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TOWARD THEORY OF *P*-ADIC VALUED PROBABILITIES

We present a short review on generalization of probability theory in that probabilities take values in the fields of p-adic numbers, $\mathbf{Q}_p$. Such probabilities were introduced to serve p-adic theoretical physics. In some quantum physical models a wave function (which is a complex probability amplitude in ordinary QM) takes vales in $\mathbf{Q}_p$ (for some prime number p) or their quadratic extensions. Such a wave function can be interpreted probabilistically in the framework of p-adic probability theory. This theory was developed by using both the frequency approach (by generalizing von Mises) and the measure-theoretic approach (by generalizing Kolmogorov). In particular, some limit theorems were obtained. However, theory of limit theorems for p-adic valued probabilities is far from being completed. Another interesting domain of research is corresponding theory of complexity. We obtained some preliminary results in this direction. However, it is again far from to be completed. Recently p-adic models of classical statistical mechanics were considered and some preliminary results about invariant p-adic valued measures for dynamical systems were obtained.

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1. Introduction

This paper is a short review on generalization of probability theory with probabilities taking values in $\mathbf{Q}_p$. The main attention will be paid to limit theorems and p-adic theory of recursive statistical tests generalizing Martin-Löf's theory. We start with recollection of formalization of theory with real valued probabilities – taking values in the segment [0,1] of the real line.

Since the creation of the modern probabilistic axiomatics by A. N. Kolmogorov [1] in 1933, probability theory was merely reduced to the theory of normalized σ-additive measures taking values in the segment [0,1] of the field of real numbers $\mathbf{R}$. In particular, the main competitor of Kolmogorov's measure-theoretic approach, von Mises' frequency approach to proba-

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bility [2], [3], practically totally disappeared from the probabilistic arena. On one hand, this was a consequence of difficulties with von Mises' definition of randomness (via place selections), see e.g., [4]–[6].[1] On the other hand, von Mises' approach (as many others) could not compete with precisely and simply formulated Kolmogorov's theory.

We mentioned von Mises' approach not only, because its attraction for applications, but also because von Mises' model with frequency probabilities played the important role in the process of formulation of the conventional axiomatics of probability theory.

We would like to mention that Kolmogorov's (as well as von Mises') assumptions were also based on a fundamental, but hidden, assumption: *Limiting behavior of relative frequencies is considered with respect to one fixed topology on the field of rational numbers* $\mathbf{Q}$*, namely, the real topology.* In particular, the consideration of this asymptotic behavior implies that probabilities belong to the field real numbers $\mathbf{R}$.

However, it is possible to study asymptotic behavior of relative frequencies (which are always rational numbers) in other topologies on field of rational numbers $\mathbf{Q}$. In this way we derive another probability-like structure that recently appeared in theoretical physics. This is so called *p-adic probability.*

We recall that p-adic numbers are applied intensively in different domains of physics – quantum logic, string theory, cosmology, quantum mechanics, quantum foundations, see, e.g., [8]–[12], dynamical systems [13], [11], [14], biological and cognitive models [11], [14]–[16].

P-adic valued probabilities were introduced in [23]–[25], [7] to serve p-adic theoretical physics [8]–[12]. In some quantum physical models [10], [11] a wave function (which is a complex probability amplitude in ordinary QM) takes vales in $\mathbf{Q}_p$ (for some prime number p) or its quadratic extensions. Such a wave function can be interpreted probabilistically in the framework of p-adic probability theory. This theory was developed by using both the frequency approach (by generalizing von Mises) and the measure-theoretic approach (by generalizing Kolmogorov). In particular, some limit theorems were obtained. However, theory of limit theorems for p-adic valued probabilities is far from being completed. Another interesting domain of research is corresponding theory of complexity. We obtained some preliminary results in this direction. However, it is again far from being completed. Recently p-adic models of classical statistical mechanics were consi-

[1] However, see also [7], where von Mises' approach was simplified, generalized, and then fruitfully applied to theoretical physics.

dered and some preliminary results about invariant p-adic valued measures for dynamical systems were obtained.

2. p-adic Numbers

The field of real numbers $\mathbf{R}$ is constructed as the completion of the field of rational numbers $\mathbf{Q}$ with respect to the metric $p(x,y) = |x-y|$, where $|\cdot|$ is the usual valuation given by the absolute value. The fields of p-adic numbers $\mathbf{Q}_p$ are constructed in a corresponding way, but by using other valuations. For a prime number p the p-adic valuation $|\cdot|_p$ is defined in the following way. First we define it for natural numbers. Every natural number n can be represented as the product of prime numbers, $n = 2^{r_2}3^{r_3}\ldots p^{r_p}\ldots$, and we define $|n|_p = p^{-r_p}$, writing $|0|_p = 0$ and $|-n|_p = |n|_p$. We then extend the definition of the p-adic valuation $|\cdot|_p$ to all rational numbers by setting $|n/m|_p = |n|_p/|m|_p$ for $m \neq 0$. The completion of $\mathbf{Q}$ with respect to the metric $\rho_p(x,y) = |x-y|_p$ is the locally compact field of p-adic numbers $\mathbf{Q}_p$.

