some function of the serial logics (as well as a functions of CL) we have to assign an argument a_i, whose value is accepted by the function, to every order $a_1,\ldots,a_n$ of arguments. It is quite easy to follow an analytical representation with the superposition of the operations $\vee$ and $\wedge$ of CL from the tabular form. The analytical representations of function of serial logic do not usually differ from those of CL. In addition to the common laws of CL the following triple of the laws whose representation is specific for the serial logic should be displayed:

Tautology: $$f^r(a,\ldots,a) = a \tag{2}$$

Commutativity: $$f^r(a_1,\ldots,a_n) = f^r(a_{i_1},\ldots,a_{i_n}) \tag{3}$$

Distributivity: $$f^r(\varphi^{q_1},\varphi^{q_2},\ldots,\varphi^{q_p}) = \varphi^{q_r},\ q_1 < q_2 < \ldots < q_p \tag{4}$$

The disjunctive expression of an arbitrary function of serial logic in the operations of CL is quite an obvious one

$$f^r(a_1,\dots,a_n)=\bigvee_{i_1\neq\dots\neq i_{n-r+1}}(a_{i_1}\wedge\dots\wedge a_{i_{n-r+1}}),a_{i_k}\in\{a_1,\dots,a_n\}. \quad (5)$$

If the set A from (1) is partially ordered as a quasi-matrix

$$A_n=\begin{Vmatrix} a_{11} & \cdots & a_{1m_1}\\ \cdots & \cdots & \cdots\\ a_{n1} & \cdots & a_{nm_n}\end{Vmatrix}=\|a_{ij}\|,\quad a_{i1}\leq\dots\leq a_{im_i},\quad i=\overline{1,n}, \quad (6)$$

then we have the generalization of serial r-operation (1) in form of the serial logical determinant (LD)

$$A_n^r\equiv|a_{ij}|_n^r=\begin{vmatrix} a_{11} & \cdots & a_{1m_1}\\ \cdots & \cdots & \cdots\\ a_{n1} & \cdots & a_{nm_n}\end{vmatrix}^r=a^r,\quad r=\overline{1,m},\quad M=\sum_{i=1}^n m_i. \quad (7)$$

In the particular case of an unordered set A_n the quasi-matrices (6) are transformed into a column $A=\begin{Vmatrix} a_1\\ \cdots\\ a_n\end{Vmatrix}$, and LD (7) becomes a usual serial r-operation (1). The algebra $\{A_n;A_n^r,\ \ r=\overline{1,m}\}$ is an algebra of the serial LD. Every possible operation in the algebra of serial LD are expressed via A_n^r, $r=\overline{1,m}$ and their superpositions. We can define LD and represent it analytically in the same way as we did with the functions of CL and serial logic; see (2). In the analytical representations, serial LD does not differ from the functions of CL and serial logic; it also obeys the laws of both logics. Also, LDs have a number of properties similar to the properties of algebraic determinants of square matrices. For example, (1) values of the LD A_n^r are non-decreasing functions of the rank r, (2) rearrangements of any two lines of the LD A_n^r do not change the values:

$$|a_{ij}+c|_n^r=|a_{ij}|_n^r+c,\ |a_{ij}\vee c|_n^r=|a_{ij}|_n^r\vee c,\ |a_{ij}\wedge c|_n^r=|a_{ij}|_n^r\wedge c, \quad (8)$$

$$|a_{ij}\cdot c|_n^r=\begin{cases} c\,|a_{ij}|_n^r, & c>0\\ c\,|a_{ij}|_n^{n-r+1}, & c<0.\end{cases} \quad (9)$$

Any serial LD can be expressed by the operations of CL:

$$\begin{vmatrix} a_{11} & \cdots & a_{1m_1}\\ \cdots & \cdots & \cdots\\ a_{n1} & \cdots & a_{nm_n}\end{vmatrix}^r=\bigvee_{\sum_{s=1}^n i_s=n+r-1}(a_{1i_1}^{m_1}\wedge\dots\wedge a_{ni_n}^{m_n}) \quad (10)$$

The entry $a_{ki_k}^{m_k}$ means that the element a_{ki_k} does not enter into conjunctions for which the conditions on $\sum i_s$ implies $i_k > m_k$. In addition to the DNF expressions of the serial r-order functions (5) and LD (10) there are exist dual CNFs. The serial LD can be decomposed on a smaller LD, e.g.:

$$|a_{ij}|_n^r = \bigvee_{i,j} a_{ij} \, |a_{ij}|_{n \backslash a_{ij}}^r . \tag{11}$$

