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HERBRAND THEOREMS: THE CLASSICAL AND INTUITIONISTIC CASES

A unified approach is applied for the construction of sequent forms of the famous Herbrand theorem for first-order classical and intuitionistic logics without equality. The forms do not explore skolemization, have wording on deducibility, and as usual, provide a reduction of deducibility in the first-order logics to deducibility in their propositional fragments. They use the original notions of admissibility, compatibility, a Herbrand extension, and a Herbrand universe being constructed from constants, special variables, and functional symbols occurring in the signature of a formula under investigation. The ideas utilized in the research may be applied for the construction and theoretical investigations of various computer-oriented calculi for efficient logical inference search without skolemization in both classical and intuitionistic logics and provide some new technique for further development of methods for automated reasoning in non-classical logics.

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1. Introduction

Herbrand's paper [1] contains a theorem called now the Herbrand theorem. This theorem permits to reduce the question of the deducibility (validity) of a formula F of first-order classical logic to the question of the deducibility (validity) of a quantifier-free ("propositional") formula F', so that the deducibility of F' can be established by means of using only propositional "calculations". When making the reduction of F to F', a certain set of terms (so-called Herbrand universe) is constructed. Different ways of construction of Herbrand universe(s) lead to different forms of the Herbrand theorem. In particular, three forms are given in [1]: A, B, and C. The Herbrand universes for B and C are defined as minimal sets of terms containing constants and functional symbols occurring in the Skolem functional form of F, and the

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unique difference between B and C consists in the ways of the skolemization of F[1].

The form A does not need the skolemization of F, and the Herbrand universe for it uses only constants, functional symbols, and certain quantifier variables from F. But its application requires checking a large number of quantifier sequences constructed from "schemes" (in the terminology of'[1]) in order to find at least one sequence satisfying a certain condition that guarantees the validity of the formula F.

Since intuitionistic logic does not keep the skolemization transformations in general, it is impossible to obtain the forms B and C for intuitionistic logic. Therefore, for intuitionistic logic, there can be made an attempt to construct a Herbrand theorem similar to the form A only, i.e. when preliminary skolemization is not obligatory and reduction of first-order investigations to propositional "calculations" is performed. Besides, it is desired to give forms of Herbrand's theorem for classical and intuitionistic cases providing the clear-cut distinction between them.

This research gives a possible decision of the problem under consideration: *the unified forms of Herbrand's theorem are formulated for classical and intuitionistic logics in a sequent form*[2]. It does not explore skolemization, have wordings on deducibility, and develops the approach suggested in [5, 6, 7] for classical logic and modified in a certain way for a tableau treatment of intuitionistic logic in [8], which permits to achieve the objective just reminded on the base of the original notions of admissibility and compatibility. Note that a similar style of inference search (not requiring skolemization and not giving Herbrand theorems at once) was exploited by the author of the paper and his coauthors in a number of sequent calculi for classical logic (see, for example, [9, 10]) and in the tableau method for intuitionistic logic from [11] in order to optimize an item-by-item examination arising when quantifier rules applications satisfying Gentzen's admissibility – i.e. to the eigenvariable condition [12] – are made[3].

Additionally note that our approach based on the notions of admissibility and compatibility shares some ideas with the papers [16] and [17]

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[1] For example, see [2] and [3] for some details relating to such a type of Herbrand's theorem.

[2] The announcement of the main results of the research was made in the slightly different form at the Kurt Gödel Centenary Symposium, Vienna, Austria, 2006 [4].

[3] S. Kanger introduced his definition of admissibility [13] which has an advantage over Gentzen's one; its modified forms were used in a number of papers concerning inference search in classical logic (see, for example, [14]) and in intuitionistic logic (see, for example, [15]).

being exploited in the original way for the construction of various computer-oriented methods for classical and non-classical logics such as matrix characterization methods [17], different modifications of the connection method (see, for example, [16], [18], [19], [20]), and ordinal sequent and tableau methods (see, for example, [21], [22]). But all these papers do not contain any direct instructions how to construct both classical and intuitionistic forms (not requiring skolemization) of Herbrand's theorem.

2. Preliminaries

We use standard terminology of first-order sequent logic without equality. The basic signature Sig_0 of the first-order language consists of a (possibly empty) set of functional symbols (including constants), a (non-empty) set of predicate symbols, and logical connectives: the quantifier symbols for the universal character $\forall$ and for existential character $\exists$ as well as the propositional symbols for the implication ($\supset$), disjunction ($\vee$), conjunction ($\wedge$), and negation ($\neg$). And $\forall x$ and $\exists x$ are called quantifiers; they are considered as a single whole. A countable set of variables is denoted by Var.

Additionally, we extend the signature Sig_0 in the following way: for any natural number (*index*) k ($k = 1, 2, \ldots$) and for any symbol $\odot$ from Sig_0, we add the *indexed symbol* ${}^k\odot$ to Sig_0 and denote the constructed extension by Sig. For example, ${}^1\vee$, ${}^3\supset$, and ${}^5\forall$ are symbols of the extended signature. These upper-left indices are used for distinguishing different copies of the same formula.

For technical purpose only, we consider that Var consists of two disjoint countable sets: Var_0 and Var'_0 ($Var = Var_0 \cup Var'_0$), where the following condition is satisfies: for any $v \in Var_0$ and any natural number (*index*) k ($k = 1, 2, \ldots$), Var'_0 contains the *indexed variable* ${}^k v$.

The notions of terms, atomic formulas, literals, formulas, free and bound variables (over both $Sig_0 \cup Var$ and $Sig \cup Var$) are defined in the usual way [23] and assumed to be known for the reader.

Sequents also are defined in the usual manner, except that their succedents and antecedents are considered as multisets (cf. [23]).

Any syntactical object over $Sig_0 \cup Var$ is called an *original* one. As usual, we assume that no two quantifiers in any formula or in any sequent have a common variable, which can be achieved by renaming bound variables.

Without loss of generality, an *initial sequent* (i.e. a sequent being investigated on deducibility) always is considered to have the form $\rightarrow F$, where F is a *closed original* formula.

Any expression constructed over $Sig \cup Var$ may be viewed as an indexed one constructed by the renaming of a certain original object (by means of adding certain upper-left indices if needed). The extension of all the necessary semantic notions to indexed terms, indexed formulas, and indexed sequents is obvious: to do this, it is enough to consider all their upper-left indices in indexed terms, formulas, and sequents to be missed.

When it is important, we write the *"indexed formula"*, *"indexed sequent"*, etc. in order to underline the fact that an appropriate formula, sequent, etc. is constructed over $Sig \cup Var$ and may contain indexed symbols.

Sometimes, the original formula $\underline{\exists y \neg \exists x P(x, f(y)) \supset \neg \forall y' \exists x' P(x', y')}$ denoted by F^* will be used in a number of examples clarifying introduced notions and obtained results (P is a predicate symbol and f is a functional symbol).

If the *principal connective* of a (indexed) formula F is $\odot$ (i.e. F has the form $F' \odot F''$ or $\odot F'$, where $\odot$ is $\supset$, $\vee$, $\wedge$, $\neg$, $\forall$, or $\exists$), then F is called *$\odot$-formula.*

As in [2], we say that an occurrence of a subformula F in a formula G is

- *positive* if F is G;
- *positive* (*negative*) if G is of the form: $G_1 \wedge G_2$, $G_2 \wedge G_1$, $G_1 \vee G_2$, $G_2 \vee G_1$, $G_2 \supset G_1$, $\forall x G_1$, or $\exists x G_1$ and F is positive (negative) in G_1;
- *negative* (*positive*) if G is of the form $G_1 \supset G_2$ or $\neg G_1$ and F is *positive (negative)* in G_1.

Further, a formula F has a *positive (negative)* occurrence in a sequent $\Gamma \rightarrow \Delta$ if F has a positive occurrence in a formula from Δ (from Γ) or if F has a negative occurrence in a formula from Γ (from Δ). Moreover, if F has the form $\forall x F'$ ($\exists x F'$) and F has a positive (negative) occurrence in a formula G or in a sequent S, then $\forall x$ ($\exists x$) is called a *positive quantifier* in G or in S, respectively; $\exists x$ ($\forall x$) is called a *negative quantifier* in G or in S, if $\exists x F'$ ($\forall x F'$) has a positive (negative) occurrence in G or in S, respectively.