The number fields $\mathbf{R}$ and $\mathbf{Q}_p$ are unique in a sense, since by Ostrovsky's theorem, see e.g., [26], $|\cdot|$ and $|\cdot|_p$ are the only possible valuations on $\mathbf{Q}$, but have quite distinctive properties. The field of real numbers $\mathbf{R}$ with its usual valuation satisfies $|n| = n \to \infty$ for valuations of natural numbers n and is said to be *Archimedian*. By a well known theorem of number theory [26] the only complete Archimedian fields are those of the real and the complex numbers. In contrast, the fields of p-adic numbers, which satisfy $|n|_p \leq 1$ for all $n \in N$, are examples of *non-Archimedian* fields.

Unlike the absolute value distance $|\cdot|$, the p-adic valuation satisfies the strong tringle inequality:

$$|x+y|_p \leq \max[|x|_p, |y|_p], x, y \in \mathbf{Q}_p.$$

Consequently the p-adic metric satisfies the strong triangle inequality $\rho_p(x,y) \leq \max[\rho_p(x,z), \rho_p(z,y)], x, y, z \in \mathbf{Q}_p$, which means that the metric ρ_p is an *ultrametric*, [26]. Write $U_r(a) = \{x \in \mathbf{Q}_p : |x-a|_p \leq r\}$, where $r = p^n$ and $n = 0, \pm 1, \pm 2, \ldots$ These are the "closed" balls in $\mathbf{Q}_p$ while the sets $\mathbf{S}_r(a) = \{x \in \mathbf{Q}_p : |x-a|_p = r\}$ are the spheres in $\mathbf{Q}_p$ of such radii r. These sets (balls and spheres) have a somewhat strange topological structure from the viewpoint of our usual Euclidian intuition: they are both open and closed at the same time, and as such are called *clopen* sets. Finally, any p-adic ball $U_r(0)$ is an additive subgroup of $\mathbf{Q}_p$, while the ball $U_1(0)$ is also a ring, which is called the *ring of p-adic integers* and is denoted by $\mathbf{Z}_p$.

The p-adic *exponential function* $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$. The series converges in $\mathbf{Q}_p$ if $|x|_p \leq r_p$, where $r_p = 1/p, p \neq 2$ and $r_2 = 1/4$. p-adic trigonometric functions $\sin x$ and $\cos x$ are defined by the standard power series. These series have the same radius of convergence r_p as the exponential series.

3. p-adic Frequency Probability Model

As in the ordinary probability theory [2], [3], the first p-adic probability model was the frequency one, [23]–[25], [7], [9], [10]. This model was based on the simple remark that relative frequencies $\nu_N = \frac{n}{N}$ always belong to the field of rational numbers $\mathbf{Q}$. And $\mathbf{Q}$ can be considered as a (dense) subfield of $\mathbf{R}$ as well as $\mathbf{Q}_p$ (for each prime number p). Therefore behaviour of sequences $\{\nu_N\}$ of (rational) relative frequencies can be studied not only with respect to the real topology on Q, but also with respect to any p-adic topology on Q. Roughly speaking a p-adic probability (as real von Mises' probability) is defined as:

$$\mathbf{P}(\alpha) = \lim_{N} \nu_N(\alpha). \tag{1}$$

Here α is some label denoting a result of a statistical experiment. Denote the set of all such labels by the symbol Ω. In the simplest case $\Omega = \{0, 1\}$. Here $\nu_N(\alpha)$ is the relative frequency of realization of the label α in the first N trials. The $\mathbf{P}(\alpha)$ is the frequency probability of the label α.

The main p-adic lesson is that it is impossible to consider, as we did in the real case, limits of the relative frequencies ν_N when the $N \to \infty$. Here the point "∞" belongs, in fact, to the real compactification of the set of natural numbers. So $|N| \to \infty$, where $|\cdot|$ is the real absolute value. The set of natural numbers $\mathbf{N}$ is bounded in $\mathbf{Q}_p$ and it is densely embedded into the ring of p-adic integers $\mathbf{Z}_p$ (the unit ball of $\mathbf{Q}_p$). Therefore sequences $\{N_k\}_{k=1}^{\infty}$ of natural numbers can have various limits $m = \lim_{k\to\infty} N_k \in \mathbf{Z}_p$.

In the p-adic frequency probability theory we proceed in the following way to provide the rigorous mathematical meaning for the procedure (1), see [24], [25]. We fix a p-adic integer $m \in \mathbf{Z}_p$ and consider the class, L_m, of sequences of natural numbers $s = \{N_k\}$ such that $\lim_{k\to\infty} N_k = m$ in $\mathbf{Q}_p$.

