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BASIC CONCEPTS OF CONTINUOUS LOGIC

In this paper, a general description of a continuous (-valued) logic is given and some problems and particulars of their solutions are discussed. Firstly, we define algebra of continuous logic and enumerate its basic unary, binary and ternary functions. All laws of continuous logic are compared to laws of discrete binary logic. We discuss how to enumerate all the functions of continuous logic with a specified number of variables and how to represent such functions in a standard form. Procedures of minimization regarding continuous logical functions and their decomposition into functions with less clarity are exploited. The procedures are compared to their counterparts from binary logic. We also tackle problems of the analysis and synthesis of continuous logical functions, and show that the problem of synthesis may not have a solution. Basics of differential and integral calculus are applied to continuous-valued logic. We demonstrate that any continuous logical function has the points where a derivative does not exist. To conclude, we briefly discuss a problem of incompleteness regarding continuous logic, application of continuous logic in mathematics, engineering and economy, give examples, draw a perspective of further development and supply an extensive bibliography of Russian works in the field.

1. Introduction

A continuous logic (CL) is a natural generalization of a discrete logic (DL). A few laws of DL take place in CL as well. A general structure of CL differs from that of DL; thus, for example an operation of negation regarding CL cannot be defined in terms of addition (as it is done in binary logic) in a consistent way. CL now forms an independent scientific discipline where theoretical and applied key points lie in such diverse fields as:

- mathematics (approximation of functions, geometry, theory of sets, theory of numbers, interval analysis);
- engineering (design of electrical circuits, synthesis of functional generators and analogue-discrete transformers, design and analysis of analogue and digital devices, diagnostics and maintenance service);
- systems (theory of service systems, pattern recognition and analysis of scenes, decision making, information processing, synchronization);

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- economy (discrete optimization, schedule theory, simulation of economic systems),
- biology (simulation of neuron systems at all levels of representation),
- sociology (simulation of the dynamics of collective behaviour);
- politology (simulations of dynamics of a society),
- history (simulation of the streams of historical events).

The new results in CL can be obtained with the help of several straightforward techniques:

- calculation of the values of logical expressions;
- equivalent conversion of logical expressions;
- unification of individual logical expressions;
- partition of common logical expressions onto individual ones.

We can also embed the algebra of CL in a more general distributive structure; in this situation all methods of the theory of structures are useful.

A range of the main problems of CL includes the following categories:

1. enumeration of all CL functions for the given number of arguments,
2. representation of CL functions in a standard form (including unambiguous representation),
3. selection of elementary CL functions;
4. minimization and decomposition of CL functions,
5. analysis and synthesis of CL functions;
6. solution of the equations and inequalities of CL,
7. differentiation and integration of CL functions,
8. verification of completeness regarding the system of CL functions.

Problems 1–4 are similar in formulation and, partially in the methods of solution, to the appropriate DL tasks. Problems 5–8 belong exclusively to CL.

2. General Description of Continuous Logic

Let $C = [A, B]$ be a closed interval such that $M = (A + B)/2$. Basic operations of CL are defined on as follows:

$$\begin{aligned} &a \vee b = \max(a, b) && \text{(Disjunction)},\\ &a \wedge b = \min(a, b) && \text{(Conjunction)},\\ &\bar{a} = 2M - a && \text{(Negation)}. \end{aligned} \tag{1}$$

The sign $\wedge$ is usually omitted.

Sometimes the following operations are used as basic ones:

- inclusion $a \supset b = (\bar{a} + b) \wedge B$,
- implication $a \to b = \bar{a} \vee b$,
- equivalence $(a \equiv b) = (a \vee \bar{b})(\bar{a} \vee b)$,
- non-equivalence $(a \not\equiv b) = (a\bar{b} \vee \bar{a}b$,
- Sheffer $a|b = \overline{ab}$,
- Webb $a \downarrow b = \overline{a \vee b}$,
- contradiction $(a \neq a) = a\bar{a}$,
- tautology $(a \equiv a) = a \vee \bar{a}$,
- prohibition $a \overline{\to} b) = a\bar{b}$.

An algebraic system of the supporting set C and the basic operations is called an algebra of CL. A CL function is a function $C^n \to C$, which is represented by the superposition of a finite number of basic operations of the CL algebra with the arguments $x_1, \ldots, x_n \in C$. The number of CL functions is finite, though the set of all functions of the sort $C^n \to C$ is infinite.

A quasi-Boolean algebra:

$$\Delta = (C; \vee, \wedge, ^{-}) \tag{2}$$

is one of the most developed and investigated algebras of CL. The functions of algebra (2) are usually tabulated. One can easily transform tabular representation into an analytical one using a method of union. A reverse transition from analytical to tabular representation is carried out by a method of partition.