The variable of a positive quantifier occurring in a formula G or in a sequent S is called a *parameter* in G or in S, respectively; the variable of a negative quantifier occurring in a formula G or in a sequent S is called a *dummy* in G or in S, respectively.

Remark. The terms "parameters" and "dummies" are taken from [13], where they are used in the analogous sense.

For F^*, we have: x and y' are dummies and x' and y are parameters.

The way of the extension of the notions of dummies and parameters to sequents and (multi)sets of formulas or of sequents is obvious.

Since the property "to be a dummy" ("to be a parameter") is invariant w.r.t. logical rules applications in sequent calculi, any parameter (dummy)

x in a formula (in a sequent, in a (multi)set of formulas or of sequents) is convenient to be written as $\overline{x}$ ($\underline{x}$).

For a formula F (for a sequent S), $\mu(F)$ ($\mu(S)$) denotes the result of the elimination of all the quantifiers from F (from S).

If F (S) is a formula (a sequent) and x is its parameter or dummy then x considered to be a parameter or a dummy in $\mu(F)$ (in $\mu(S)$).

Convention. For any expression Ex over $Sig \cup Var$ and any index k, the notation ${}^kEx$ denotes the following expression: we delete all the upper-left indices in logical connectives, parameters, and dummies of Ex and add k as a upper-left index to all the symbols of the result of the deletion except for constant, for dummies, for predicate, and for functional symbols.

Therefore, we have for F^*: ${}^1F^*$ is ${}^1\exists {}^1\overline{y} {}^1\neg {}^1\exists {}^1\underline{x} P({}^1\underline{x}, f({}^1\overline{y}))^1 \supset {}^1\neg {}^1\forall {}^1y'$ ${}^1\exists {}^1x'\ P({}^1\overline{x}', {}^1\underline{y}')$ and $\mu({}^1F^*)$ is ${}^1\neg P({}^1\underline{x}, f({}^1\overline{y}))^1 \supset {}^1\neg\ P({}^1\overline{x}', {}^1\underline{y}')$.

We use slight modifications of the cut-free calculi LK and LJ [12] for technical purposes only. In this connection, we suppose a reader to know all the notions relating to deducibility in Gentzen (sequent) calculi. Draw you attention to the fact that all the *inference trees* in calculi under consideration are understood in the usual sense and grow "from top to bottom" by applying inference rules to an input sequent and afterwards to its "heirs", and so on. Additionally remind that any inference tree having only leaves with axioms is called a *proof tree.*

3. Admissibility and compatibility

Let F be a formula. By (i, F), we denote the i-th occurrence of its subformula if F is read from left to right. We write $(i, F) \sqsubseteq_F (j, F)$ if and only if (j, F) is a subformula of (i, F). Obviously, the relation $\sqsubseteq_F$ is partial ordered.

If (i, F) is the occurrence of a $\odot$-formula, where $\odot$ is a logical connective (a propositional connective or a quantifier), we also refer to this occurrence as to ${}_i\odot$-occurrence in F.

If a formula F has ${}_i\odot$- and ${}_j\odot'$-occurrences of its subformulas ($i \neq j$) and ${}_i\odot$-occurrence $\sqsubseteq_F$ ${}_j\odot$-occurrence, then ${}_j\odot'$ is said to be in the *scope* of ${}_i\odot$; this fact is denoted by ${}_i\odot \lessdot_F {}_j\odot'$. If $\odot$ ($\odot'$) is $\forall x$ or $\exists x$, we always write ${}_ix \lessdot_F {}_j\odot'$ (${}_i\odot \lessdot_F {}_jx$); and when $\odot'$ ($\odot$) is $\forall y$ or $\exists y$, we write ${}_ix \prec_F {}_jy$ (${}_iy \prec_F {}_jx$) underlining the fact that $\lessdot_F$ is restricted only in the case of the consideration of quantifiers variables.

Obviously, for any formula F, $\prec_F \subset \lessdot_F$ and the relations $\prec_F$ and $\lessdot_F$ are irreflexive and transitive.

We also *extend the (transitive and irreflexive) relations* $\prec_F$ *and* $\lessdot_F$

determined by an original formula F to the case of indexed symbols in the following way: for any natural numbers i and j we have ${}^{i}x \prec_F {}^{j}y$ and ${}^{i}\odot \lessdot_F {}^{j}\odot'$ if and only if $x \prec_F y$ and $\odot \lessdot_F \odot'$ respectively, where x, y, $\odot$, and $\odot'$ occur in F.

Moreover, any occurrence ${}_{i}\odot$ of a symbol $\odot$ in a formula F is treated as a new symbol. Therefore, ${}_{i}\odot$ and ${}_{j}\odot$ $(i \neq j)$ are different symbols denoting the same logical "operation" $\odot$. That is why ${}_{i}\odot$- and ${}_{j}\odot$-occurrences in F can be considered as different subformulas of F $(i \neq j)$, if needed.

The *extensions of* $\prec_F$ and *of* $\lessdot_F$ to the case of an original and indexed sequent S ($\prec_S$ and $\lessdot_S$) are obvious. The same relates to the case of a set Ξ of original and indexed formulas or of sequents ($\prec_\Xi$ and $\lessdot_\Xi$).

All the above-given extensions do not lead to confusion since we already begin all our investigations with an original formula or with an original sequent.

For the formula F^*, we have the following numbering of its logical connectives, predicate symbols and variables:

${}_1\exists\ {}_1\overline{y}\ {}_2\neg\ {}_3\exists\ {}_3\underline{x}\ {}_4P({}_3\underline{x}, f({}_1\overline{y}))\ {}_5\supset\ {}_6\neg\ {}_7\forall\ {}_7\underline{y}'\ {}_8\exists\ {}_8\overline{x}'\ {}_9P({}_8\overline{x}', {}_7\underline{y}')$.

Therefore, ${}_5\supset$ is the minimal element of $\lessdot_{F^*}$, ${}_1\overline{y} \prec_{F^*} {}_3\underline{x}$, ${}_2\neg \lessdot_{F^*} {}_3\underline{x}$; the variables ${}_3\underline{x}$ and ${}_8\overline{x}'$ are not comparable, the same concerns ${}_3\underline{x}$ and ${}_7\neg$ as well.

For the formula ${}_1\exists\ {}_1{}^3\overline{y}\ {}_2\neg\ {}_3\exists\ {}_3\underline{x}\ {}_4P({}_3\underline{x}, f({}_1{}^3\overline{y}))\ {}_5\supset {}_6\neg\ {}_7\forall\ {}_7\underline{y}'\, {}_8P({}^1\overline{x}', {}_7\underline{y}')$, being the result of the elimination of ${}_8\exists\ {}_8\overline{x}'$ in F^* and subsequent indexing of $\overline{y}$ and $\overline{x}'$, we have in particular that ${}_1{}^3\overline{y} \prec_{F^*} {}_3\underline{x}$ and the variables ${}_3\underline{x}$ and ${}^1\overline{x}'$ are not comparable.

A *substitution*, σ, is a finite mapping from variables to terms denoted by $\sigma = \{x_1 \mapsto t_1, \ldots, x_n \mapsto t_n\}$, where variables $x_1, \ldots, x_n$ are pairwise different and x_i is distinguished from t_i for all $i = 1 \ldots n$. For an expression Ex and a substitution σ, the result of the application of σ to the expression of Ex is understood in the usual sense; it is denoted by $Ex \cdot \sigma$.

For any set Ξ of expressions, $\Xi \cdot \sigma$ denotes the set obtained by the application of σ to every expression in Ξ. If Ξ is a set of (at least two) expressions and $\Xi \cdot \sigma$ is a singleton, then σ is called a *unifier* of Ξ. The notion of a *most general simultaneous unifier (mgsu)* of a set of expressions also is understood in the usual sense.