Let us consider the fixed sequence of natural numbers $s \in L_m$. We define a p-adic s-probability

$$\mathbf{P}(\alpha) = \lim_{k\to\infty} \nu_{N_k}(\alpha), s = \{N_k\}.$$

This is the limit of relative frequencies with respect to the fixed sequence $s = \{N_k\}$ of natural numbers. For any subset A of the set of labels Ω, we define its s-probability as

$$\mathbf{P}(A) = \lim_{k\to\infty} \nu_{N_k}(A), s = \{N_k\},$$

where $\nu_{N_k}(A)$ is the relative frequency of realization of labels α belonging to the set A in the first N trials. As $\mathbf{Q}_p$ is an additive topological semigroup (as well as $\mathbf{R}$), we obtain that the p-adic probability is additive:

Theorem 3.1

$$\mathbf{P}(A_1 \cup A_2) = \mathbf{P}(A_1) + \mathbf{P}(A_2), A_1 \cap A_2 = \emptyset. \tag{2}$$

As $\mathbf{Q}_p$ is even an additive topological group (as well as $\mathbf{R}$), we get that

Theorem 3.2

$$\mathbf{P}(A_1 \setminus A_2) = \mathbf{P}(A_1) - \mathbf{P}(A_1 \cap A_2). \tag{3}$$

Trivially, for any sequence $s = \{N_k\}$, $\mathbf{P}(\Omega) = \lim_{k\to\infty}\nu_{N_k}(\Omega) = 1$, as $\nu_N(\Omega) = \frac{N}{N} = 1$ for any N. As $\mathbf{Q}_p$ is a multiplicative topological group (as well as $\mathbf{R}$), we get (see von Mises [2], [3] for the real case and [7] for the p-adic case) Bayes' formula for conditional probabilities:

Theorem 3.3

$$\mathbf{P}(A|B) = \lim_{k\to\infty} \frac{\nu_{N_k}(A \cap B)}{\nu_{N_k}(A)} = \frac{\mathbf{P}(A \cap B)}{\mathbf{P}(A)}, \quad \mathbf{P}(A) \neq 0. \tag{4}$$

As we know, frequency probability played the crucial role in conventional probability theory in determination of the range of values (namely, the segment [0,1]) of a probabilistic measure, see remarks on von Mises' theory in Kolmogorov's book [1]. Frequencies always lie between zero and one. Thus their limits (with respect to the real topology) belong to the same range.

In the p-adic case we can proceed in the same way. Let $r \equiv r_m = \frac{1}{|m|_p}$ (where $r = \infty$ for $m = 0$). We can easily get, see [23], [24], that for the p-adic frequency s-probability, $s \in L_m$, the values of $\mathbf{P}$ always belong to the p-adic ball $U_r(0) = \{x \in \mathbf{Q}_p : |x|_p \leq r\}$. In the p-adic probabilistic model such a ball $U_r(0)$ plays the role of the segment [0,1] in the real probabilistic model.

4. Measure-Theoretic Approach

As in the real case, the structure of an additive topological group of $\mathbf{Q}_p$ induces the main properties of probability that can be used for the axiomatization in the spirit of Kolmogorov, [1]. Let us fix $r = p^{\pm l}, l = 0, 1, \ldots,$ or $r = \infty$.

Axiomatics. *Let Ω be an arbitrary set (a sample space) and let F be a field of subsets of Ω (events). Finally, let $\mathbf{P} : F \to U_r(0)$ be an additive function (measure) such that $\mathbf{P}(\Omega) = 1$. Then the triple $(\Omega, F, \mathbf{P})$ is said to be a p-adic r-probabilistic space and $\mathbf{P}$ p-adic r-probability.*

Following to Kolmogorov we should find some technical mathematical restriction on $\mathbf{P}$ that would induce fruitful integration theory and give the possibility to define averages. Kolmogorov (by following Borel, Lebesque, Lusin, and Egorov) proposed to consider the σ-additivity of measures and the σ-structure of the field of events. Unfortunately, in the p-adic case the situation is not so simple as in the real one. One could not just copy Kolmogorov's approach and consider the condition of σ-additivity. There is, in fact, a No-Go theorem, see, e.g., [27]:

Theorem 4.1

All σ-additive p-adic valued measures defined on σ-fields are discrete.

Here the difficulty is not induced by the condition of σ-additivity, but by an attempt to extend a measure from the field F to the σ-field generated by F. Roughly speaking there exist σ-additive "continuous" $\mathbf{Q}_p$-valued measures, but they could not be extended from the field F to the σ-field generated by F. Therefore it is impossible to choose the σ-additivity as the basic integration condition in the p-adic probability theory.

The first important condition (that was already invented in the first theory of non-Archimedian integration of Monna and Springer [28]) is *boundedness*: $||A||_{\mathbf{P}} = \sup\{|\mathbf{P}(A)|_p : A \in F\} < \infty$.

Of course, if $\mathbf{P}$ is a p-adic r-probability with $r < \infty$, then this condition is fulfilled automatically. It is nontrivial only if the range of values of a p-adic probability is unbounded in $\mathbf{Q}_p$.[2] We pay attention to one important particular case in that the condition of boundedness alone implies

[2] In the frequency formalism this corresponds to considering of p-adic (frequency) s-probabilities for $s \in L_0$; e.g., $s = \{N_k = p^k\}$. In this case $m = \lim_{k\to\infty} p^k = 0$.

fruitful integration theory. Let Ω be a compact zero-dimensional topological space.[3] Then the integral

$$E\xi = \int_{\Omega} \xi(\omega)\mathbf{P}(d\omega)$$

is well defined for any continuous function $\xi : \Omega \to \mathbf{Q}_p$. For example, this theory works well for the following choice: Ω is the ring of q-adic integers Z_q, and $\mathbf{P}$ is a bounded p-adic r-probability, $r < \infty$. The integral is defined as the limit of Riemannian sums [28], [27].