The number $P(n)$ of n-ary CL functions in quasi-Boolean algebra grows quite fast when n increases: $P(0) = 2$, $P(1) = 6$, $P(2) = 84$, $P(3) = 43918$. For $n = 0$ the functions are constant:

$$y_0 = A, \quad y_1 = B. \tag{3}$$

For $n = 1$ there are constants y_0, y_1 and 4 functions, essentially depending on argument x:

$$y_2 = x, \quad y_3 = \bar{x}, \quad y_4 = x \vee \bar{x}, \quad y_5 = x\bar{x}. \tag{4}$$

For $n = 2$ there are constants (3), 8 functions (4), depending on one argument (x_1 or x_2), and 10 functions, depending on two arguments:

$$\begin{aligned} &y_{10} = x_1 \vee x_2,\ y_{11} = x_1x_2,\ y_{12} = (x_1 \vee \bar{x}_2)(\bar{x}_1 \vee x_2),\\ &y_{13} = x_1\bar{x}_2 \vee \bar{x}_1x_2,\ y_{14} = \overline{x_1x_2},\ y_{15} = \overline{x_1 \vee x_2},\ y_{16} = \bar{x}_1 \vee x_2,\\ &y_{17} = x_1 \vee \bar{x}_2,\ y_{18} = \bar{x}_1x_2,\ y_{19} = x_1\bar{x}_2. \end{aligned} \tag{5}$$

There are also 64 functions depending on 2 arguments. They can be obtained by the superposition of the previous 20 functions or via the compilation of the function value tables and the consequent transition to analytical representation. For $n = 3$ CL functions include all previous functions depending on, at most, 2 arguments, and all functions essentially depending on 3 arguments. The most common ternary functions include the following:

Disjunction and conjunction (maximum and minimum):

$$y = x_1 \vee x_2 \vee x_3 = \max(x_1, x_2, x_3), \;\; y = x_1 x_2 x_3 = \min(x_1, x_2, x_3), \quad (6)$$

median and median negation (inversion):

$$\begin{aligned} y &= \mathrm{med}(x_1, x_2, x_3) = x_1 x_2 \vee x_1 x_3 \vee x_2 x_3, \\ y &= \overline{\mathrm{med}}(x_1, x_2, x_3) = \bar{x}_1 \bar{x}_2 \vee \bar{x}_1 \bar{x}_3 \vee \bar{x}_2 \bar{x}_3, \end{aligned} \quad (7)$$

Sheffer and Webb functions:

$$y = \overline{x_1 x_2 x_3}, \quad y = \overline{x_1 \vee x_2 \vee x_3}, \quad (8)$$

and elementary three-place disjunction and conjunction:

$$y = x_1 \vee \bar{x}_1 \vee x_2 \vee \bar{x}_2 \vee x_3 \vee \bar{x}_3, \quad y = x_1 \bar{x}_1 x_2 \bar{x}_2 x_3 \bar{x}_3. \quad (9)$$

It is possible to obtain other functions depending essentially on 3 arguments by the superposition of the functions listed above or by compiling the table of values followed by a transition to analytical representation. Any set of CL functions of a great number of arguments is generated in the same way.

Notice that the number $P(n)$ of CL functions of n arguments grows with n sufficiently faster than the number $Q(n)$ of functions of binary logic. For example, $Q(0) = 2$, $Q(1) = 4$, $Q(2) = 16$, $Q(3) = 256$. Therefore we are not able to apply exhaustive techniques to the investigation of CL functions, as it is done with binary functions. Thus we should stick to the analysis of typical CL functions.

Basic operations of CL were analyzed by R. McNaughton; the general description of CL and its mathematical apparatus is elaborated on by S. A. Ginsburg, V. I. Levin and P. N. Shimbirev. A review of their work can be found in [1, 2, 4, 6, 7, 9, 11, 12, 14, 21].

3. Some Examples of Applications regarding Continuous Logic

Example 1 (geometry). Given a piece-wise linear function $y = f(x)$ formed of two linear functions $y = ax+b$ and $y = cx+d$, the first function is accepted on the left side of the intersection of graphs of these two functions; the second function works on the right side. Using graphs of $y = ax + b$ and $y = cx + d$ we can check that only two possibilities for the formation of $f(x)$ are correct: we can use either the lower of the lines (concave function $f(x)$) or the upper of the lines (convex function $f(x)$). Therefore we obtain an analytic form of the piece-wise linear function $f(x)$:

$$y = (ax + b) \overset{\vee}{\wedge} (cx + d),$$

where the operation of CL disjunction $\vee$ is applied when the graph of $f(x)$ is concave and the operation of CL conjunction $\wedge$ is used when it is convex.

The analytical representation of piece-wise linear and piece-wise non-linear functions in terms of CL is developed in the work of E. I. Berkovich, S. A. Ginsburg and V. I. Levin [1, 6, 12].