For any formula F (for any sequent S, for any set Ξ of formulas or sequents), each substitution σ induces a (possibly empty) relation $\ll_{F,\sigma}$ $(\ll_{S,\sigma}, \ll_{\Xi,\sigma})$ as follows: $y \ll_{F,\sigma} x$ $(y \ll_{S,\sigma} x,\ y \ll_{\Xi,\sigma} x)$ if and only if there exists $x \mapsto t \in \sigma$ such that x is a dummy in F $(S,\ \Xi)$, the term t contains y, and y is a parameter in F $(S,\ \Xi)$. Obviously, $\ll_{F,\sigma}$ $(\ll_{S,\sigma},\ \ll_{\Xi,\sigma})$ is an irreflexive relation (i.e. a relation that does not have any pair $\langle z, z\rangle$).

For example, consider the substitution $\sigma^* = \{\underline{x} \mapsto \overline{x}', \underline{y}' \mapsto f(\overline{y})\}$, where $\underline{x}$ and $\underline{y}'$ are dummies in F^* and $\overline{x}'$ and $\overline{y}$ are parameters in F^*. Then $\overline{x}' \ll_{F,\sigma} \underline{x}$ and $\overline{y} \ll_{F,\sigma} \underline{y}'$.

For a substitution σ and for an original formula F (for an original sequent S, for a set Ξ of expressions), $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$, $\lhd_{\Xi,\sigma}$) denotes the transitive closure of $\prec_F \cup \ll_{\{F\},\sigma}$ (of $\prec_S \cup \ll_{S,\sigma}$, of $\prec_\Xi \cup \ll_{\Xi,\sigma}$). At the same time, $\blacktriangleleft_{F,\sigma}$ ($\blacktriangleleft_{S,\sigma}$, $\blacktriangleleft_{\Xi,\sigma}$) denotes the transitive closure of $\lessdot_F \cup \ll_{\{F\},\sigma}$ (of $\lessdot_S \cup \ll_{S,\sigma}$, of $\lessdot_\Xi \cup \ll_{\Xi,\sigma}$).

We extend the notions of $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$, $\lhd_{\Xi,\sigma}$) and $\blacktriangleleft_{F,\sigma}$ ($\blacktriangleleft_{S,\sigma}$, $\blacktriangleleft_{\Xi,\sigma}$) to corresponding indexed units in the same way that was used when defining $\prec_F$ ($\prec_S$, $\prec_\Xi$) and $\lessdot_F$ ($\lessdot =_S$, $\lessdot_\Xi$).

A substitution σ is *admissible* (cf. [17]) for a formula F (for a sequent S, for a set Ξ of expressions) if and only if for every $x \mapsto t \in \sigma$, x is a dummy in F (in S, in Ξ), and $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$, $\lhd_{\Xi,\sigma}$) is an irreflexive relation.

For the above-given formula F^* and substitution σ^*, we have: ${}_1\overline{y} \lhd_{F^*,\sigma^*} {}_7\underline{y}' \lhd_{F^*,\sigma^*} {}_8\overline{x}' \lhd_{F^*,\sigma^*} {}_3\underline{x}$. Thus, σ^* is admissible substitution for F^*.

If $\sigma' = \{{}_7\underline{y}' \mapsto {}_8\overline{x}'\}$, then ${}_8\overline{x}' \ll_{\sigma'} {}_7\underline{y}'$. Since ${}_7\underline{y}' \prec {}_8\overline{x}'$, ${}_8\overline{x}' \lhd_{F^*,\sigma'} {}_8\overline{x}'$. Therefore, σ' is a reflexive relation and it is not admissible for F^*.

Obviously, $\lhd_{F,\sigma} \subseteq \blacktriangleleft_{F,\sigma}$ ($\lhd_{S,\sigma} \subseteq \blacktriangleleft_{S,\sigma}$, $\lhd_{\Xi,\sigma} \subseteq \blacktriangleleft_{\Xi,\sigma}$). Therefore, the following facts hold on the base of the definitions.

Proposition 1

The relation $\blacktriangleleft_{F,\sigma}$ as well as $\blacktriangleleft_{S,\sigma}$ and $\blacktriangleleft_{\Xi,\sigma}$ are irreflexive (antisymmetric) if and only if $\lhd_{F,\sigma}$ as well as and $\lhd_{S,\sigma}$ and $\lhd_{\Xi,\sigma}$ are irreflexive (antisymmetric) relations. Moreover, the irreflexivity of $\blacktriangleleft_{F,\sigma}$ (of $\blacktriangleleft_{S,\sigma}$, of $\blacktriangleleft_{\Xi,\sigma}$) implies the antisymmetry of $\blacktriangleleft_{F,\sigma}$ (of $\blacktriangleleft_{S,\sigma}$, of $\blacktriangleleft_{\Xi,\sigma}$) and vise versa.

This proposition permits the investigation of the irreflexivity (or the antisymmetry) of $\lhd$ to replace by the investigation of the irreflexivity (or the antisymmetry) of $\blacktriangleleft$ and vice versa.

Let F be a formula and ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ a sequence of all its logical connectives occurrences being maybe indexed. Let Tr_F be an inference tree for the initial sequent $\to F$ such that if $\alpha_{Tr_F}({}_{j_1 i}\odot_1)$ denotes an inference rule application eliminating the occurrence ${}_{j_1}\odot_1$ in F then Tr can be constructed in accordance with the order determined by the sequence $\alpha_{Tr_F}({}_{j_1}\odot_1), \ldots, \alpha_{Tr_F}({}_{j_r}\odot_r)$. In this case, ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ is called a *proper sequence* for Tr_F. (It is obvious that there may exist a connectives occurrences sequence for a formula F such that the sequence is not proper for any Tr_F. Besides, it must be clear that there may exist more than one proper sequence for an inference tree Tr_F in the case of the existence of one for F.)

Let F be a formula and Tr_F an inference tree for the initial sequent $\rightarrow F$. The tree Tr_F is called *compatible* with a substitution σ if and only if there exists a proper sequence ${}_{j_1}\odot_1, \ldots, {}_{j_r}\odot_r$ for Tr_F such that for any natural numbers m and n, the property $m < n$ implies that the ordered pair $\langle {}_{j_n}\odot_n, {}_{j_m}\odot_m \rangle$ does not belong to $\blacktriangleleft_{F,\sigma}$.

The results of the next section demonstrate the importance of the notion of compatibility for the intuitionistic case, while it is redundant for classical one as a whole and must be "transformed" into the notion of admissibility. There are several examples clarifying the role of compatibility in the next section.

4. Herbrand theorems

This section contains the main results of the paper, which condense the investigations presented in [7, 11, 8] in a unified form. Their proofs are omitted here; they contains in the subsequent sections. Besides, here we restrict ourselves only by considering *original* syntactical units for clearness of all the necessary constructions and theorems. Additionally note that without loss of generality, we are interested in establishing the deducibility of a sequent of the form $\rightarrow F$, where F is a closed formula.

Let F be a formula and $F_1, \ldots, F_n$ its variants. If $F_1, \ldots, F_n$ does not have any bound variables in pairs, then $F_1 \wedge \ldots \wedge F_n$ ($F_1 \vee \ldots \vee F_n$) is called a *variant $\wedge$-duplication* (a *variant $\vee$-duplication*).

Herbrand extension. Let G be a formula, F its subformula, and H a variant $\wedge$-duplication ($\vee$-duplication) of F not having common variables with G. Then the result of the replacement of F by H in G is called a *one-step Herbrand extension of* G. Further, the result $HE(G)$ of a finite sequence of one-step extensions consequently applied to G, then to a one-step Herbrand extension of G, and so on is called a *Herbrand extension* of G. If $HE(G)$ is generated by means of only $\wedge$-extensions, H is called an *intuitionistic Herbrand extension* of G.

Herbrand quasi-universe. Let F be a formula. Then $HQ(F)$ denotes the following minimal set of terms (called a *Herbrand quasi-universe*): (i) every constant and every parameter occurring in F belong to $HQ(F)$ (if there is no constant in F then the special constant $c_0 \in HQ(F)$); (ii) if f is a k-ary functional symbol and terms $t_1, \ldots, t_k \in HQ(F)$ then $f(t_1, \ldots, t_k) \in HQ(F)$.