In the equation (11), term $|a_{ij}|_{n \backslash a_{ij}}^r$ is LD, obtained from the LD $|a_{ij}|_n^r$ by the elimination of the element a_{ij} (by a logical addition of the element a_{ij}). There are also other decompositions of LD, in particular the decompositions minimal on complexity and so-called block decompositions. Their sequential use allows, alongside with the obvious formula (10), to open LD. The complexity of disclosure of serial LD by using (10) is polynomial on n: $N \approx n^r/(r-1)!$ (r is fixed); the complexity of sequential decomposition of LD onto blocks has $O(rM)$ bound. The calculating of serial LDs A_n^r by sequential ordering of their elements is even easier. An appropriate algorithm has a complexity $N = (n-1)r$; however, it requires to store the sub-products. A complexity of the approximation of a value of serial LD is significantly smaller because

$$\bigwedge_{i=1}^{n} a_{i]d_i[}^{m_i} \leq |a_{ij}|_n^r \leq \bigvee_{i=1}^{n} a_{i[l_i]}, \; d_i = (n+r-1)m_i/M, \; l_i = rm_i/M, \tag{12}$$

where $]a[$ and $[a]$ are the rounds up to the nearest integer downwards and up.

Order logic and the order LD are used in analytical expressions of processes in high-dimensional systems, when the same low-dimensional processes are well elaborated analytically with the help of CL. Thus, e.g., in the example 2, part 1, of the paper we saw that the process on output of two-input automaton is expressed using operations of CL. This means that the process on the output of a multi-input automaton might be expressed analytically by the order LD. Actually, two LDs are enough: (i) LD, any i-th row of which represent moments when signal is changed, $0 \to 1$, on i-th input; (ii) LD, any row of which consists of the moments of signal changing, $1 \to 0$, on i-th input; the moments are arranged in ascending order.

For order logic, the order LD and their applications to a simulation of applied systems see [1–3, 6, 7]; numerous results of the approach can be found in the proceedings of the conferences on continuous-valued logic [9–15].

3. Logical determinants with the restrictions on a sum of elements

Let us consider a rectangular matrix with real elements

$$A_n = \left\| \begin{matrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right\| = \|a_{ij}\| . \qquad (13)$$

We are interested in the various sums $\sum_q^{/} a_{ij}$, $q = 1, 2, \ldots$, of the matrix elements which include exactly one element from each column (we can include as many elements as necessary from each row) and the functions like those below:

$$A^{1\vee} \equiv \overset{1}{\left| \begin{matrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right|^{\vee}} = \overset{1}{|a_{ij}|^{\vee}} = \bigvee_q \sum\nolimits_q^{/} a_{ij},$$

$$A^{1\wedge} \equiv \overset{1}{\left| \begin{matrix} a_{11} & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right|^{\wedge}} = \overset{1}{|a_{ij}|^{\wedge}} = \bigwedge_q \sum\nolimits_q^{/} a_{ij}, \qquad (14)$$

Let us consider a possibility of the elementary LD with limitations of the 1st sort for a matrix A. In the particular case of the single non-zero column of the matrix A, we see that LD $A^{1\vee}$ passes into the CL disjunction of the elements of this column; the LD $A^{1\wedge}$ is transformed correspondingly into their conjunction. The analytical representation of LD with the 1st sort restrictions is different, in general, of the functions of CL, both serial logic and serial LD, because the algebraic operation + can be used in addition to the CL operations $\wedge$ and $\vee$. The expressions of LD with limitations of the 1st sort obey all laws of CL, and, moreover, have a number of properties similar to the properties of algebraic determinants. Thus, the rearrangement of any two lines (columns) of the LDs $A^{1\vee}$ or $A^{1\wedge}$ does not change its value. We also have

$$|ca_{ij}|^{\vee} = c\,|a_{ij}|^{\vee}, \quad |ca_{ij}|^{\wedge} = c\,|a_{ij}|^{\wedge}, \quad c > 0;$$

$$|ca_{ij}|^{\vee} = c\,|a_{ij}|^{\wedge}, \quad |ca_{ij}|^{\wedge} = c\,|a_{ij}|^{\vee}, \quad c < 0;$$

$$|a_{ij} + c|_{n\times m}^{\vee} = |a_{ij}|_{n\times m}^{\vee} + cm,$$

$$|a_{ij} + c|_{n\times m}^{\wedge} = |a_{ij}|_{n\times m}^{\wedge} + cm;$$

$$|c - a_{ij}|_{n\times m}^{\overset{\vee}{\wedge}} = cm - |a_{ij}|_{n\times m}^{\overset{\wedge}{\vee}} . \qquad (15)$$

The LD $A^{1\vee}$ can be decomposed on smaller LDs, on column or the collections of columns

$$A^{1\vee} = \bigvee_{i=1}^{n} a_{ij} + A_{j}^{1\vee}, \quad A^{1\vee} = A_{\{j_1,\ldots,j_r\}}^{1\vee} + A_{j_1,\ldots,j_r}^{1\vee} \tag{16}$$