Example 2 (theory of discrete automata). Let us consider an automaton with two binary inputs x_1, x_2 and one binary output y; the automaton implements a Boolean function:

$$y = x_1 \,\&\, x_2, \quad x_1, x_2, y \in \{0, 1\}.$$

The inputs are determined by the binary processes of the form:

$$x_1(t) = \begin{cases} 0, & t < a, \\ 1, & t \geq a, \end{cases} \quad x_2(t) = \begin{cases} 1, & t < b, \\ 0, & t \geq b. \end{cases}$$

Our purpose here is to analyze the binary process $y(t)$ on an automaton's output and its reaction on the given input processes. Let $1(A, B)$ be a binary process of impulses in the time interval (A, B). The reaction of an automaton equals the impulse $1(a, b)$ when $b \geq a$ and it is the constant 0 when $b < a$. We can interpret the constant 0 as an impulse with a coincidental end and beginning. Then the reaction can be written in the terms of CL as follows:

$$y(t) = \left\{ \begin{array}{rl} 1(a, b) & \text{if } b \geq a \\ 0 = 1(a, a) & \text{if } b < a \end{array} \right| = 1(a, a \vee b).$$

Analytical theory of the processes in discrete automata was developed by V. I. Levin in [2, 4, 6, 14].

Example 3 (optimization). Let us imagine three vacancies and three candidates to fill these vacancies. Let a_{ij} be an efficiency of ith candidate to jth position. Our aim here is to distribute the positions between the candidates in such a manner that all positions are occupied, all the candidates are accepted and an integral efficiency is maximal. Obviously, every distribution of the positions between the candidates has its own sum of elements regarding the matrix $A = \|a_{ij}\|$; this sum includes exactly one element from every column and every row. Thus, we need to find a maximal sum $A^{\vee}$ of the elements of the matrix A. The sum has the following general form:

$$A^{\vee} = (a_{11} + a_{22} + a_{33}) \vee (a_{11} + a_{23} + a_{32}) \vee (a_{12} + a_{21} + a_{33}) \vee \\ \vee (a_{12} + a_{23} + a_{31}) \vee (a_{13} + a_{21} + a_{32}) \vee (a_{13} + a_{22} + a_{31}).$$

An algorithm of exhaustive search can be employed. The expression simplified with the help of law (20) looks like this:

$$A^{\vee} = \{a_{11} + [(a_{22} + a_{33}) \vee (a_{23} + a_{32})]\} \vee \\ \vee \{a_{12} + [(a_{21} + a_{33}) \vee (a_{23} + a_{31})]\} \vee \\ \vee \{a_{13} + [(a_{21} + a_{32}) \vee (a_{22} + a_{31})]\}.$$

It is three operations less than the previous expression. Therefore, we reduced the complexity of an exhaustive search.

The methods of optimization using CL were designed by V. I. Levin in [6, 9, 14–20].

Numerous examples of these techniques can be found in [1] (approximation of functions and the design of electrical circuits), [3] (design of digital devices), [5] (set theory and decision theory), [6] (design of analogue and digital devices, simulation of tools, queue theory, pattern recognition), [7] (fault-tolerance and diagnostics, technical servicing), [8, 13] (control, decision making), [9] (simulation and optimization of economic systems), [10–12, 17, 19] (synthesis of functional generators, design of analogue and hybrid devices), [15, 16, 18, 20] (simulation of economic systems, social groups, societies and historical events).

4. Laws of Continuous Logic

There is a straightforward generalization regarding CL in the frame of DL for the continuous interval C:

$a \vee a = a,\ aa = a$	(Tautology)	(10)
$a \vee b = b \vee a,\ ab = ba$	(Commutative)	(11)
$(a \vee b) \vee c = a \vee (b \vee c),\ (ab)c = a(bc)$	(Associative)	(12)
$a(b \vee c) = ab \vee ac,\ a \vee bc = (a \vee b)(a \vee c)$	(Distributive)	(13)
$\overline{a \vee b} = \bar{a}\bar{b},\ \overline{ab} = \bar{a} \vee \bar{b}$	(de Morgan)	(14)
$a \vee ab = a,\ a(a \vee b) = a$	(Absorption)	(15)

$\bar{\bar{a}} = a$ (Double negation) (16)

$aA = A,\ aB = a,\ a \vee A = a,\ a \vee B = B$ (Operation with constants) (17)

$a\bar{a}(b \vee \bar{b}) = a\bar{a},\ a\bar{a} \vee (b \vee \bar{b}) = b \vee \bar{b}$ (Kleene) (18)

The laws of contradiction and the eliminated third of DL are replaced by

$$a\bar{a} = M - |a - M|,\ a \vee \bar{a} = M + |a - M|. \quad (19)$$

As soon as the operations of CL are applied to a continuum it is quite reasonable to combine them with algebraic operations over continuous variables.

Instead of addition and multiplication we can use new distributive laws of CL, which combine either disjunction or conjunction with addition:

$$\begin{aligned} a + (b \vee c) = (a + b) \vee (a + c),\ a + (b \wedge c) = (a + b) \wedge (a + c), \\ a - (b \vee c) = (a - b) \wedge (a - c),\ a - (b \wedge c) = (a - b) \vee (a - c). \end{aligned} \quad (20)$$

A similar law works when disjunction and conjunction are coupled with multiplication.