In other words, $HQ(F)$ can be considered as a minimal set of terms constructing from constants and parameters occurring in F with the help of functional symbols of F with arities more that 0.

So, pLK and pLJ denote the propositional parts of LK and LJ, respectively, which means that pLK and pLJ do not contain quantifier rules, as well as $(Con \rightarrow)$ and $(\rightarrow Con)$ (see the next section) when antecedents and succedents of sequents are identified with multisets.

Theorem 1 (Sequent form of Herbrand's theorem for classical logic)

For a formula F, the sequent $\rightarrow F$ is deducible in the calculus LK (in any sequent calculus coextensive with LK) if and only if there are an Herbrand extension $HE(F)$ and a substitution σ of terms from the Herbrand quasi-universe $HQ(F)$ for all the dummies of $HE(F)$ such that

(i) there exists a proof tree $Tr_{\mu(HE(F))\cdot\sigma}$ for $\rightarrow \mu(HE(F))\cdot\sigma$ in pLK and
(ii) σ is an admissible substitution for $HE(F)$.

For intuitionistic logic, Theorem 1 transforms to the following form.

Theorem 2 (Sequent form of Herbrand's theorem for intuitionistic logic)

For a formula F, the sequent $\rightarrow F$ is deducible in the calculus LJ (in any sequent calculus coextensive with LJ) if and only if there are an intuitionistic Herbrand extension $HE(F)$ and a substitution σ of terms from the Herbrand quasi-universe $HQ(HE(F))$ for all the dummies of $HE(F)$ such that

(i) there exists a proof tree $Tr_{\mu(HE(F))\cdot\sigma}$ for $\rightarrow \mu(HE(F))\cdot\sigma$ in pLJ,
(ii) σ is an admissible substitution for $HE(F)$, and
(iii) $Tr_{\mu(HE(F))\cdot\sigma}$ is compatible with σ.

Draw your attention to the fact that Theorems 1 and 2 are distinguished by only the existence of (iii) in Theorem 2 (and by the calculi LK and LJ). The requirement (iii) is essential for intuitionistic logic. It is easy to check this fact with the help of the following simple examples.

Example 1. Let we have the sequent S: $\rightarrow F$, where F is $\neg\forall x P(x) \supset \exists y \neg P(y)$ ($\rightarrow F$ is deducible in LK and is not deducible in LJ). Obviously, for any intuitionistic Herbrand extension $HE(F)$, $\mu(HE(F))$ has the form $\neg(P(\overline{x}_{1,1}) \wedge \ldots \wedge P(\overline{x}_{1,p_1})) \wedge \ldots \wedge \neg(P(\overline{x}_{r,1}) \wedge \ldots \wedge P(\overline{x}_{r,p_r})) \supset \neg P(y)$ and Herbrand quasi-universe for it is equal to $\{c_0, \overline{x}_{1,1}, \ldots, \overline{x}_{r,p_r}\}$.

For $\rightarrow \mu(HE(F))$, any substitution $\sigma_{i,j} = \{y \mapsto \overline{x}_{i,j}\}$, where i and j are any natural numbers not exceeding r and p_r respectively, leads to a possibility to construct a proof tree $Tr_{i,j}$ for the selected extension $\mu(HE(F))$. (Obviously, the substitution $\{y \mapsto c_0\}$ does not have such a property.) It is easy to check the admissibility of $\sigma_{i,j}$ for $HE(F)$ and the absence of compatibility of $Tr_{i,j}$ with $\sigma_{i,j}$ regardless of the selection of $Tr_{i,j}$ and $\sigma_{i,j}$.

As a result, we have the deducibility of S in LK by Theorem 1 and the non-deducibility of S in LJ by Theorem 2. (When constructing any proof tree for S in LJ, any relation $\blacktriangleleft_{HE(F),\sigma_{i,j}}$ requires the application of the rule eliminating the first negation of F on the second step of deducing the proof tree, which is impossible to do in LJ for S.)

Example 2. If we slightly modify Example 1, taking $\rightarrow \exists x \neg P(x) \supset \neg \forall y P(y)$ as S, we have for S: $HQ(S) = \{c_0, \overline{x}\}$ and the substitution $\{y \mapsto \overline{x}\}$ is admissible for S. In this case, any proof tree for $\rightarrow \neg P(\overline{x}) \supset \neg P(\overline{x})$ is compatible with $\{y \mapsto \overline{x}\}$. Thus, S is deducible in LJ (and, of course, in LK).

Example 3. If we take $\rightarrow \forall y \exists x P'(y,x) \supset \exists y_1 \forall x_1 P'(x_1, y_1)$ as a sequent S, we have: $HQ(S) = \{x, x_1\}$ and for the substitution $\sigma = \{y \mapsto x_1, y_1 \mapsto x\}$, the sequent $\rightarrow P'(x_1, x) \supset P'(x_1, x)$ is deducible in pLK. Unfortunately, σ is not admissible for S and we cannot say anything about the deducibility of S even in LK. But it is easy to show that the construction of any Herbrand extension of $\forall y \exists x P'(y,x) \supset \exists y_1 \forall x_1 P'(x_1, y_1)$ cannot lead to an admissible substitution for any Herbrand extension and any its proof tree. Therefore, S is not deducible neither in LK nor in LJ.

As you can see, the above-given examples demonstrate that in comparison with Theorem 1, the grave disadvantage of Theorem 2 consists in the existence of the condition (iii) requiring a certain form of a proof tree for a sequent $\mu(\rightarrow F) \cdot \sigma$ in pLJ (or in its any analogue coextensive with pLJ): it must be *compatible* with σ, which does not permit any order of propositional rules applications leading to Tr while, in the classical case, any proof tree Tr for a sequent $\mu(\rightarrow F) \cdot \sigma$ in pLK admits any order of propositional rules applications leading to Tr.

Finally, note that since LK and LJ are sound and complete calculi, the obtained results permit to reduce the investigation the semantic characterization of classical and/or intuitionistic validity (of first-order formulas) to propositional deducibility satisfying certain conditions. Additionally, pay your attention to the fact that Theorem 1 can easily be transformed into some of sequent forms of Herbrand's theorem given in [7] for classical logic.

The rest of the paper is devoted to proving the Herbrand theorems. Only the syntactical approach is used at that.

5. Calculi LG, LK', and LJ'

For proving the main results, it is convenient to transform the cut-free calculi LK and LJ from [12] into calculi LK' and LJ' over $Sig \cup Var$ that are

$$\frac{\Gamma, A^k{\wedge}B \to \Delta}{\Gamma, A, B \to \Delta}\ (\wedge \to) \qquad \frac{\Gamma \to A^k{\wedge}B, \Delta}{\Gamma \to A, \Delta \quad \Gamma \to B, \Delta}\ (\to \wedge)$$

$$\frac{\Gamma, A^k{\vee}B \to \Delta}{\Gamma, A \to \Delta \quad \Gamma, B \to \Delta}\ (\vee \to)$$

$$\frac{\Gamma \to A^k{\vee}B, \Delta}{\Gamma \to A, \Delta}\ (\to \vee_1) \qquad \frac{\Gamma \to A^k{\vee}B, \Delta}{\Gamma \to B, \Delta}\ (\to \vee_2)$$

$$\frac{\Gamma, A^k \supset B \to \Delta}{\Gamma \to A, \Delta \quad \Gamma, B \to \Delta}\ (\supset\to) \qquad \frac{\Gamma \to A^k \supset B, \Delta}{\Gamma, A \to B, \Delta}\ (\to\supset)$$

$$\frac{\Gamma, {}^k\neg A \to \Delta}{\Gamma \to A, \Delta}\ (\neg \to) \qquad \frac{\Gamma \to {}^k\neg A, \Delta}{\Gamma, A \to \Delta}\ (\to \neg)$$

$$\frac{\Gamma, {}^k\forall {}^k\underline{x} A({}^k\underline{x}) \to \Delta}{\Gamma, A(t_k) \to \Delta}\ (\forall \to) \qquad \frac{\Gamma \to {}^k\forall {}^k\overline{x} A({}^k\overline{x}), \Delta}{\Gamma \to A({}^k\overline{x}), \Delta}\ (\to \forall)$$