The same decomposition takes place for the LD $A^{1\wedge}$. In (16), $A_{j_1,\ldots,j_r}^{1\vee}$ is LD, derived from $A^{1\vee}$ by the exclusion of $j_1,\ldots,j_r$ columns (logical complement of $j_1,\ldots,j_r$ columns), $A_{\{j_1,\ldots,j_r\}}^{1\vee}$ is LD, formed by the columns $j_1,\ldots,j_r$ (minor from these columns). The sequential decomposition LD, which is an agreement with (1), has the following obvious expression:

$$A^{1\vee} = \sum_{j=1}^{m} \bigvee_{k=1}^{n} a_{kj}, \qquad A^{1\wedge} = \sum_{j=1}^{m} \bigwedge_{k=1}^{n} a_{kj}, \tag{17}$$

The complexity of the decomposition is $N = (n-1)(m-1)$.

Now we operate with the sums of the elements of a matrix (13) like the one below

$$\sum\nolimits_{q}^{\prime\prime} a_{ij}, \quad q = 1, 2, \ldots,$$

which includes exactly one element from each column and p_i elements from the ith row; $b_i \leq p_i \leq c_i$, $i = \overline{1,n}$, $\sum\limits_{i=1}^{n} p_i = m$. So, we have

$$A^{2\vee} \equiv \left| \begin{matrix} a_{11} & \overset{2}{\cdots} & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right|^{\vee} = \left| \overset{2}{a}_{ij} \right|_{(b_i,c_i)}^{\vee} = \bigvee_{q} \sum\nolimits_{q}^{//} a_{ij},$$

$$A^{2\wedge} \equiv \left| \begin{matrix} a_{11} & \overset{2}{\cdots} & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right|^{\wedge} = \left| \overset{2}{a}_{ij} \right|_{(b_i,c_i)}^{\wedge} = \bigwedge_{q} \sum\nolimits_{q}^{//} a_{ij} \tag{18}$$

Let us consider LD with limitations of the 2nd sort for a matrix A. Such LD only when $\sum\limits_{i=1}^{n} c_i \geq m$; there is a difference from LD (14). In the specific case of single non-zero column in the matrix A, the LD $A^{2\vee}$ is transformed into the CL disjunction, and the LD $A^{2\wedge}$ is transformed into the CL conjunction of elements of this column. In a common case, the analytical representation of LD (18) differs from the CL functions of both serial logic and serial LD; besides the CL operation $\vee$ and $\wedge$ include the operation $+$. The expressions of LD with the limitations of the 2nd sort obey all laws of CL but have also a number of specific properties reminding properties of the algebraic determinants. Thus, for example: (1) rearrangements of two lines together with their limitations do not change the value of LD; (2) lines, the area of which limitations is empty, can be eliminated without change of the value of LD. The LDs $A^{2\vee}$, $A^{2\wedge}$ can be decomposed on smaller LDs from any column

$$A^{2\vee} = \bigvee_{i=1}^{n} (a_{ij} + A_{ij}^{2\vee}), \quad A^{2\wedge} = \bigwedge_{i=1}^{n} (a_{ij} + A_{ij}^{2\wedge}). \tag{19}$$

In (19), $A_{ij}^{2\vee}$ is LD, obtained from the LD $A_{ij}^{2\vee}$ by the exclusion of jth column and shift of an interval $[b_i, c_i]$ of values of the parameter p_i along one position to the left (but not further than 0); this is called a logical adjunct (complement) of an element a_{ij} in the LD $A^{2\vee}$. The logical addition of element a_{ij} in the LD $A^{2\wedge}$ is similar to (19). We can also implement decomposition on the collection of columns

$$\begin{aligned} A^{2\vee} &= \bigvee_{k} (A^{2\vee}_{\{j_1,\ldots,j_r\},U_k} + A^{2\vee}_{j_1,\ldots,j_r,V_k}), \\ A^{2\wedge} &= \bigwedge_{k} (A^{2\wedge}_{\{j_1,\ldots,j_r\},U_k} + A^{2\wedge}_{j_1,\ldots,j_r,V_k}) \end{aligned} \tag{20}$$

In 1st formula of (20), $A^{2\vee}_{\{j_1,\ldots,j_r\},U_k}$ is LD, formed from the columns $j_1, \ldots, j_r$ of LD $A^{2\vee}$, with the restrictions on the number of elements in different lines by area U_k (minor from these columns). $A^{2\vee}_{j_1,\ldots,j_r,V_k}$ is LD, obtained from LD $A^{2\vee}$ by the exclusion of columns $j_1, \ldots, j_r$ and restrictions on numbers of elements in different lines by area V_k (logical complement of the specified minor in the LD $A^{2\vee}$). The sum of U_k and V_k for any k gives us the restriction on number of elements in lines of the whole LD $A^{2\vee}$. The definitions for the terms of the 2nd formula of (20) are similar. Subsequent decomposition of the LD $A^{2\vee}$, $A^{2\wedge}$, with usage of the formulas (19) and (20) produces simple LDs, which can be calculated directly. Complexity of the calculation of $n \times m$-size LD with the limitations of the 2nd sort is $N < 3n^{m-1}$.