A law of descent of negation on addends is as follows:

$$\overline{a + b} = \bar{a} - b = \bar{b} - a \quad (21)$$

The operations of CL are expressed with the help of addition and multiplication, as well as two auxiliary functions: $I(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$ and $|x|$. Therefore disjunction and conjunction in CL are expressed as

$$a \vee b = 0{,}5[a + b + |a - b|], \quad a \wedge b = 0{,}5[a + b - |a - b|]. \quad (22)$$

The possibility to express CL operations via algebraic operations (see (1) and (22)) indicates a connection between algebra and logic.

If we consider the generalization of CL itself, CL will be a special case of distributive structure with a pseudo-complement, i.e. with the operation of negation, which is not a complement because contradiction and excluded middle laws do not take place. In this context the values of a continuous CL variable (which belongs to the interval $[A, B]$) can be interpreted similarly to a variable of DL: the boundary value $x = A$ ($x = B$) represents the statements “absolutely false” (“absolutely true”), and the intermediate values x, $A < x < B$, measure the truth values of other statements.

Some basic laws of CL for the particular case of $C = [0, 1]$ were indicated by R. McNaughton. These laws were investigated in a general manner by S. A. Ginsburg, V. I. Levin and E. I. Berkovich. A review of these results can be found in [1, 2, 4, 6, 9, 14–16, 21].

5. Enumeration and the Standardization of Continuous-Valued Logical Functions

The enumeration of all CL functions for a fixed number of arguments and the representation of CL functions in standard form are the two most typical problems of CL. The problem relating to the enumeration of all CL functions in algebra (2) requires a specification of the appropriate analytical expressions. This can be done using the following two steps:

1. Enumeration of the tables of function values (the tables for all functions have an identical order of the following of argument ordering variants $x_1, \ldots, x_n$ and their negations $\bar{x}_1, \ldots, \bar{x}_n$, with various distributions of function values equal x_i or $\bar{x}_i$);
2. Transition from the tables to appropriate analytical expressions by a method of uniting.

Unfortunately this approach is not appropriate for $n \geq 3$. Therefore in practice we are limited by the defined classes of CL functions, which can be selected from the corresponding similar functions of DL, simplicity in the deriving of formulas and their practical importance.

Being the standard forms of CL, the functions of algebra (2) obey disjunctive and conjunctive normal forms (DNF and CNF). These forms differ from similar forms of binary DL because their elementary conjunctions (disjunctions) may include, together with argument x_i, its negation $\bar{x}_i$. The transition from any analytical representation of CL function to its DNF or CNF is similar to the transformation in binary DL. In the case of DNF the transition consists of (1) descent of negations on more simple expressions according to laws (14), (16); (2) disclosure of brackets in agreement with law (13). For CNF we have (1) the same descent of negations; (2) placement of brackets in accordance to law (13).

Example 4. Let us transform the analytical representation of CL to its DNF:

$$\begin{aligned}(x_1x_2 \vee \bar{x}_2x_3)\overline{\bar{x}_1x_4} &= (x_1x_2 \vee \bar{x}_2x_3)(x_1 \vee \bar{x}_4) = \\ &= x_1x_2 \vee x_1\bar{x}_2x_3 \vee x_1x_2\bar{x}_4 \vee \bar{x}_2x_3\bar{x}_4 = \\ &= x_1x_2 \vee x_1\bar{x}_2x_3 \vee \bar{x}_2x_3\bar{x}_4.\end{aligned}$$

We can accept canonical forms of DNF and CNF as unambiguous standard forms of the functions of CL. DNF unambiguously represents the CL function if it is a deadlock disjunction of non-decomposable elementary conjunctions. In turn, the elementary conjunction is non-decomposable in a disjunction of conjunctions when it is fundamental, i.e. consistent (does not contain simultaneously x_i and $\bar{x}_i$) or inconsistent but contains all the arguments of a given function, in direct x_i or inverse $\bar{x}_i$ form. From this point we can apply the following algorithm for the reduction of some DNF of a function of CL to canonical DNF:

1. Select in DNF all fundamental conjunctions;
2. represent each non-fundamental conjunction k (i.e. a conjunction which is contradictory and does not contain all of the arguments of the functions) by a disjunction of the fundamental conjunctions (to do this we can combine its conjunction with a suitable disjunction $x_j \vee \bar{x}_j$ because it does not change the value of k that includes $x_i\bar{x}_i \leq M$, for $x_j \vee \bar{x}_j \geq M$) and to open the brackets;
3. eliminate the smaller conjunction of each pair of comparable (in the sense of ratio $\leq$) conjunctions of DNF. The resultant canonical DNF is similar in sense but not form to the complete DNF of a Boolean function.