$$\frac{\Gamma, {}^k\exists {}^k\overline{x} A({}^k\overline{x}) \to \Delta}{\Gamma, A({}^k\overline{x}) \to \Delta}\ (\exists \to) \qquad \frac{\Gamma \to {}^k\exists {}^k\underline{x} A({}^k\underline{x}), \Delta}{\Gamma \to A(t_k), \Delta}\ (\to \exists)$$

$$\frac{\Gamma, A \to \Delta}{\Gamma, A, {}^l A \to \Delta}\ (Con \to) \qquad \frac{\Gamma \to A, \Delta}{\Gamma \to A, {}^l A, \Delta}\ (\to Con)$$

$$\frac{\Gamma, A \to A, \Delta}{}\ (Ax)$$

In (Ax), A is an atomic formula. In $(Con \to)$ and $(\to Con)$, l denotes a new index w.r.t. an inference tree constructed before applications of $(Con \to)$ and $(\to Con)$. The rules $(\to \forall)$ and $(\exists \to)$ do not introduce any confusion w.r.t. the parameter ${}^k\overline{x}$ due to the convention about bound variables in an initial sequent. The term t_k satisfies the eigenvariable condition for ${}^k\underline{x}$ in both $(\to \exists)$ and $(\forall \to)$; moreover it contains only constants and functional symbols occurring in an initial sequent. This is the unique restriction for converting LG into LK'. But in the case of LJ', we must additionally require that the succedent of any sequent does not contain more than one formula; therefore, the rule $(\to Con)$ is redundant as well as Δ is an empty multiset in (Ax) and in all the rules eliminating logical connectives in succedents.

Figure 1: Calculi LG

adopted for using multisets as a succedent and an antecedent of any sequent. This permits to use the usual contraction rules (denoted by $(\to Con)$ and $(Con \to)$ here) as the only structural rules of LK' and LJ' since we have no restrictions to the number of formulas in succedents and antecedents of axioms in the case of classical logic and in the case of intuitionistic logic, we require the succedent of any sequent to contain no more than one formula.

The calculi LK' and LJ' are convenient to be determined with the help of a special calculus LG: LK' and LJ' are constructed from it by putting certain restrictions on LG.

Fig. 1 contains the description of LG. Remind that *inference trees in LG are applied "from top to bottom"* in order to attempt to construct a proof tree beginning with a sequent under consideration and finishing by axioms.

Another *distinctive feature* of LG concerns its axioms. Since we consider that any initial sequent (i.e. the sequent being investigated on deducibility) has the form $\rightarrow F$, where F is a closed formula, the process of the construction of a proof tree for it begins with the sequent $\rightarrow {}^1F$ called a *starting sequent* for LG and for calculi described below.

For the calculus LG, we have the following obvious result.

Proposition 2

For a formula F, a starting sequent $\rightarrow^1 F$ is deducible in the calculus LG in a way satisfying the restrictions for LJ' (for LK') if and only if an initial sequent $\rightarrow F$ is deducible in the calculus LJ (in LK) (in any sequent calculus coextensive with LJ (LK)) in the usual sense.

The process of inferring in LG is not essentially distinguished from inference search in both LK and LJ. That is why there are no examples for it; moreover, LG uses Gentzen's notion of admissibility, i.e. it contains the quantifier rules with the eigenvariable condition that leads to the great inefficiency of logical search. The calculus LB from the next section contains a certain idea how we can improve the efficiency of logical inference search in sequent (and tableau) calculi. (Also see [7], where some discussion of different notions of admissibility is made.)

6. Proofs of main results

To prove our main results, in this paper we follow the ideas of obtaining the Herbrand theorems suggested in [7, 8]. That is why we simply formulate "key" propositions giving the schemes of their proofs. So, all considerations concern the case of indexed formulas in general.

6.1. Basic calculus LB

An *equation* is an unordered pair of terms s and t written as $s \approx t$. Assume L is an atomc formula of the form $R(t_1, \ldots, t_n)$ and M is an atomic formula of the form $R(s_1, \ldots, s_n)$ where R is a predicate symbol and

$t_1, \dots, t_n, s_1, \dots, s_n$ are terms. Then $\Sigma(L, M)$ denotes the set of equations $\{t_1 \approx s_1, \dots, t_n \approx s_n\}$. In this case, L and M are said to be *equal modulo* $\Sigma(L, M)$ ($L \approx M$ *modulo* $\Sigma(L, M)$).

The calculus LB without the quantifier rules, as well as without $(Con \rightarrow)$ and $(\rightarrow Con)$ is denoted by pLB. (Note that de facto pLB is pLK when succedents and antecedents of sequents are considered as multisets.)

Proposition 3

For a closed formula F, the starting sequent $\rightarrow^1 F$ is deducible in the calculus LG satisfying the restrictions for LJ' (for LK') if and only if there exists an inference tree $Tr_{^1F}$ in LB such that below-given (1), (2), and (3) ((1) and (2)) hold:

(1) all the leaves of $Tr_{^1F}$ are quasi-axioms and there exists the mgsu σ of the sets of equations from all the quasi-axioms of Tr,
(2) σ is admissible for the set of all the sequences of $Tr_{^1F}$, and
(3) $Tr_{^1F}$ is compatible with σ.

A *proof* of Prop. 3 can be obtained on the base of the following arguments in a way analogous to the one used when proving Prop. 2 in [7] and adapted to the consideration of both the classical and intuitionistic cases (cf. [8]).

First of all note that for proving the proposition, the item (2) can not be taken into consideration on account of Prop. 1. It is present in the proposition since the examination of compatibility for the classical case becomes redundant as a whole; it is replaced by the verification of only admissibility as its part. The validity of this also is confirmed by the results of [7].

For proving the *sufficiency*, let $Tr_{^1F}$ be a proof tree for a starting sequent $\rightarrow {}^1F$ in the calculus LB, σ a substitution that unifies all the equations of leaves of $Tr_{^1F}$ compatible with σ. Without loss of generality, we can assume that terms of σ do not contain dummies (otherwise, they could be replaced by a constant, say, c_0) and that for any dummy from $Tr_{^1F}$, there exists the term t such that $x \mapsto t \in \sigma$.

Since $\blacktriangleleft_{\Xi,\sigma}$ is an antisymmetric relation (see Prop. 1), the following statement has place: $\blacktriangleleft_{\Xi,\sigma}$ can be completed to a *strict linear order relation* $\blacktriangleleft^+{}_{\Xi,\sigma}$ (that is $\blacktriangleleft^+{}_{\Xi,\sigma}$ is determined for any different pair of elements from $\blacktriangleleft_{\Xi,\sigma}$ and $\odot \blacktriangleleft_{\Xi,\sigma} \odot'$ implies $\odot \blacktriangleleft^+{}_{\Xi,\sigma} \odot'$).