Now we will consider every possible sum $\sum_q^{///} a_{ij}$, $q = 1, 2, \ldots$, of the elements of matrix A (13) of sizes $n \times n$, a sum includes equally one unit from each row and each column. Thus we obtain the following functions:

$$A^{3\vee} \equiv \left| \begin{matrix} a_{11} & \overset{3}{\cdots} & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{matrix} \right|^{\vee} = \left| \overset{3}{a_{ij}} \right|^{\vee} = \bigvee_{q} \sum\nolimits_q^{///} a_{ij},$$

$$A^{3\wedge} \equiv \begin{vmatrix} & 3 & \\ a_{11} & \cdots & a_{1n} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nn} \end{vmatrix}^{\wedge} = \left|\overset{3}{a_{ij}}\right|^{\wedge} = \bigwedge_q \sum\nolimits_q^{///} a_{ij} \tag{21}$$

The special cases of LD with the restrictions of the 2nd sort (18) are called LDs with the restrictions of the 3rd sort for a matrix A. They always exist. The analytical representation of LD (21) includes (in a common case, in addition to the operations CL $\vee$ and $\wedge$) the operation $+$, which is different of the CL functions of serial logic and serial LD. In special cases of a single non-zero column or single non-zero line in a matrix A the LD $A^{3\vee}$ passes into a disjunction, and the LD $A^{3\wedge}$ is transformed into a conjunction of the elements of the given column (the given line). LDs of the 3rd sort $A^{3\vee}$, $A^{3\wedge}$ are special cases of LD of the 2nd sort $A^{2\vee}$, $A^{2\wedge}$ (18) for $m = n$ and the restrictions: $b_i = p_i = c_i = 1$, $i = \overline{1,n}$. Therefore LDs of the 3rd sort (21) have properties of LD of the 2nd sort. Moreover they have properties of LD with the restrictions of the 1st sort, see (4), and also a number of some specific properties, for example:

$$A^{3\vee} = (A^{3\mathrm{T}})^{\vee}, \quad A^{3\wedge} = (A^{3\mathrm{T}})^{\wedge}, \tag{22}$$

where “T” means transposition of a matrix. The LD (21) can be decomposed onto the smaller LDs from any column or any line:

$$\begin{aligned} A^{3\vee} &= \bigvee_{i=1}^{n} (a_{ij} + A_{ij}^{3\vee}), \quad j = \overline{1,n}, \\ A^{3\vee} &= \bigvee_{j=1}^{n} (a_{ij} + A_{ij}^{3\vee}), \quad i = \overline{1,n}. \end{aligned} \tag{23}$$

The same can be done for $A^{3\wedge}$. In (23), $A_{ij}^{3\vee}$ is LD, obtained from the LD $A^{3\vee}$ by the elimination of the ith line and jth column, intersection of which hosts the element a_{ij} (a logical complement of the element a_{ij} in LD $A^{3\vee}$). The decomposition of (23) onto the collections of lines and columns is similar:

$$\begin{aligned} A^{3\vee} &= \bigvee_{D_r} (A_{\{D_r,B_r\}}^{3\vee} + A_{D_r,B_r}^{3\vee}), \quad B_r \subset \{1,\dots,n\}, \\ A^{3\vee} &= \bigvee_{B_r} (A_{\{D_r,B_r\}}^{3\vee} + A_{D_r,B_r}^{3\vee}), \quad D_r \subset \{1,\dots,n\} \end{aligned} \tag{24}$$

The same can be done for $A^{3\wedge}$. In (24) $A_{\{D_r,B_r\}}^{3\vee}$ is LD, obtained by the selection, in the LD $A^{3\vee}$, the elements being on the intersection of lines

$D_r = (i_1, \dots, i_r)$ and columns $B_r = (j_1, \dots, j_r)$; $A^{3\vee}_{D_r,B_r}$ is LD, obtained from the LD $A^{3\vee}$ by eliminating the lines D_r and columns B_r. The sequential decomposition of LD (21) allows us to calculate them. Thus, the least complexity turns out with the use of decompositions (24).