Example 5. Let us transform DNF of a CL function to its canonical DNF:

$$y = x_2x_4 \vee x_1\bar{x}_2x_3\bar{x}_4 \vee x_1x_2\bar{x}_2x_3\bar{x}_3.$$

The first two conjunctions are fundamental. The third conjunction is not: when multiplied as $x_4 \vee \bar{x}_4$ it is transformed to $x_1x_2\bar{x}_2x_3\bar{x}_3x_4 \vee x_1x_2\bar{x}_2x_3\bar{x}_3\bar{x}_4$. These newly emerged fundamental conjunctions are absorbed by the first two fundamental conjunctions of DNF of y. Eventually we have the canonical DNF:

$$y = x_2x_4 \vee x_1\bar{x}_2x_3\bar{x}_4.$$

Any CL function different from a fundamental conjunction is decomposable, i.e. a class of non-decomposable (elementary) CL functions in algebra (2) consists of only fundamental conjunctions.

The problems of representation and enumeration of CL functions were discussed by C. M. Clark, D. Dubois, H. Prade, A. Kandel, V. I. Levin, M. Mukaidono and P. N. Shimbiriev (see [3, 5, 6, 9, 11–13]). The standard representation of CL functions was tackled by F. P. Preparata, A. Kandel, D. Dubois, H. Prade, V. I. Levin and P. N. Shimbiriev (see reviews in [3, 6, 9, 11].

6. Minimization and the Decomposition of Continuous-Valued Logical Functions

The minimization of CL functions, similarly to the minimization of DL functions, aims to produce a form with a minimal number of variables.

A procedure of the minimization of the functions of CL in algebra (2) is developed only for such functions as represented in DNF. It deals with the search for the DNF with a minimal number of the entries of $x_i, \bar{x}_i$. The procedure of minimization regarding CL functions can be described in terms similar to the minimization of a Boolean function:

1. Make a search for all the fundamental conjunctions of the CL function f (these conjunctions play a role in the elementary conjunctions of the complete DNF of a Boolean function) and a representation f in canonical deadlock form;
2. search for all the simple implicants of the function (as usual, an implicant of the function f is thought of as an elementary conjunction k such that $k \leq f$; the implicant k is called simple if it is not absorbed by other implicants);
3. compute a minimal covering of the set of fundamental conjunctions by a set of simple implicants; for examples, with the help of tables of implicants.

Steps (1) and (2) are specific for CL functions. Step (1) is discussed in Section 7 of the paper. As for step (2), it is based on the content of consensus of elementary conjunctions k_i: if $k_1 = x_i a$, $k_2 = \bar{x}_i b$, where a, b are conjunctions of other characters, then consensus of k_1 and k_2 is represented in the sets of such contradictory conjunctions that (i) ab, (if it is inconsistent); (ii) conjunctions $x_i \bar{x}_i ab$, $i = \overline{1, n}$ (if ab is consistent). If k_1, k_2 are non-representable in indicated form with any i then consensus equals 0.

Example 6. For elementary conjunctions $k_1 = x_1 \bar{x}_2 x_3$, $k_2 = x_2 \bar{x}_3$ the consensus is

$$\{x_1 x_2 \bar{x}_2, x_1 x_3 \bar{x}_3\}.$$

For elementary conjunctions $k_1 = x_1 x_2 x_3$, $k_2 = x_2 \bar{x}_3$ the consensus is

$$\{x_1 x_2 x_3 \bar{x}_3, x_1 x_2 \bar{x}_2, x_1 \bar{x}_1 x_2\}.$$

We can search for all the simple implicants of CL function f, represented in deadlock DNF $f = \bigvee_i k_i$ by using the following algorithm:

1. For some pair k_i, k_j a consensus is formed;

2. all conjunctions obtained at step 1 are added to the disjunction $\bigvee_i k_i$;
3. all conjunctions k_a included in other conjunctions k_b (i.e. $k_a \le k_b$) are eliminated.

Steps 1–3 are repeated for new pairs k_i, k_j until the form of the function f remains unchanged. The final expression $f = \bigvee_i \tilde{k}_i$ in terms of conjunctions $\tilde{k}_i$ does contain all the simple implicants of the function f.

The number of CL functions grows enormously when the number of arguments increases, which is reflected in the complexity of minimization; therefore, the problem of decomposition regarding CL functions begins to play a very important role. The decomposition of the CL function $f(x)$, $x = (x_1, \ldots, x_n)$, represents f as a composition of several CL functions with a smaller number of arguments:

$$f(x) = F[f_m(x^m), \ldots, f_1(x^1), x^i], \quad x^i \subset x, \quad i = \overline{0, m}. \tag{23}$$

If the intersection of the sets x^i, $i = \overline{0, m}$ is empty, the decomposition is called a separating decomposition, otherwise this is a non-separating decomposition. The representation in (23) with $m = 1$ is called a simple decomposition. This is presently the only known algorithm in the search for simple decomposition of the CL function in algebra (2).