But in general boundedness alone does not imply fruitful integration theory. We should consider another condition, namely *continuity* of $\mathbf{P}$. The most general continuity condition was proposed by A. van Rooij [27].[4]

Definition 4.1

A p-adic valued measure that is bounded, continuous, and normalized is called p-adic probability measure.

Everywhere below we consider p-adic probability spaces endowed with p-adic probability measures.

Let $(\Omega, F, \mathbf{P})$ be a p-adic probabilistic space. *Random variables* $\xi : \Omega \to \mathbf{Q}_p$ are defined as $\mathbf{P}$-integrable functions.

As the frequency p-adic probability theory induces, see [7], (as a Theorem) Bayes' formula for conditional probability, we can use (4) as the definition of conditional probability in the p-adic axiomatic approach (as it was done by Kolmogorov in the real case).

Example 4.1. (p-adic valued uniform distribution on the space of q-adic sequences). Let p and q be two prime numbers. We set $X_q = \{0, 1, \ldots, q-1\}, \Omega_q^n = \{x = (x_1, \ldots, x_n) : x_j \in X_q\}, \Omega_q^\star = \bigcup_n \Omega_q^n$ (the space of finite sequences), and

$$\Omega_q = \{\omega = (\omega_1, \ldots, \omega_n, \ldots) : \omega_j \in X_q\}$$

(the space of infinite sequences). For $x \in \Omega_q^n$, we set $l(x) = n$. For $x \in \Omega_q^\star, l(x) = n$, we define a cylinder U_x with the basis x by $U_x = \{\omega \in \Omega_q : \omega_1 = x_1, \ldots, \omega_n = x_n\}$. We denote by the symbol F_{cyl} the field of subsets of Ω_q generated by all cylinders. In fact, the F_{cyl} is the collection of all finite unions of cylinders.

[3] There exists a basis of neighborhoods that are open and closed at the same time.

[4] We remark that in many cases continuity coincides with σ-additivity.

First we define the uniform distribution on cylinders by setting $\mu(U_x) = 1/q^{l(x)}, x \in \Omega_q^\star$. Then we extend μ by additivity to the field F_{cyl}. Thus $\mu : F_{\text{cyl}} \to \mathbf{Q}$. The set of rational numbers can be considered as a subset of any $\mathbf{Q}_p$ as well as a subset of $\mathbf{R}$. Thus μ can be considered as a p-adic valued measure (for any prime number p) as well as the real valued measure. We use symbols $\mathbf{P}_p$ and $\mathbf{P}_\infty$ to denote these measures. The probability space for the uniform p-adic measure is defined as the triple

$$\mathcal{P} = (\Omega, F, \mathbf{P}), \text{ where } \Omega = \Omega_q, F = F_{\text{cyl}} \text{ and } \mathbf{P} = \mathbf{P}_p.$$

The $\mathbf{P}_p$ is called a *uniform p-adic probability distribution.*

The uniform p-adic probability distribution is a probabilistic measure iff $p \neq q$. The range of its values is a subset of the unit p-adic ball.

Remark 4.1. Values of $\mathbf{P}_p$ on cylinders coincide with values of the standard (real-valued) uniform probability distribution (Bernoulli measure) $\mathbf{P}_\infty$. Let us consider, the map $j_\infty(\omega) = \sum_{j=0}^\infty \frac{\omega_j}{2^{j+1}}$. The j_∞ maps the space Ω_q onto the segment [0,1] of the real line $\mathbf{R}$ (however, j_∞ is not one to one correspondence). The j_∞-image of the Bernoulli measure is the standard Lebesque measure on the segment [0,1] (the uniform probability distribution on the segment [0,1]).

Remark 4.2. The map $j_q : \Omega_q \to \mathbf{Z}_q, j_q(\omega) = \sum_{j=0}^\infty \omega_j q^j$, gives (one to one!) correspondence between the space of all q-adic sequences Ω_q and the ring of q-adic integers Z_q. The field F_{cyl} of cylindrical subsets of Ω_q coincides with the field $B(\mathbf{Z}_q)$ of all clopen (closed and open at the same time) subsets of $\mathbf{Z}_q$. If Ω_q is realized as $\mathbf{Z}_q$ and F_{cyl} as $B(Z_q)$, then μ_p is the p-adic valued Haar measure on $\mathbf{Z}_q$. The use of the topological structure of $\mathbf{Z}_q$ is very fruitful in the integration theory (for $p \neq q$). In fact, the space of integrable functions $f : \mathbf{Z}_q \to \mathbf{Q}_p$ coincides with the space of continuous functions (random variables) $C(\mathbf{Z}_q, \mathbf{Q}_p)$, see [28], [27], [26], [7].