Now we can consider the main matrix A (13) and the matrix of restrictions $B = \|b_{ij}\|$; both matrices have the same order. Let us operate with the sums of elements of both matrices of sorts $\sum_q^{/} a_{ij}$, $\sum_q^{/} b_{ij}$, $q = 1, 2, \dots$ (as well as in LD of the 1st sort of matrices A, B). Functions of the form

$$A^{4\vee} \equiv \left| \begin{matrix} a_{11} & \overset{4}{\cdots} & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right|^{\vee} = \left| \overset{4}{a_{ij}} \right|^{\vee} = \bigvee_q \sum\nolimits_q^{/} \underset{\sum_q^{/} b_{ij} \le b}{a_{ij}} ,$$

$$A^{4\wedge} \equiv \left| \begin{matrix} a_{11} & \overset{4}{\cdots} & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm} \end{matrix} \right|^{\wedge} = \left| \overset{4}{a_{ij}} \right|^{\wedge} = \bigwedge_q \sum\nolimits_q^{/} \underset{\sum_q^{/} b_{ij} \le b}{a_{ij}} \tag{25}$$

are called LDs with the restrictions of the 4th sort for a matrix A. Such LDs exist only for $B^{1\wedge} \le b$. In the specific case of single non-zero column in a matrix A the LD $A^{4\vee}$ is transformed into disjunction and the LD $A^{4\wedge}$ becomes the CL conjunction of some elements of this column (satisfying the restrictions). In general, the analytical expression of LD (25) includes the CL operations $\vee$ and $\wedge$ as well as the operation $+$. The expressions of LD of the 4th sort obey all laws of CL. These LD have properties (15) of LD of the 1st sort and the following specific properties:

1) rearrangements of two tines (columns) of a main matrix A simultaneously with rearrangement of appropriate lines (columns) of a matrix of limitations B does not change the value of LD;
2) the common for all elements of a column addend can be taken out for the sign of LD;
3) LD with column from equal elements is equal to sum of this element and the CL disjunction of logical adjuncts of all elements of a column.

Here, the logical adjunct $A^{4\overset{\vee}{\wedge}}_{ij}$ of the element a_{ij} in LD $A^{4\overset{\vee}{\wedge}}$ is LD obtained from $A^{4\overset{\vee}{\wedge}}$ by the elimination of the jth column and decrease of the boundary b on b_{ij}. Concerning the LDs $A^{4\vee}$ and $A^{4\wedge}$, we can say that it is possible to decompose them onto smaller LDs from any column which agrees to the 1st formula of (23), and also to decompose them onto any collection of columns. Using decompositions allows us to calculate LD.

The LD with restriction on sum of elements are used in the analytical representations of algorithms of static optimization, such as problems of optimal distribution of limited resources. Such algorithms can be expressed by the disjunction and conjunction of CL because there we compute maximum and minimum of two variables, i.e. make elementary acts of optimization. The importance of LD in this case lies in a compact description of the algorithms when problems are high-dimensional; the different types of the restrictions on sums of elements of LD correspond to different classes of static optimization. For example, the LD with the restriction of the 3rd sort (21) is an analytical representation of algorithm for the problem on assignment of n candidates on n positions, when a matrix of efficiency of the candidates i for the positions j is $|a_{ij}|$, i.e. every candidate receives a position, and every vacancy is filled by a candidate. In this situation, the decomposition of LD (23) allows us to decrease the problem of dimension from $n \times n$ to $(n-1) \times n$ or $n \times (n-1)$. The LDs with restrictions on sum of elements and their application to the problems of static optimization are invented by V. I. Levin in [2, 3, 6, 7].

4. Logical determinants with a domain restriction

In the rectangular $n \times m$-size matrix A (13) we shall consider every possible descending step paths from a block (1, 1) into a block (n, m). Let us designate by $\sum_q^1 a_{ij}$ a sum of elements of the matrix A along the path of the length q, and $\sum_q^2 a_{ij}$ a sum of elements outside the path. Functions of the form

$$A^{\vee} \equiv |a_{ij}|^{\vee} = \bigvee_q \sum\nolimits_q^1 a_{ij}, \quad A^{\wedge} \equiv |a_{ij}|^{\wedge} = \bigwedge_q \sum\nolimits_q^2 a_{ij} \tag{26}$$

are called disjunctive and conjunctive LDs of a matrix A. They are elementary LDs with the restrictions on area of creation of sums of elements. The function

$$A^{+} \equiv |a_{ij}|^{+} = \sum_{i=1}^{n} \sum_{j=1}^{m} a_{ij} \tag{27}$$

Any matrix A always has the LD $A^{\vee}$, $A^{\wedge}$, and A^{+}. In the case of single non-zero element in the q-length path of a matrix A, the LD $A^{\vee}$ is transformed into the CL disjunction and LD $A^{\wedge}$ becomes a conjunction of the elements of this matrix. The analytical representation of the LDs $A^{\vee}$ and $A^{\wedge}$ contains the CL operations of disjunction and conjunction and algebraic addition; the LD A^{+} has only addition. The expressions LD $A^{\vee}$ and $A^{\wedge}$ satisfy to the laws of CL. Additionally, $A^{\vee}$, $A^{\wedge}$ and A^{+} have a number of specific properties similar to the properties of algebraic determinants:

$$|a_{ij}+c|^{\vee}_{n\times m} = |a_{ij}|^{\vee}_{n\times m} + c(m+n-1);$$

$$|a_{ij}+c|^{\wedge}_{n\times m} = |a_{ij}|^{\wedge}_{n\times m} + c(mn-m-n+1);$$

$$|a_{ij}+c|^{+}_{n\times m} = |a_{ij}|^{+}_{n\times m} + cmn;$$

$$|a_{ij}c|^{\vee} = c\,|a_{ij}|^{\vee}, \quad |a_{ij}c|^{\wedge} = c\,|a_{ij}|^{\wedge}, \quad c>0;$$

$$|a_{ij}c|^{\vee} = c\left(|a_{ij}|^{\vee}-\Delta\right), \quad |a_{ij}c|^{\wedge} = c\left(|a_{ij}|^{\wedge}+\Delta\right), \quad c<0, \qquad (28)$$

where $\Delta = \bigvee\limits_{q}\sum_q^1 a_{ij} - \bigwedge\limits_{q}\sum_q^1 a_{ij}$; $\begin{vmatrix} a_{11}+c & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm}+d \end{vmatrix}^{\vee} = |a_{ij}|^{\vee} + c + d$; $\begin{vmatrix} a_{11}+c & \cdots & a_{1m} \\ \cdots & \cdots & \cdots \\ a_{n1} & \cdots & a_{nm}+d \end{vmatrix}^{\wedge} = |a_{ij}|^{\wedge}$; $|a_{ij}c|^{+} = c\,|a_{ij}|^{+}$, $A^{\vee}+A^{\wedge}=A^{+}$.

The rearrangement of each pair of symmetric lines and each pair of symmetric columns does not change the value of LD $A^{\vee}$ and $A^{\wedge}$. The LDs $A^{\vee}$ and $A^{\wedge}$ can be decomposed on angular elements a_{11} and a_{nm} as

$$A^{\vee} = a_{11} + (A^{\vee}_{1,-} \vee A^{\vee}_{-,1}), \quad A^{\vee} = a_{nm} + (A^{\vee}_{m,-} \vee A^{\vee}_{-,n}). \qquad (29)$$

This is similar to $A^{\wedge}$. In (29) $A^{\vee}_{i,-}$ is LD obtained from $A^{\vee}$ by the elimination of the ith line (logical complement of the ith line), and $A^{\vee}_{-,j}$ is LD, obtained from $A^{\vee}$ by the elimination of the jth column (logical complement of the jth column). The decompositions of the LDs $A^{\vee}$ and $A^{\wedge}$ onto a collection of elements of lines and columns are possible. Consistently decomposing the LDs $A^{\vee}$ and $A^{\wedge}$ in formulas (29) we can calculate LD. But practically it is more convenient to use the formula

$$A^{\vee}_{rk} = (A^{\vee}_{r,k-1} \vee A^{\vee}_{r-1,k}) + a_{rk}, \qquad (30)$$

where $A^{\vee}_{rk}$ is LD of a type $A^{\vee}$ with r first lines and k first columns of a matrix A. The required LD $A^{\vee}$ is $A^{\vee}_{nm}$. Therefore, to compute $A^{\vee}$ we can calculate the angular element $A^{\vee}_{nm}$ in the matrix $A^{*} = \|A^{\vee}_{rk}\|$ associated with the matrix A. To do it we can use a recurrent relation (30) and wave algorithms of sequential finding of elements of matrix A^{*}, which starts in the left upper corner of the matrix and stops in the right lower corner. The complexity of such a calculation of LD $A^{\vee}$ has an upper bound $O(mn)$.

The LD A^+ is calculated directly from (27); its complexity is also $O(mn)$. The LD $A^\wedge$ is calculated from (28) through $A^\vee$ and A^+.

The LDs with domain restrictions are used in analytical representations of problems of dynamic optimization, such as schedule theory; in this situations LD allows to make a compact description of optimality condition even in the cases of high dimensions. The theory of LD with domain restrictions are elaborated in the works [2, 3, 6, 7].