Due to the statement, the principle connective of ${}^1F$ is the first element of the linear order $\blacktriangleleft^+{}_{\Xi,\sigma}$; moreover we can construct the tree $Tr_{^1F}$ applying rules of LB that subsequently eliminate the logical connectives in sequents

$$\frac{\Gamma, A^k \wedge B \to \Delta}{\Gamma, A, B \to \Delta}\ (\wedge \to) \qquad \frac{\Gamma \to A^k \wedge B, \Delta}{\Gamma \to A, \Delta \quad \Gamma \to B, \Delta}\ (\to \wedge)$$

$$\frac{\Gamma, A^k \vee B \to \Delta}{\Gamma, A \to \Delta \quad \Gamma, B \to \Delta}\ (\vee \to)$$

$$\frac{\Gamma \to A^k \vee B, \Delta}{\Gamma \to A, \Delta}\ (\to \vee_1) \qquad \frac{\Gamma \to A^k \vee B, \Delta}{\Gamma \to B, \Delta}\ (\to \vee_2)$$

$$\frac{\Gamma, A^k \supset B \to \Delta}{\Gamma \to A, \Delta \quad \Gamma, B \to \Delta}\ (\supset \to) \qquad \frac{\Gamma \to A^k \supset B, \Delta}{\Gamma, A \to B, \Delta}\ (\to \supset)$$

$$\frac{\Gamma, {}^k\neg A \to \Delta}{\Gamma \to A, \Delta}\ (\neg \to) \qquad \frac{\Gamma \to {}^k\neg A, \Delta}{\Gamma, A \to \Delta}\ (\to \neg)$$

$$\frac{\Gamma, {}^k\forall {}^k\underline{x} A({}^k\underline{x}) \to \Delta}{\Gamma, A({}^k\underline{x}) \to \Delta}\ (\forall \to) \qquad \frac{\Gamma \to {}^k\forall {}^k\overline{x} A({}^k\overline{x}), \Delta}{\Gamma \to A({}^k\overline{x}), \Delta}\ (\to \forall)$$

$$\frac{\Gamma, {}^k\exists {}^k\overline{x} A({}^k\overline{x}) \to \Delta}{\Gamma, A({}^k\overline{x}) \to \Delta}\ (\exists \to) \qquad \frac{\Gamma \to {}^k\exists {}^k\underline{x} A({}^k\underline{x}), \Delta}{\Gamma \to A({}^k\underline{x}), \Delta}\ (\to \exists)$$

$$\frac{\Gamma, A \to \Delta}{\Gamma, A, {}^lA \to \Delta}\ (Con \to) \qquad \frac{\Gamma \to A, \Delta}{\Gamma \to A, {}^lA, \Delta}\ (\to Con)$$

$$\frac{\Gamma, A \to A', \Delta}{}\ (QuAx)$$

In $(Con \to)$ and $(\to Con)$, l denotes a new index w.r.t. an inference tree constructed before applications of $(Con \to)$ and $(\to Con)$. In a quasi-axiom $(QuAx)$, $A \approx A'$ *modulo* $\Sigma(A, A')$, where A and A' are atomic formulas. In compliance with the above-given convention about parameters and dummies, ${}^k\overline{x}$ always is considered as a parameter in the rules $(\to \forall)$ and $(\exists \to)$; ${}^k\underline{x}$ always is considered as a dummy in the rules $(\to \exists)$ and $(\forall \to)$. There are no other restrictions in the classical case. But in the intuitionistic case, the succedent of any sequent does not contain more than one formula and, therefore, the rule $(\to Con)$ is redundant as well as Δ is an empty multiset in $(QuAx)$ and in all the rules eliminating logical connectives in succedents.

Figure 2: Calculi LB

under consideration in accordance with strict linear order $\blacktriangleleft_{\Xi,\sigma}$; such a possibility is guaranteed by the statement and by that any rule of any sequent calculus under consideration always eliminates the principle connective of a formula in a sequent which the rule is applied to).

Since the calculi LG and LB are distinguished by the quantifier rules only, it is easy to transform Tr_{1F} to a tree Tr'_{1F} for the starting sequent $\to {}^1F$ by replacing applications of the quantifier rules $(\forall \to)$ and $(\to \exists)$ of

LB by appropriate applications of quantifier rules $(\forall \to)$ and $(\to \exists)$ of LG in accordance with the following instructions: in $Tr_{^1F}$, every rule application

$$\frac{\Gamma, {}^k\forall^k\underline{x}A({}^k\underline{x}) \to \Delta}{\Gamma, A({}^k\underline{x}) \to \Delta}\ (\forall \to) \qquad \Big(\frac{\Gamma \to {}^k\exists^k\underline{x}A({}^k\underline{x}), \Delta}{\Gamma \to A({}^k\underline{x}), \Delta}\ (\to \exists)\Big)$$

is replaced by the rule application

$$\frac{\Gamma, {}^k\forall^k\underline{x}A({}^k\underline{x}) \to \Delta}{\Gamma, A(t_k) \to \Delta}\ (\forall \to) \qquad \Big(\frac{\Gamma \to {}^k\exists^k\underline{x}A({}^k\underline{x}), \Delta}{\Gamma \to A(t_k), \Delta}\ (\to \exists)\Big)$$

where ${}^kx \mapsto t_k \in \sigma$. Note that in two last rules, t_k satisfies the eigenvariable condition for ${}^kx$ in Tr'_{1F} because of the compatibility of $Tr_{^1F}$ with σ (and, as a result, the admissibility of σ for the set of all the sequents in $Tr_{^1F}$). Thus, Tr'_{1F} is a proof tree or $\to^1 F$ in LG and the sufficiency is proved.

To prove the *necessity*, we must perform the "converse" transformation. This means that in a proof tree Tr'_{1F} for a starting sequent $\to {}^1F$, every application of the rule $(\forall \to)$ $((\to \exists))$:

$$\frac{\Gamma, {}^k\forall^k\underline{x}A({}^k\underline{x}) \to \Delta}{\Gamma, A(t_k) \to \Delta}\ (\forall \to) \qquad \Big(\frac{\Gamma \to {}^k\exists^k\underline{x}A({}^k\underline{x}), \Delta}{\Gamma \to A(t_k), \Delta}\ (\to \exists)\Big)$$

is replaced by the rule application:

$$\frac{\Gamma, {}^k\forall^k\underline{x}A({}^k\underline{x}) \to \Delta}{\Gamma, A({}^k\underline{x}) \to \Delta}\ (\forall \to) \qquad \Big(\frac{\Gamma \to {}^k\exists^k\underline{x}A({}^k\underline{x}), \Delta}{\Gamma \to A({}^k\underline{x}), \Delta}\ (\to \exists)\Big)$$

"preserving" all the other rules application without any modification, which leads to an inference tree $Tr_{^1F}$ in LB containing only leaves with quasi-axioms.

If σ' is determined as the set containing all the ${}^kx \mapsto t_k$ from Tr'_{1F} and only them, then obviously, σ' is a unifier of the sets of equations from all the quasi-axioms of $Tr_{^1F}$. By the main property of unifiers, we conclude that there is a mgsu σ of the sets of equations from all the quasi-axioms of $Tr_{^1F}$. Since Tr'_{1F} is a proof tree in LG and the eigenvariable condition is satisfied for every ${}^kx \mapsto t_k \in \sigma'$, the substitution σ is admissible for the set of all the sequents in $Tr_{^1F}$ and, therefore, $Tr_{^1F}$ is compatible with σ by Prop. 1. This *finishes* the proof of Prop. 3.

Example 4. Let us establish the deducibility of the stating sequent $\to {}^1F^*$ in LG (denoted by S^*) by constructing a tree $Tr_{^1F^*}$ satisfying Prop. 3.

$$\underline{\to {}^1\exists^1\overline{y}\,{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y}))\ {}^1\supset{}^1\neg{}^1\forall^1\underline{y}'\,{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}')}\qquad (\to {}^1F^*)$$

$$\underline{{}^1\exists^1\overline{y}\,{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})) \to {}^1\neg{}^1\forall^1\underline{y}'\,{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}')}\qquad (\text{by } (\to\supset))$$

$$\underline{{}^1\exists^1\overline{y}\,{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})), {}^1\forall^1\underline{y}'\,{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \to}\qquad (\text{by } (\to \neg))$$

$$\underline{{}^1\exists^1\overline{y}\,{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})), {}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \to}\qquad (\text{by } (\forall \to)$$

$$\underline{{}^1\neg{}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y})), {}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \to}\qquad (\text{by } (\exists \to)$$

$$\underline{{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \to {}^1\exists^1\underline{x}P({}^1\underline{x}, f({}^1\overline{y}))}\qquad (\text{by } (\neg \to)$$

$$\underline{{}^1\exists^1\overline{x}'P({}^1\overline{x}',{}^1\underline{y}') \to P({}^1\underline{x}, f({}^1\overline{y}))}\qquad (\text{by } (\to \exists))$$

$$P({}^1\overline{x}',{}^1\underline{y}') \to P({}^1\underline{x}, f({}^1\overline{y}))\qquad ((QuAx), \text{ by } (\exists \to))$$

Draw your attention to the fact that the order of quantifier rule applications in LB is immaterial and has no influence on the final result on deducibility.