5. *p*-adic Limit Theorems

5.1. *p*-adic Asymptotics of Bernoulli Probabilities

Everywhere in this section p is a prime number distinct from 2. We start with considering the classical Bernoulli scheme (in the conventional probabilistic framework) for random variables $\xi_j(\omega) = 0, 1$ with probabilities $1/2, j = 1, 2, \ldots$. First we consider a finite number n of random variables: $\xi_1(\omega), \ldots, \xi_n(\omega)$. A sample space corresponding to these random variables can be chosen as the space $\Omega_2^n = \{0, 1\}^n$. The probability of an event A is defined as

$$\mathbf{P}^{(n)} = \frac{|A|}{|\Omega_2^n|} = \frac{|A|}{2^n},$$

where the symbol $|B|$ denotes the number of elements in a set B. The typical problem of ordinary probability theory is to find the asymptotic behavior of the probabilities $\mathbf{P}^{(n)}(A), n \to \infty$. It was the starting point of the theory of limit theorems in conventional probability theory.

But the probabilities $\mathbf{P}^{(n)}(A)$ belong to the field of rational numbers $\mathbf{Q}$. We may study behavior of $\mathbf{P}^{(n)}(A)$, not only with respect to the usual real metric $\rho_\infty(x, y)$ on $\mathbf{Q}$, but also with respect to an arbitrary metric $\rho(x, y)$ on $\mathbf{Q}$. We have studied the case of the p-adic metric on $\mathbf{Q}$, see [29], [30]. We remark that $\mathbf{P}^{(n)}(A) = \sum_{x \in A} \mu(U_x)$, where μ is the uniform distribution on Ω_2. By realizing μ as the (real valued) probability distribution $\mathbf{P}_\infty$ we use the formalism of conventional probability theory. By realizing μ as the p-adic valued probability distribution $\mathbf{P}_p$ we use the formalism of p-adic probability theory.

What kinds of events A are naturally coupled to the p-adic metric? Of course, such events must depend on the prime number p. As usual, we consider the sums

$$S_n(\omega) = \sum_{k=1}^{n} \xi_n(\omega).$$

We are interested in the following question. Does p divide the sum $S_n(\omega)$ or not? Set $A(p, n) = \{\omega \in \Omega_2^n : p \text{ divides the sum } S_n(\omega)\}$. Then $\mathbf{P}^{(n)}(A(p, n)) = L(p, n)/2^n$, where $L(p, n)$ is the number of vectors $\omega \in \Omega_2^n$ such that p divides $|\omega| = \sum_{j=1}^{n} \omega_j$. As usual, denote by $\bar{A}$ the complement of a set A. Thus $\bar{A}(p, n)$ is the set of all $\omega \in \Omega_q^n$ such that p does not divide the sum $S_n(\omega)$. We shall see that the sets $A(p, n)$ and $\bar{A}(p, n)$ are asymptotically symmetric from the p-adic point of view:

$$\mathbf{P}^{(n)}(A(p, n)) \to \tfrac{1}{2} \quad \text{and} \quad \mathbf{P}^{(n)}(\bar{A}(p, n)) \to \tfrac{1}{2} \tag{5}$$

in the p-adic metric when $n \to 1$ in the same metric. Already in this simplest case we shall see that the behavior of sums $S_n(\omega)$ depends crucially on the choice of a sequence $s = \{N_k\}_{k=1}^{\infty}$ of natural numbers. A limit distribution of the sequence of random variables $S_n(\omega)$, when $n \to \infty$ in the ordinary sense, does not exist. We have to describe all limiting distributions for different sequences s converging in the p-adic topology.

Let $(\Omega, F, \mathbf{P})$ be a p-adic probabilistic space and $\xi_n : \Omega \to \mathbf{Q}_p (n = 1, 2, \ldots)$ be a sequence of equally distributed independent random variables,

$\xi_n = 0, 1$ with probability $1/2$.[5] We start with the following result that can be obtained through purely combinatorial considerations (behavior of binomial coefficients C_m^r in the p-adic topology).

Theorem 5.1

Let $m = 0, 1, \ldots, p^s - 1 (s = 1, 2, \ldots), r = 0, \ldots, m$, and $l \geq s$. Then

$$\lim_{n \to m} \mathbf{P}(\omega : S_n(\omega) \in U_{1/p^l}(r)) = \frac{C_m^r}{2^m}.$$

Formally this theorem can be reformulated as the following result for the convergence of probabilistic distributions: *The limiting distribution on* $\mathbf{Q}_p$ *of the sequence of the sums* $S_n(\omega)$*, where* $n \to m$ *in* $\mathbf{Q}_p$*, is the discrete measure* $\kappa_{1/2,m} = 2^{-m} \sum_{r=0}^{m} C_m^r \delta_m$.

We consider the event $A(p, n, r) = \{\omega : S_n(\omega) = pi + r\}$ for $r = 0, 1, \ldots, p - 1$. This event consists of all ω such that the residue of $S_n(\omega)$ mod p equals to r. Note that the set $A(p, n, r)$ coincides with the set $\{\omega : S_n(\omega) \in U_{1/p}(r)\}$.

Corollary 5.1

Let $n \to m$ in $\mathbf{Q}_p$, where $m = 0, 1, \ldots, p - 1$. Then the probabilities $\mathbf{P}^{(n)}(A(p, n, r))$ approach $C_m^r / 2^m$ for all residues $r = 0, \ldots, m$.