5. Hybrid continuous-valued logics

CL is hybrid if it includes non-logical operation in addition to CL operations. We shall consider two two-place operations

$$\begin{aligned} \vee_p(x_1, x_2) &= x_1 \cdot p + x_2 \cdot \bar{p}, \\ \wedge_p(x_1, x_2) &= x_1 \cdot \bar{p} + x_2 \cdot p, \end{aligned} \qquad x_1, x_2 \in C, \quad p \in \{0, 1\}, \qquad (31)$$

where $p = p(y_1, \ldots, y_n)$ is m-place binary predicate with real values of predicate variables $y_1, \ldots, y_m$. In (31) we added two arithmetical operations $\cdot$ $+$, in addition to the negation operation $\bar{p}$. The predicate p in (31) is a control parameter, the choice of which value defines a choice of the concrete function $f(x_1, x_2)$. Thus, for $p = 1(x_1 - x_2)$, the functions $\vee_p$ and $\wedge_p$ are transformed into the CL disjunction and conjunction. A number of possible choices of the functions f increases with a superposition of functions (31) onto subject and predicate variables x_i and y_i. The algebra $\{C, \vee_p, \wedge_p, \bar{\ }\}$ is called a predicate algebra of choice. There are several laws which take place in the algebra $\{C, \vee_p, \wedge_p, \bar{\ }\}$ and which are analogous with the laws of the algebra of CL:

$$\begin{aligned} &\text{Tautology:} && \vee(x, x) = \wedge(x, x) = x, \\ &\text{Commutativity:} && \vee(x_1, x_2) = \wedge(x_2, x_1), \\ &\text{Distributivity:} && f[{}^\vee_\wedge x_1, x_2] = {}^\vee_\wedge [f(x_1), f(x_2)], \end{aligned} \qquad (32)$$

However, there are also several laws, which are specific only for this algebra.

If we allow the variable x_i to take as continuous values from the set $C = [A, B]$ as discrete values a hybrid logic emerges. It can be based on the threshold operations, which transform continuous variables into discrete variables, and anti-threshold operations, which implement inverse transformation. Thus, for two continuous and one discrete variables these operations are as follows:

$$P(x_1, x_2) = \left\{ \begin{array}{l} 1, x_1 \geq x_2 \\ 0, x_1 < x_2 \end{array} \right| , \quad D_1(y) = \left\{ \begin{array}{l} x_1, y = 1 \\ x_2, y = 0 \end{array} \right| ,$$

$$D_2(y) = \left\{ \begin{array}{l} x_2, y = 1 \\ x_1, y = 0 \end{array} \right| \begin{array}{l} x_1, x_2 \in C \\ y \in \{0, 1\}. \end{array} \tag{33}$$

The algebra $\{C \cup \{0;1\}; P, D_1, D_2\}$ is called an algebra of hybrid logic. Any function derived by a superposition of operations (33) is called a function of hybrid logic. The base operations of hybrid logic and CL are connected by the following ratio

$$D_1 P(x_1, x_2) = x_1 \vee x_2, \quad D_2 P(x_1, x_2) = x_1 \wedge x_2. \tag{34}$$

Also, hybrid logic can be based on the CL operations $\vee$, $\wedge$, $\overline{}$. In that case, it satisfies the CL laws.

If we combine the CL operations $\vee$ and $\wedge$ with the arithmetic operations $+$, $-$, $\cdot$, $:$, the so-called logic-arithmetic algebra may be constructed. Any function $C^n \to C$ as a superposition of six determined operations is called a logic-arithmetic function. In logic-arithmetic algebra the laws of arithmetic, CL and mixed logic-arithmetic laws may be used. For example:

$$\begin{array}{l} a + (b \vee c) = (a + b) \vee (a + c), a + (b \wedge c) = (a + b) \wedge (a + c), \\ a - (b \vee c) = (a - b) \wedge (a - c), a - (b \wedge c) = (a - b) \vee (a - c). \end{array} \tag{35}$$

It is more difficult to define the laws containing operations $\cdot$, $:$. Let symbols $\underset{k}{\vee}$ and $\underset{k}{\wedge}$ mean the operations $\vee$ and $\wedge$, respectively, when the constant k is positive and they mean $\wedge$ and $\vee$, respectively, when the constant k is negative. Then the following laws hold

$$k \cdot (a \vee b) = k \cdot a \underset{k}{\vee} k \cdot b, \quad k \cdot (a \wedge b) = k \cdot a \underset{k}{\wedge} k \cdot b. \tag{36}$$

The predicate algebra of choice solves a problem of simulation of discontinuous functions and facilitate analysis and synthesis of analog and digital devices. This algebra is developed by L. I. Volgin [4]. Hybrid logic is applied in the investigation of the devices, which process a mixture of continuous and discrete signals, e.g. analog-digital encoders. Hybrid logic, in the form discussed in this paper, is offered by P. N. Shimbiriev [5]. The logic-arithmetic algebra aims to describe such systems, which implement both continuous-logic and arithmetic operations, e.g. electric systems and economic systems. This algebra is offered by E. I. Berkovich and V. I. Levin in [2, 3, 7].

6. Complex continuous-valued logics

CL is complex if continuous-valued operations are performed over complex numbers, matrices or intervals.