The problems of minimization and decomposition of CL functions were investigated by A. Kandel, D. Dubois, H. Prade, N. P. Shimbiriev (see [3, 9, 11, 12]).

7. Analysis and Synthesis of Continuous-Valued Logical Functions

Analysis and synthesis of CL functions is quite different from those problems relating to DL. Let be a range of values of a vector of arguments $x = (x_1, \ldots, x_n)$, D_f be a range of values of the CL function $f(x)$; there is also a one-to-one correspondence:

$$(x \in D_x) \Leftrightarrow (f(x) \in D_f). \tag{24}$$

The analysis of function f is converted to the following problem: given range D_f and function $f(x)$, find the range D_x in accordance with (24). The synthesis of the function $f(x)$ can be thought of as the following: given ranges D_x and D_f, construct the CL function f which realizes the correspondence in (24). Most methods of the analysis are developed for special cases when f is either a many-placed disjunction or conjunction, and D_f is a half-interval or interval. They are based on the following equivalencies:

$$\begin{aligned}&\Big(\bigvee_{i=1}^{n} x_i \geq a\Big) \Leftrightarrow (x_1 \geq a \text{ or } \dots \text{ or } x_n \geq a);\\&\Big(\bigvee_{i=1}^{n} x_i \leq b\Big) \Leftrightarrow (x_1 \leq b, \dots, x_n \leq b);\\&\Big(\bigwedge_{i=1}^{n} x_i \geq a\Big) \Leftrightarrow (x_1 \geq a, \dots, x_n \geq a);\\&\Big(\bigwedge_{i=1}^{n} x_i \leq b\Big) \Leftrightarrow (x_1 \leq b \text{ or } \dots \text{ or } x_n \leq b);\end{aligned} \tag{25}$$

In general, with the arbitrary CL function f and its range D_f, we should apply formal methods: divide D_f on sub-range, half-intervals, make a decision for each appropriate inequality (see section 8) and unify the results. Sometimes the analysis of the function $f(x)$ may be understood as the search for given f and ranges $D_{x_1}, \dots, D_{x_n}$ for the arguments $x_1, \dots, x_n$ (components in aggregate range D_x) of the appropriate range D_f (24). This task is an inverse of the previous one; it is based on the following equivalencies:

$$\begin{aligned}&(a \leq x_1 \leq b, c \leq x_2 \leq d) \Leftrightarrow (a \vee c \leq x_1 \vee x_2 \leq b \vee d, ac \leq x_1 x_2 \leq bd),\\&(a \leq x \leq b) \Leftrightarrow (2M - b \leq \bar{x} \leq 2M - a)\end{aligned} \tag{26}$$

The task of the synthesis of the CL function in a common case has no unique solution. An algorithm for the exact solution is unknown. One of the possible methods might be as follows:

1. Discard the requirement $x \leq D_x$;
2. select any standard function $f(x)$;
3. analyze $f(x)$ for the given condition $f \in D_f$, select an appropriate condition for $x : x \in D'_x$;
4. if $D_x \subseteq D'_x$, then $f(x)$ is a solution of this task; otherwise, make a transition to the following function $f(x)$ and continue.

Such an exhaustive search is unrealistic for large n, therefore we can reject the requirement $x \in D_x$, and set up the following problem of the synthesis: construct function $f(x)$ on the given range D_f such that $f(x) \in D_f$. But almost any (except constants) function $f(x)$ of CL with suitable x can accept any value in C. Therefore we have the following problem: Find, according to (24), the range D_x on the given range D_f and selected function $f(x)$.

8. Solution of Equations and Inequalities of CL

Equations and inequalities of CL bear the same sense as the equations and inequalities of DL. However they must be treated in a different way because they do relate to continuous sets. An equation of CL is:

$$f(a, x) \leqq F(a, x), \tag{27}$$

where f and F are given CL functions, $a = (a_1, \dots, a_k)$ is a vector of parameters, $x = (x_1, \dots, x_n)$ is a vector of unknowns. An individual solution of the inequality (27) names any vector x, for which the equality is fair. Equations and inequalities in CL are classified on the number of unknowns n and on the complexity of CL functions of the left and right parts represented in standard deadlock DNF. Now we can fill the inequality with one unknown in the standard form:

$$ax \vee a'\bar{x} \vee bx\bar{x} \vee c \leqq dx \vee d'\bar{x} \vee lx\bar{x} \vee e \tag{28}$$

A maximal number of unknowns and their negations in one elementary conjunction of a standardized inequality is called an order I of the inequality; thus, for example, $I = 2$ for the equation (28). The equations with $I = 1$ are called linear, and those with $I \geq 2$ are nonlinear. A general form of a simple equation with n unknowns in the standard form looks like this:

$$\left(\bigvee_{i=1}^{n} a_i x_i\right) \vee \left(\bigvee_{i=1}^{n} a'_i \bar{x}_i\right) \vee c \leq \left(\bigvee_{i=1}^{n} d_i x_i\right) \vee \left(\bigvee_{i=1}^{n} d'_i \bar{x}_i\right) \vee e \tag{29}$$

The inequalities of CL can be subdivided into ones containing negations of unknowns and those which do not contain them. The main method of the solution of inequalities of CL is the sequential partition of their right-hand and left-hand parts allowing replacing an input inequality by equivalent association of systems of the more simple equations and inequalities.