The constructed tree contains the only one list being a quasi-axiom. For its left atomic formula A and right atomic formula A', have $A \approx A'$ *modulo* $\Sigma(A, A')$, where $\Sigma(A, A') = \{{}^1\overline{x}' \approx {}^1\underline{x}, {}^1\underline{y}' \approx f({}^1\overline{y})\}$

The substitution $\sigma^* = \{{}^1\underline{x} \mapsto {}^1\overline{x}', {}^1\underline{y}' \mapsto f({}^1\overline{y})\}$ is the unique mgsu of $\Sigma(A, A')$. In section 3 it was proven that σ^* is an admissible substitution for F^* and, as a result, for the set of all the sequents of $Tr_{{}^1F^*}$. Obviously, $Tr_{{}^1F^*}$ is compatible with σ^*. By Prop. 3, the starting sequent S^* is deducible in LG satisfying the restrictions for LJ' (and, of course, for LK'). Therefore, the initial sequent $\to F^*$ is deducible in both LJ and LK.

Example 5. If we consider the formula ${}^1F^{**}$: ${}^1\neg\ {}^1\forall^1\overline{y}\ {}^1\exists^1\underline{x}\ P({}^1\underline{x}, f({}^1\overline{y}))$ ${}^1\supset\ {}^1\exists^1\underline{y}'\ {}^1\neg\ {}^1\exists^1\overline{x}'\ P({}^1\overline{x}',{}^1\underline{y}')$ instead of ${}^1F^*$, it is easy to construct an inference tree $Tr_{{}^1F^{**}}$ for the starting sequent $\to {}^1F^{**}$ such that (1) and (2) from Prop. 2 hold. But it is impossible for the item (3) to take place for both $Tr_{{}^1F^{**}}$ and any other inference tree for $\to {}^1F^{**}$ independent of a generated mgsu at that. By Prop. 3 and Prop. 2, we have that the initial sequent $\to F^{**}$ is deducible in LK but not deducible in LJ.

Example 6. If we consider the formula ${}^1F^{***}$: ${}^1\forall^1\overline{y}\ {}^1\neg{}^1\forall^1\underline{x}\ P({}^1\underline{x}, f({}^1\overline{y}))$ ${}^1\supset\ {}^1\neg{}^1\forall^1\underline{y}'\ {}^1\exists^1\overline{x}'\ P({}^1\overline{x}',{}^1\underline{y}')$, we can prove that for any inference tree $Tr_{{}^1F^{***}}$ for the starting sequent $\to {}^1F^{***}$ it is impossible to construct a mgsu admissible for all the sequents of $Tr_{{}^1F^{***}}$. Therefore, by Prop. 3 and Prop. 2, the initial sequent $\to F^{***}$ cannot be deduced even in LK.

6.2. Convolution of inference trees

This section explores the idea suggested in [7] and modified for the "convolution" ("reduction") of a inference tree in LB into a certain sequent

$$\frac{\Gamma, \iota(A) \wedge \iota(B) \to \Delta}{\Gamma, \iota(A), \iota(B) \to \Delta}\ (\uparrow \wedge \to) \qquad \frac{\Gamma \to \iota(A) \wedge \iota(B), \Delta}{\Gamma \to \iota(A), \Delta \quad \Gamma \to \iota(B), \Delta}\ (\uparrow \to \wedge)$$

$$\frac{\Gamma, \iota(A) \vee \iota(B) \to \Delta}{\Gamma, \iota(A) \to \Delta \quad \Gamma, \iota(B) \to \Delta}\ (\uparrow \vee \to)$$

$$\frac{\Gamma \to \iota(A) \vee \iota(B), \Delta}{\Gamma \to \iota(A), \Delta}\ (\uparrow \to \vee_1) \qquad \frac{\Gamma \to \iota(A) \vee \iota(B), \Delta}{\Gamma \to \iota(B), \Delta}\ (\uparrow \to \vee_2)$$

$$\frac{\Gamma, \iota(A) \supset \iota(B) \to \Delta}{\Gamma \to \iota(A), \Delta \quad \Gamma, \iota(B) \to \Delta}\ (\uparrow \supset \to) \qquad \frac{\Gamma \to \iota(A) \supset \iota(B), \Delta}{\Gamma, \iota(A) \to \iota(B), \Delta}\ (\uparrow \to \supset)$$

$$\frac{\Gamma, \neg\iota(A) \to \Delta}{\Gamma \to \iota(A), \Delta}\ (\uparrow \neg \to) \qquad \frac{\Gamma \to \neg\iota(A), \Delta}{\Gamma, \iota(A) \to \Delta}\ (\uparrow \to \neg)$$

$$\frac{\Gamma, \forall^k \underline{x}\iota(A(^k\underline{x})) \to \Delta}{\Gamma, \iota(A(^k\underline{x})) \to \Delta}\ (\uparrow \forall \to) \qquad \frac{\Gamma \to \forall^k \overline{x}\iota(A(^k\overline{x})), \Delta}{\Gamma \to \iota(A(^k\overline{x}))\Delta}\ (\uparrow \to \forall)$$

$$\frac{\Gamma, \exists^k \overline{x}\iota(A(^k\overline{x})) \to \Delta}{\Gamma, \iota(A(^k\overline{x})) \to \Delta}\ (\uparrow \exists \to) \qquad \frac{\Gamma \to \exists^k \underline{x}\iota(A(^k\underline{x})), \Delta}{\Gamma \to \iota(A(^k\underline{x})), \Delta}\ (\uparrow \to \exists)$$

$$\frac{\Gamma, \iota(A) \wedge {}^k\iota(A) \to \Delta}{\Gamma, \iota(A), {}^k\iota(A) \to \Delta}\ (\uparrow Con \to) \qquad \frac{\Gamma \to \iota(A) \vee {}^k\iota(A), \Delta}{\Gamma \to \iota(A), {}^k\iota(A), \Delta}\ (\uparrow \to Con)$$

There are no restrictions in the classical case. But in the intuitionistic case, the succedent of any sequent does not contain more than one formula and, therefore, the rule $(\to Con)$ is redundant as well as Δ is an empty multiset in $(QuAx)$ and in all the rules "restored" logical connectives in succedents when reading rules "from bottom to up".

Figure 3: Convolution Calculus for LB

that can be modified after this into a sequent deduced by applying the only (propositional) rules of pLB. This gives us a possibility to obtain the main results of the paper – the "syntactical" forms of Herbrand theorems.

Let Tr be an inference tree for a starting sequent ${}^1S_0$ of the form $\to^1 F$ in the calculus LB, where F is an original closed formula. To every sequent S in Tr, we assign the sequent $\iota_{Tr}(S)$ or simply $\iota(S)$ (an analogue of the expression called the *spur* of S in [7]), as follows:

– If S is a leaf of Tr, then $\iota(S)$ is the result of deleting all the upper-left indices in logical connectives occurring in S.

– If S is not a leaf node and spurs are assigned to all its successors in an inference rule of LB, we assign $\iota(S)$ to S in accordance with the rules of the convolution calculus. Note that if a rule R of the calculus LB is applied to the sequent S in Tr ("from top to bottom"), the spur is assigned to S as prescribed by the rule R of the convolution calculus applied "bottom up".

Further, $\iota(Tr)$ denotes the result of the replacement of every sequent S from Tr by $\iota(S)$.

The following properties of proof trees can easily be proved by induction on the number of rules applications.

Proposition 4

Let Tr be an inference tree for a stating sequent ${}^1S_0$ in the calculus LB. Suppose all the leaves of Tr are quasi-axioms and there exists the mgsu σ of all the sets of equations from the quasi-axioms of Tr. If σ' denotes the result of the deletion all the upper-left indices in σ then the following properties hold w.r.t. $\iota(Tr)$, $\iota_{Tr}({}^1S_0)$, σ, and σ':

1) $\iota(Tr)$ is an inference tree for the initial (original) sequent $\iota_{Tr}({}^1S_0)$ in LB;

2) All the leaves of $\iota(Tr)$ are quasi-axioms containing only equations unified by the substitution σ';

3) $\mu(\iota(Tr)) \cdot \sigma'$ can be considered as an inference tree in the calculus pLB for the initial sequent $\mu(\iota_{Tr}({}^1S_0)) \cdot \sigma'$;

4) σ is admissible for the set of all the sequents of Tr if and only if σ' is admissible for $\iota_{Tr}({}^1S_0)$;

5) $\iota(Tr)$ is compatible with σ' if and only if Tr is compatible with σ.