In particular, as $A(p, n) \equiv A(p, n, 0)$, we get (5). What happens in the case $m \geq p$? We have only the following particular result:

Theorem 5.2

Let $n \to p$ in $\mathbf{Q}_p$ and $r = 0, 1, 2, \ldots, p$. Then

$$\lim_{n \to p} \mathbf{P}(\omega : S_n(\omega) \in U_{1/p^l}(r)) = \frac{C_p^r}{2^p},$$

where $s \geq 2$ for $r = 0, p$ and $s \geq 1$ for $r = 1, \ldots, p - 1$.

5.2. Laws of Large Numbers

We now study the general case of dichotomous equally distributed independent random variables: $\xi_n(\omega) = 0, 1$ with probabilities q and $q' = 1 - q$, $q \in \mathbf{Z}_p$. We shall study the weak convergence of the probability distributions

[5] Here $1/2$ is considered as a p-adic number. In the conventional theory $1/2$ is considered as a real number.

$\mathbf{P}_{S_{N_k}}$ for the sums $S_{N_k}(\omega)$. We consider the space $C(\mathbf{Z}_p, \mathbf{Q}_p)$ of continuous functions $f : \mathbf{Z}_p \to \mathbf{Q}_p$. We will be interested in convergence of integrals

$$\int_{\mathbf{Z}_p} f(x) d\mathbf{P}_{S_{N_k}}(x) \to \int_{\mathbf{Z}_p} f(x) d\mathbf{P}_S(x), f \in C(\mathbf{Z}_p, \mathbf{Q}_p),$$

where $\mathbf{P}_S$ is the limiting probability distribution (depending on the sequence $s = \{N_k\}$). To find the limiting distribution $\mathbf{P}_S$, we use the method of characteristic functions. We have for characteristic functions

$$\phi_{N_k}(z, q, a) = \int_{\Omega} \exp\{z S_{N_k}(\omega)\} d\mathbf{P}(\omega) = (1 + q'(e^z - 1))^{N_k}.$$

Here z belong to a sufficiently small neighborhood of zero in the $\mathbf{Q}_p$; see [10] for detail about the p-adic method of characteristic functions. Let a be an arbitrary number from $\mathbf{Z}_p$. Let $s = \{N_k\}_{k=1}^{\infty}$ be a sequence of natural numbers converging to a in the $\mathbf{Q}_p$. Set $\phi(z, q, a) = (1 + q'(e^z - 1))^a$. This function is analytic for small z. It is easy to see that the sequence of characteristic functions $\{\phi_{N_k}(z, q, a)\}$ converges (uniformly on every ball of a sufficiently small radius) to the function $\phi(z, q, a)$. Unfortunately, we could not prove (or disprove) a p-adic analogue of Levy's theorem. Therefore in the general case the convergence of characteristic functions does not give us anything. However, we shall see that we have Levy's situation in the particular case under consideration: There exists a bounded probability measure distribution, denoted by $\kappa_{q,a}$, having the characteristic function $\phi(z, q, a)$ and, moreover, $\mathbf{P}_{S_{N_k}} \to \mathbf{P}_S = \kappa_{q,a}$, $N_k \to a$.

We start with the first part of the above statement. Here we shall use Mahlers integration theory on the ring of p-adic integers, see e.g., [26], [27], [9], [10]. We introduce a system of binomial polynomials: $C(x, k) = C_x^k = \frac{x(x-1)\dots(x-k+1)}{k!}$ (that are considered as functions from $\mathbf{Z}_p$ to $\mathbf{Q}_p$). Every function $f \in C(\mathbf{Z}_p, \mathbf{Q}_p)$ is expanded into a series (a Mahler expansion, see [40]) $f(x) = \sum_{k=0}^{\infty} a_k C(x, k)$. It converges uniformly on $\mathbf{Z}_p$. If μ is a bounded measure on $\mathbf{Z}_p$, then

$$\int_{\mathbf{Z}_p} f(x) \mu(dx) = \sum a_k \int_{\mathbf{Z}_p} C(x, n) \mu(dx).$$

Therefore to define a p-adic valued measure on $\mathbf{Z}_p$ it suffices to define coefficients $\int_{\mathbf{Z}_p} C(x, n) \mu(dx)$. A measure is bounded iff these coefficients are bounded. Using the Mahler expansion of the function $\phi(z, q, a)$, we obtain

$$\lambda_m(q, a) = \int_{\mathbf{Z}_p} C(x, m) \kappa_{q,a}(dx) = (1 - q)^m C(a, m).$$

As $|C(a,m)|_p \leq 1$ for $a \in \mathbf{Z}_p$, we get that the distribution $\kappa_{q,a}$ (corresponding to $\phi(z,q,a))$ is bounded measure on $\mathbf{Z}_p$. Set $\lambda_{mn}(q,a) = \int_\Omega C(S_n(\omega),m)dP(\omega)$. We find

$$\lambda_{mN_k}(q,a) = (1-q)^m C_{N_k}^m.$$

Thus $\lambda_{mN_k}(q,a) \to \lambda_m(q,a), N_k \to a$. This implies the following limit theorem.