Example 7. Let us consider equation (27), where the last operation on the left-hand side is the disjunction of CL:

$$f_1(a, x) \vee f_2(a, x) \leqq F(a, x).$$

Using a definition of CL disjunction we can subdivide this equation into an equivalent union of two following systems of equations:

$$\left\{ \begin{array}{l} f_1(a, x) \geq f_2(a, x) \\ f_1(a, x) \leqq F(a, x) \end{array} \right\} \cup \left\{ \begin{array}{l} f_1(a, x) < f_2(a, x) \\ f_2(a, x) \leqq F(a, x) \end{array} \right\}$$

Here, the newly obtained equation is simpler than the original one because it contains fewer operations in one of its parts. The simplification can be continued with the right-hand part of the equation, etc. The process will be finished when we obtain indivisible equations and inequalities that represent a solution of a given equation. Cases when the last operation of the left-hand part or the right-hand part of a given equation is a conjunction can be considered by analogy.

The theory, together with the solution techniques, was developed by V. I. Levin. The most detailed investigation of the problem can be found in [2]. We also recommend the reviews in [4–6, 9].

9. Differentiation and Integration of Continuous-Valued Logical Functions

Functions of DL, defined on discrete arguments, cannot be differentiated or integrated. CL functions have continuous-valued arguments. Therefore they can be differentiated and integrated. However, it is difficult to differentiate the functions of CL because they always have some points at which they break, where a derivative does not exist. Let us call a CL function derived as a superposition of operations $\vee$ and $\wedge$ of the arguments x_i (their negations) as a function of the first and the second sort, respectively. Point $x = (x_1, \ldots, x_n)$ is called a half-regular point of the function of the first sort, if it has an ε-vicinity in the constant ordering of $x_1, \ldots, x_n$. The point $x = (x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n)$ is called a regular point of the function of the second sort, if it has an ε-vicinity in the constant ordering of $x_1, \bar{x}_1, \ldots, x_n, \bar{x}_n$. It is necessary and sufficient for point x to have its coordinates strictly ordered in values to be a half-regular (or regular) point. The following theorems are very important:

1. Any CL function of the first sort has in each half-regular point the single derivative for any argument with values from the set $\{0, 1\}$;
2. any CL function of the second sort has in each regular point the single derivative for any argument with values from the set $\{1, -1, 0\}$;
3. any CL function f in each point of existence of its derivatives $\partial f/\partial x_i$, $i = \overline{1, n}$, has no more than one non-zero derivative.

The main method of differentiation of CL functions lies in their sequential partition with obtaining a collection of simpler expressions, correct in their sub-ranges, and their differentiation. If necessary, some general rules of differential calculus (as, e.g. a derivative of a sum and a product) can be used as well.

Some examples of the derivatives of CL functions are shown below:

$$\begin{aligned} &x'_x = 1, \quad (\bar{x})'_x = -1, \quad (x \vee \bar{x})'_x = 1(x - M) - 1(M - x), \\ &(x\bar{x})'_x = 1(M - x) - 1(x - M), \quad x \neq M; \\ &(x_1 \vee x_2)'_{x_1} = 1(x_1 - x_2), \quad (x_1 x_2)'_{x_1} = 1(x_2 - x_1), \quad x_1 \neq x_2. \end{aligned} \tag{30}$$

Here $1(x)$ is a single function. The condition $x \neq M$ excludes the irregular point $x = M$, where the third and fourth derivatives do not exist. The differential calculation in CL may be a source of new laws. They emerge, in particular, with differentiation of the laws of a CL algebra and can be considered as the differential equations defining various functions of CL. For example:

$$(x_1 \vee x_2)'_{x_1} + (x_1x_2)'_{x_1} = 1, \quad (x_1 \vee x_2)'_{x_1} \cdot (x_1x_2)'_{x_1} = 0. \tag{31}$$

The system in (31) of the two differential equations determines two functions of CL: disjunction and conjunction, which are solutions of the system.

When differentiating functions with a large number of arguments, it is reasonable to transform them to standard forms where a differentiate variable is selected. This form can be easily differentiated. Thus, for example, for standard forms (in the class of DNFs) with the selected variable

$$ax \vee d, \quad b\bar{x} \vee d \tag{32}$$

the derivatives are as follows:

$$\begin{aligned} (ax \vee d)'_x &= I(ax - d) \cdot I(a - x), \quad ax \neq d, \quad x \neq a; \\ (b\bar{x} \vee d)'_x &= I(bx - d) \cdot I(\bar{x} - b), \quad b\bar{x} \neq d, \quad \bar{x} \neq b. \end{aligned} \tag{33}$$

For functions of CL we can define the high-order derivatives, e.g. 2nd order and 3rd order derivatives. In this case any function of the 1st order has a derivative of higher orders, which equal 0; the same takes place for the 2nd order function.