CL can be generalized to complex-valued logic assuming that basic set C in the quasi-Boolean algebra $\{C, \vee, \wedge, \bar{\ }\}$ is a field of complex-valued numbers. In this case, the center M of the set C is a point $M = 0$ and operation of negation of a complex number a is defined as follows

$$\bar{a} = 2M - a = -a, \quad a \in C. \tag{37}$$

The operations of disjunction $\vee$ and conjunction $\wedge$ of CL are defined in this case in a standard manner: $\vee = \max$, $\wedge = \min$. Since max and min for complex-valued numbers are not defined we have

$$a \vee b = 0,5[a + b + |a - b|], \qquad a \wedge b = 0,5[a + b - |a - b|], \tag{38}$$

where all operations on the right sides of the equations in (38) are generalized to the complex-valued case. The designed complex-valued quasi-Boolean algebra obeys the laws of tautology, commutativity, double negation and descent of negation on addends as well as laws (35) and (38). However, it does not obey the laws of absorptions, of Kleene, operations with constants, of excluded middle, of contradiction, of associative and of de Morgan.

The generalization of CL to a matrix case is also possible. In this case variables are represented by rectangular matrices $A = \|a_{ij}\|$ and $B = \|b_{ij}\|$ of identical dimensions, whose elements a_{ij} and b_{ij} take values from a segment C. In this case, all logical operations are defined item-by-item, for example:

$$A \vee B = \|a_{ij} \vee b_{ij}\|, \quad A \wedge B = \|a_{ij} \wedge b_{ij}\|, \quad \bar{A} = \|\overline{a_{ij}}\|. \tag{39}$$

Therefore all laws of scalar CL are transferred to matrix CL.

If the operations of CL are implemented over random variables from the set C we have probabilistic interpretation of CL. All laws of deterministic CL are well preserved in the probabilistic CL. However all functions of CL such as a superposition of logical operations over random variables become stochastic. The main problem here is to find a probability distribution and moments of the given functions CL, which arguments are distributed by the given laws. If X_i, $i = \overline{1, n}$, is a random variable distributed with the law $F_i(X)$, a disjunction $\bigvee_{i=1}^{n} X_i$, and conjunction $\bigwedge_{i=1}^{n} X_i$ are distributed by the laws

$$F^{\vee}(x) = \prod_{i=1}^{n} F_i(x), \quad F^{\wedge}(x) = 1 - \prod_{i=1}^{n} [1 - F_i(x)]. \tag{40}$$

The distribution of the arbitrary function φ can be found as

1. an expression φ is resulted in a disjunctive normal form;
2. inequalities $\varphi < x$ are calculated by the method of partition; they are resulted in a sort of union of not-intersected systems of inequalities of the arguments $X_1, \ldots, X_n$,
3. a law of distribution $F_\varphi(X) = P(\varphi < X)$ of summation of probabilities of indicated systems of inequalities equal to integrals from products of density function $f_i(x)$ of values x_i is calculated.

If the operations of CL are implemented over interval variables $\tilde{a} = [a_1, a_2]$ from the set C we obtain the so-called interval CL.

The operations can be defined as follows

$$\tilde{a} \vee \tilde{b} = \{a \vee b | a \in \tilde{a}, b \in \tilde{b}\}, \quad \tilde{a} \wedge \tilde{b} = \{a \wedge b | a \in \tilde{a}, b \in \tilde{b}\} \tag{41}$$

In interval CL all laws of standard CL are preserved. However, all CL functions, such as a superposition of logical operations over intervals, become interval functions. Thus, the main problem here is to find an interval of values of the functions using the interval of the values of their variables:

$$\begin{aligned} [a_1, a_2] \vee [b_1, b_2] &= [a_1 \vee b_1, a_2 \vee b_2], \\ [a_1, a_2] \wedge [b_1, b_2] &= [a_1 \wedge b_1, a_2 \wedge b_2], \\ [\overline{a_1, a_2}] &= [\overline{a_2}, \overline{a_1}]\,. \end{aligned} \tag{42}$$

The functions $C^n \rightarrow C$, represented in various areas by the various forms of the given algebra CL, are called piecewise functions of CL. The following generalization can be investigated with the help of the serial LDs.

Complex CL is used in investigation of aperiodic processes in electric circuits. This logic, together with matrix CL, is developed by E. I. Berkovich (see review in [7]). Probabilistic and interval CL, aimed to explore systems with uncertain parameters and noise, are invented by V. I. Levin [1, 2, 6–8].

7. Conclusion

The generalizations of continuous-valued logics discussed in the paper can be developed further, however even indicate here fields of logical determinants, algebra of choice, hybrid logics, as well as matrix, probabilistic and interval logics, still bear a full potential for new discoveries. The detailed discussion of the subject, including extensive bibliography, can be found in [1–8].

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