Theorem 5.3 (p-adic Law of Large Numbers)

The sequence of probability distributions $\{\mathbf{P}_{S_{N_k}}\}$ converges weakly to $\mathbf{P}_S = \kappa_{q,a}$, when $N_k \to a$ in $\mathbf{Q}_p$.

One might say that in the p-adic case there is no the law of large numbers in the ordinary meaning. We could not consider asymptotics for $n \to \infty$. Only properly selected subsequencies of natural numbers generate fruiful asymptotics of probabilities. However, even such a weakened law of large numbers may serve for applications. In physics sometimes one selects special sequences of observation times; similar subselections may appear in other statistical applications.

5.3. The Central Limit Theorem

Here we restrict our considerations to the case of symmetric random variables $\xi_n(\omega) = 0, 1$ with probabilities 1/2. We study the p-adic asymptotic of the normalized sums

$$G_n(\omega) = \frac{S_n(\omega) - ES_n(\omega)}{\sqrt{DS_n(\omega)}}, \tag{6}$$

Here $ES_n = n/2, D\xi_n = E\xi^2 - (E\xi)^2 = 1/4$ and $DS_n = n/4$. Hence

$$G_n(\omega) = \frac{S_n(\omega) - n/2}{\sqrt{n}/2} = \sum_{j=1}^{n} \frac{2\xi_n}{\sqrt{n}} - \sqrt{n}.$$

By applying the method of characteristic functions we can find the characteristic function of the limiting distribution. Let us compute the characteristic function of random variables $G_n(\omega)$:

$$\psi_n(z) = (\cosh\{z/\sqrt{n}\})^n.$$

Set $\psi(z,a) = (\cosh\{z/\sqrt{a}\})^a, a \in \mathbf{Z}_p, a \neq 0$. This function belongs to the space of locally analytic functions. There exists the p-adic analytic generalized function, see [10] for detail, γ_a with the Borel-Laplace transform $\psi(z,a)$.

Unfortunately, we do not know so much about this distribution (an analogue of Gaussian distribution?). We only proved the following theorem:

Theorem 5.4

The γ_1 is the bounded measure on $\mathbf{Z}_p$.

Open Problems:

1). Boundedness of γ_a for $a \neq 1$.
2). Weak convergence of $\mathbf{P}_{G_n}$ to $\mathbf{P}_G = \gamma_a$ (at least for $a = 1$).

6. *p*-adic Valued Probabilities – Coupling with Statistics

In fact, Kolmogorov's probability theory has two (more or less independent) counterparts:
(a) axiomatics (a mathematical representation);
(i) interpretation (rules for application).

The first part is the measure-theoretic formalism. The second part is a mixture of frequency and ensemble interpretations: "... we may assume that to an event A which has the following characteristics: (a) one can be practically certain that if the complex of conditions $\sum$ is repeated a large number of times, N, then if n be the number of occurrences of event A, the ratio n/N will differ very slightly from $\mathbf{P}(A)$; (b) if $\mathbf{P}(A)$ is very small, one can be practically certain that when conditions $\sum$ are realized only once the event A would not occur at all", [1].

As we have already noticed, (a) and (i) are more or less independent. Therefore Kolmogorov's measure-theoretic formalism, (a), is used successfully, for example, in the subjective probability theory.

In practice we apply Kolmogorov's (conventional) interpretation, (i), in the following way. First of all we have to fix $0 < \epsilon < 1$, *significance level.* If the probability $\mathbf{P}(A)$ of some events A is less than ϵ, this event is considered as practically impossible. We now generalize the conventional interpretation of probability to the case of $\mathbf{Q}_p$-valued probabilities. First of all we have to fix some neighborhood of zero, V, *significance neighborhood.*

If the probability $\mathbf{P}(A)$ *of some event* A *belongs to* V*, this event is considered as practically impossible.*

Since the group $\mathbf{Q}_p$ is metrizable, then the situation is similar to the standard (real) probability. We choose $\epsilon > 0$ and consider the ball

$$V_\epsilon = \{x \in \mathbf{Q}_p : \rho_p(0, x) < \epsilon\}.$$

If $\rho_p(0, \mathbf{P}(A)) < \epsilon$, then the event A is considered as practically impossible.

Let us borrow some ideas from statistics. We are given a certain sample space Ω with an associated distribution $\mathbf{P}$. Given an element $\omega \in \Omega$, we want to test the hypothesis "ω belongs to some reasonable majority". A reasonable majority $\mathcal{M}$ can be described by presenting *critical regions* $\Omega^{(\epsilon)}(\in F)$ of the significance level $\epsilon, 0 < \epsilon < 1 : \mathbf{P}(\Omega^{(\epsilon)}) < \epsilon$. The complement $\bar{\Omega}^{(\epsilon)}$ of a critical region $\Omega^{(\epsilon)}$ is called $(1-\epsilon)$ confidence interval. If $\omega \in \Omega^{(\epsilon)}$, then the hypothesis '$\omega$ belongs to majority $\mathcal{M}$' is rejected with the significance level ϵ. We can say that ω fails the test to belong to $\mathcal{M}$ at the level of critical region $\Omega^{(\epsilon)}$.