It is possible to integrate CL functions as functions of continuous variables. The function may be decomposed into the collection of more simple expressions, correct in their sub-ranges, which are integrated. If necessary, such usual rules of integration as integral of a sum, subdivision of integration interval, etc., can be used. The obtained integrals always exist because CL functions are continuous.

Differential and integral calculus of CL functions was investigated in detail by E. I. Berkovich and V. I. Levin [12].

10. Completeness in Continuous-Valued Logic

In CL, as well as in DL, there is a problem of completeness. A system of CL functions $\{f_1, \dots, f_m\}$ is a complete system (basis) in R class, if any function from R can be represented by a superposition of the functions $f_1, \dots, f_m$. In contrast with DL, where R is a given and the basis is unknown, in CL the basis is usually a given, and R class has to be found. The following examples seem to be useful:

1. The system $\{\vee, \wedge\}$ is the basis for class R_1 of the functions $C^n \to C$ which accept the value of one of the arguments;
2. the system $\{\vee, \wedge, ^-\}$ is the basis for class R_2 of the functions $C^n \to C$ which accept the value of one of the arguments or its negation;

3. the systems $\{\overline{x_1 x_2}\}$ and $\{\overline{x_1 \vee x_2}\}$ are the basis for class R_1;
4. the systems $\{\overline{x_1 x_2}, ^-\}$ and $\{\overline{x_1 \vee x_2}, ^-\}$ are the basis for class R_2;
5. the system $\{\vee, \wedge, \supset\}$ is the basis for class R_3 of such functions $C^n \to C$ which can be represented in the following form:

$$y = \left[A \vee \left(b_0 + \sum_{i=1}^{n} b_i x_i\right)\right] \wedge B, \quad \text{where } b_0, \ldots, b_n \text{ are integer.} \qquad (34)$$

The classes R_1, R_2, R_3 are different subsets of the continuous set of all CL functions. Mathematically, these classes are quite narrow. However their practical importance cannot be overestimated because the elementary operations of CL (disjunction, conjunction etc.) are similar to the processes of real systems. This adequacy together with the completeness of CL operations lies in the basis of numerous applications of CL in the investigation of mathematical, engineering, economical, social and other phenomena. The problems of completeness regarding CL functions were investigated by R. McNaughton, F. P. Preparata and V. I. Levin. An overview of the results can be found in [5, 6, 9, 21].

11. Conclusions

We may predict that in the future the main attention regarding the theory of CL will be, apparently, given to the development of new generalizations of CL. The good old traditional tasks should not be overlooked either. In the field of enumeration of CL functions, in search for their representations and minimal forms, more effective solutions will be found. Great progress is expected in the applications of CL concerning operation research, simulation of complex economic systems and neuronal structures, description and the analysis of the processes in sociology and history.

It should be noted that in addition to CL, discussed here in this paper, there is another continuous logic. This is the $\aleph_0$-valued logic of Lukasiewicz, defined during the interval $[0, 1]$. This logic has basic operations which are similar to those of CL. However, as it was demonstrated by R. McNaughton [21], but not for all tuples of arguments, for which CL functions are defined, it is possible to determine corresponding functions of Lukasiewicz logic. In the works of V. I. Levin [2, 3, 6, 7, 9] it was proved that by simulating applied systems with CL we must define the operations on the interval $[A, B]$, where $A < 0$, $B > 1$. Therefore, the theoretical and applied potentials of CL are wider than those of Lukasiewicz logic.

The further particulars on the subjects, mentioned in our review, can be found in the publications [1–13]. A lot of applied results were discussed at the conferences held in Penza and Ulianovsk [14–20]. Let us now briefly discuss some Russian works in the field [1, 2, 4, 6–12]. The early book [1] considers the basic principles of continuous-valued logic applied to problems of function approximation, synthesis of functional schemes and the design of electrical circuits. A theory of CL, including equations and inequalities of CL and their application to automata theory and the design of digital devices is proposed in [2, 4]. The book [6] considers CL, its generalizations and application to automata theory, information processing, reliability theory, decision theory, and optimization. A theory of the reliability of engineering systems, based on CL, is built in [7]. The collection of papers [8] includes various fuzzy logics and their application to artificial intelligence. The mathematical apparatus of CL, its generalization and its application to scheduling, optimization and simulation of economic systems can be found in [9]. The hybrid systems, derived from CL and DL and algebraic structures, form a subject of the monograph [11]; here we can also find the minimization of CL functions and its application to functional synthesis. Some basic results of CL and its applications in mathematics, economic, engineering, system theory and biology are presented in [12].

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