Example 7. For $\rightarrow^1 F^*$ and σ^* from the Example 4, we have: $\iota_{Tr}(\rightarrow^1 F^*)$ is $\exists y \neg \exists x P(x, f(y)) \supset \neg \forall y' \exists x' P(x', y')$, $\mu(\iota_{Tr}(\rightarrow^1 F^*)$ is $\neg P(\underline{x}, f(\overline{y})) \supset \neg P(\overline{x}', \underline{y}')$, and $\sigma^{*\prime} = \{\underline{x} \mapsto \overline{x}', \underline{y}' \mapsto f(\overline{y})\}$.

For $Tr_{{}^1F^*}$, $\iota(Tr_{\rightarrow Tr_{{}^1F^*}})$ presents the tree

$$\begin{array}{ll}
\underline{\rightarrow \exists \overline{y} \neg \exists \underline{x} P(\underline{x}, f(\overline{y})) \supset \neg \forall \underline{y}' \exists \overline{x}' P(\overline{x}', \underline{y}')} & (\rightarrow F^*) \\
\underline{\exists \overline{y} \neg \exists \underline{x} P(\underline{x}, f(\overline{y})) \rightarrow \neg \forall \underline{y}' \exists \overline{x}' P(\overline{x}', \underline{y}')} & (\text{by } (\rightarrow\supset)) \\
\underline{\exists \overline{y}^1 \neg \exists \underline{x} P(\underline{x}, f(\overline{y})), \forall \underline{y}' \exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow} & (\text{by } (\rightarrow \neg)) \\
\underline{\exists \overline{y} \neg \exists \underline{x} P(\underline{x}, f(\overline{y})), \exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow} & (\text{by } (\forall \rightarrow) \\
\underline{\neg \exists \underline{x} P(\underline{x}, f(\overline{y})), \exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow} & (\text{by } (\exists \rightarrow) \\
\underline{\exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow \exists \underline{x} P(\underline{x}, f(\overline{y}))} & (\text{by } (\neg \rightarrow) \\
\underline{\exists \overline{x}' P(\overline{x}', \underline{y}') \rightarrow P(\underline{x}, f(\overline{y}))} & (\text{by } (\rightarrow \exists)) \\
P(\overline{x}', \underline{y}') \rightarrow P(\underline{x}, f(\overline{y})) & ((QuAx), \text{ by } (\exists \rightarrow))
\end{array}$$

The tree $\iota(Tr_{\to Tr_{1F^*}})$ can be transformed into the tree $\mu(\iota(Tr_{\to Tr_{1F^*}}))$:

$$\cfrac{\cfrac{\cfrac{\to\neg P(\underline{x}, f(\overline{y}))\supset\neg P(\overline{x}', \underline{y}') \quad (\to \iota F^*)}{\neg P(\underline{x}, f(\overline{y}))\to\neg P(\overline{x}', \underline{y}') \quad (\text{by } (\to\supset))}}{{}^1\neg P(\underline{x}, f(\overline{y})), P(\overline{x}', \underline{y}')\to \quad (\text{by } (\to\neg))}}{P(\overline{x}', \underline{y}')\to P(\underline{x}, f(\overline{y})) \quad ((QuAx), \text{ by } (\neg\to))}$$

It is easy to check that items (1)–(5) from Prop. 4 take place w.r.t. $\iota_{Tr}(\to^1 F^*)$, $\mu(\iota(Tr_{\to Tr_{1F^*}}))$, σ^*, and $\sigma^{*\prime}$.

6.3. Proving Herbrand theorems

"Summarizing" the results of section 6.1 and 6.2, we obtain the "key statement" leading to the Herbrand theorems.

Proposition 5 ("Key statement")

A sequent ${}^1S_0$ of the form $\to^1 F$ is deducible in the calculus LG, where F is a closed original formula, if and only if there are an inference tree Tr for ${}^1S_0$ and a substitution σ of terms from the Herbrand quasi-universe $HQ(\iota_{Tr}({}^1S_0))$ for all the dummies in $\iota_{Tr}({}^1S_0)$ such that

- (i) $\mu(\iota(Tr))\cdot\sigma$ is a proof tree in the calculus pLB for the initial sequent $\mu(\iota_{Tr}({}^1S_0))\cdot\sigma$;
- (ii) σ is an admissible substitution for $HE(F)$;
- (iii) the tree $\mu(\iota(Tr))\cdot\sigma$ is a compatible with σ.

Indeed, Prop. 3 and 4 provide the truth of items (i), (ii), and (iii) since $\mu(\iota_{Tr}({}^1S_0))$ coincides with $HE(F)$. The requirement about taking terms from $HQ(\iota_{Tr}(S))$ is provided by the subformula property of LB (relating to terms of the quantifier rules) and the mgsu properties.

Example 8. Let ${}^1S_0$ denote the sequent $\to {}^1F^*$ and σ the substitution σ^* from the Example 7. If $\mu(\iota(Tr))$ is the tree $\mu(\iota(Tr_{\to Tr_{1F^*}}))$ taking from the Example 7 too, then all the items of Prop. 5 hold. Thus, the sequent is deducible in LG.

Now, it is easy to convert the "key statement" to the Herbrand theorems.

First of all note that the construction of sequent $\iota_{Tr}({}^1S_0)$ for a sequent ${}^1S_0$ of the form $\to^1 F$, in accordance with the convolution, calculus produces a certain Herbrand extension $HE(F)$ being the succedent of $\iota_{Tr}({}^1S_0)$.

Moreover, when constructing a tree $\iota(Tr)$ for $\iota_{Tr}({}^1S_0)$ by means of only rules of the calculus LJ', we obtain that $\iota_{Tr}({}^1S_0)$ represents itself an intuitionistic Herbrand extension. Therefore, Theorem 2 holds in accordance with Prop. 2, 3, and 5.

The proof of Theorem 1 can be obtained in the same way, if we take into account the fact that checking compatibility is unnecessary for classical logic, since any step of the reduction of any formula to its prenex normal form have no influence on the deducibility of the formula in LK'. (Obviously, $\iota_{Tr}({}^1S_0)$ represents itself a "full" Herbrand extension, which cannot obligatory be intuitionistic.)

If we compare the results of this paper with the results of some of papers being reminded in the introduction, we observe that in contrast to [17] and to papers based on it, Prop. 1 asserts that for checking admissibility in both the classical and the intuitionistic cases, we can restrict ourselves only by consideration of quantifier structures of a formula (of a sequent) investigating on deducibility, i.e. by examining $\lhd_{F,\sigma}$ ($\lhd_{S,\sigma}$) only but not $\blacktriangleleft_{F,\sigma}$ ($\blacktriangleleft_{S,\sigma}$).

7. Conclusion

The paper presents author's results on Herbrand theorems for the sequent form of first-order classical and intuitionistic logics. The sequent formalism under consideration permitted to present the unified approach to wording and proving the Herbrand forms suggested here. Besides, it also gave a possibility to achieve enough general considerations: many famous Herbrand theorem forms for classical logic can be produced as its applications. Additionally note that obtained proofs have a transparent character and are connected with deducibility only.

The approach suggested in the paper is based on the development of the special technique of inference search in sequent calculi that has relatively high efficiency in comparison with the traditional methods based on Gentzen's or Kanger's notions of admissibility of substitutions. It can be improved in the direction of the optimization of rules applications, for example, in ways similar to that was explored for constructing the sequent calculi in [7] for classical logic and in [11, 8] for intuitionistic logic.

Another positive feature of the approach is that it may give a possibility for redescribing matrix characterization methods ([17]) and different modifications of the connection method ([16], [18], [19]) in its notions and notations. It may also be applied for constructing efficient methods of inference search in modal and other logics.

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