$\mathbf{Q}_p$-statistical machinery works in the same way. We consider significance levels V given by neighborhoods of zero in $\mathbf{Q}_p$. Thus we consider critical regions $\Omega^{(V)}(\in F)$:

$$\mathbf{P}(\Omega^{(V)}) \in V.$$

If $\omega \in \Omega^{(V)}$, then the hypothesis "ω belongs to majority $\mathcal{M}$" (represented by the statistical test $\{\Omega^{(V)}\}$) is rejected with the significance level V. Since $\mathbf{Q}_p$ is metrizable, then we can always choose $V = V_\epsilon, \epsilon > 0$.

Of course, the strict mathematical description of the above statistical considerations can be presented in the framework of Martin-Löf [6], [4], [7] statistical tests. We remark that such a p-adic framework was already developed in [7]. We emphasize some similarities and differences between real and p-adic theories:

In the p-adic case (as in the real case) it is possible to enumerate effectively all p-adic tests for randomness.

However, a universal p-adic test for randomness does not exist [7].

We now define $\mathbf{Q}_p$-random sequences, namely sequences

$$\omega = (\omega_1, \ldots, \omega_N, \ldots), \omega_j = 0, 1,$$

that are random with respect to a $\mathbf{Q}_p$-valued probability distribution in the same way as in the real Martin-Löf approach.

The general scheme of the application of $\mathbf{Q}_p$-valued probabilities is the same as in the ordinary case:

1) we find initial probabilities;
2) then we perform calculations by using calculus of $\mathbf{Q}_p$-valued probabilities;
3) finally, we apply the above interpretation to resulting probabilities.

The main question is "How can we find initial probabilities?" Here the situation is more or less similar to the situation in the ordinary probability theory. One of possibilities is to apply the frequency arguments (as R. von Mises). We have already discussed such an approach for p-adic probabilities. Another possibility is to use subjective approach to probability. I think that everybody agrees that there is nothing special in segment $[0,1]$ as the set of labels for the measure of belief in the occurrence of some event. In the same way we can use, for example, the segment $[-1,1]$ (signed probability) or the unit complex disk (complex probability) or the set of p-adic integers $\mathbf{Z}_p$ (p-adic probability). Since $\mathbf{Q}_p$ is a field we can apply the machinery of Bayesian probabilities and, finally, use our interpretation of probabilities to make a statistical decision. The third possibility is to use symmetry arguments, Laplacian approach. For example, by such arguments we can choose (in some situations) the uniform $\mathbf{Q}_p$-valued distribution.

Example 6.1. (A p-adic statistical test) Theorem 5.1. implies that, for each p-adic sphere $\mathbf{S}_{1/p^l}(r)$, where l, r, m were done in Theorem 3.1:

$$\lim_{k\to\infty} \mathbf{P}(\{\omega \in \Omega_2 : S_{N_k}(\omega) \in \mathbf{S}_{1/p^l}(r)\}) = 0,$$

for each sequence $s = \{N_k\}, N_k \to m, k \to \infty$. We can construct a statistical test on the basis of this limit theorem (as well as any other limit theorem). Let $s = \{N_k\}, N_k \to m$, be a fixed sequence of natural numbers. For any $\epsilon > 0$, there exists k_ϵ such that, for all $k \geq k_\epsilon$,

$$|\mathbf{P}(\{\omega \in \Omega_2 : S_{N_k}(\omega) \in \mathbf{S}_{1/p^l}(r)\})|_p < \epsilon.$$

We set $\Omega^{(\epsilon)} = \bigcup_{k\geq k_\epsilon} \{\omega \in \Omega_2 : S_{N_k}(\omega) \in \mathbf{S}_{1/p^l}(r)\}$. We remark that

$$|\mathbf{P}(\Omega^{(\epsilon)})|_p < \epsilon.$$

We now define reasonable majority of outcomes as sequences that do not belong to the sphere $\mathbf{S}_{\frac{1}{p^l}}(r)$, "nonspherical majority." Here the set $\Omega^{(\epsilon)}$ is the critical region on the significance level ϵ.

Suppose that a sequence ω belongs to the set $\Omega^{(\epsilon)}$. Then the hypothesis "ω belongs nonspherical majority" must be rejected with the significance level ϵ. In particular, such a sequence ω is not random with respect to the uniform p-adic distribution on Ω_2. If, for some sequence of 0 and 1, $\omega = (\omega_j)$ we have $\omega_1 + \ldots + \omega_{N_k} - r = \alpha \mod p^l, \alpha = 1, \ldots, p-1$, for all $k \geq k_\epsilon$, then it is rejected.

The simplest test is given by $m = 1, r = 0, N_k = 1 + p^k$ and $\omega_1 + \ldots + \omega_{N_k} = \alpha \mod p, \alpha = 1, \ldots, p-1$.

At the end we recall some recent applications of p-adic numbers: a) in cognitive science and neurophysiology [31], [32]; b) logical foundation of p-adic probability [33]; c) modeling disordered systems (spin glasses) [34]–[36]; d) p-adic cosmology and quantum physics [37].

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