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NON-ARCHIMEDEAN FOUNDATIONS OF MATHEMATICS

Finite foundations of mathematics developed by D. Hilbert are presently considered in computer science as an original mathematical canon. Nevertheless, transfinite foundations of mathematics proposed by G. Cantor can also be urgent for soft computing. In this paper I consider some perspectives of transfinite foundations, namely I propose non-Archimedean foundations of mathematics and non-Archimedean multiple-validity. Further, I construct a logical language with non-Archimedean valued semantics.

1. Finite and Transfinite Foundations of Mathematics

In the foundational views of late-19th century mathematicians we can observe two approaches to foundations of mathematics, which are presently called finite and transfinite foundations. According to the first approach developed, e.g. by Kronecker and Brouwer, only objects that are "*intuitively present as immediate experience prior to all thought*" [4] are interpreted as initial objects of algebra and analysis. These objects are considered to be natural numbers. This means that all the mathematical operations must be defined on finite sets or on sets of potential attainable objects, i.e. they must have a reduction to operations on natural numbers. As a result, completed infinite totalities must be rejected in mathematical research. What comes to mind here the Kronecker's famous aphorism: "*Die ganze Zahl schuf der liebe Gott, alles Übrige ist Menschenwerk*" ("*God created natural numbers, all others are fashioned by human beings*").

For instance, David Hilbert considered natural numbers to be the finitary numerals, those which have no meaning, i.e., they do not stand for abstract objects, but can be operated on and compared. According to Hilbert, knowledge of their properties and relations is intuitive and unmediated by logical inference.

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In the transfinite approach to foundations of mathematics created by Georg Cantor, operations on completed infinite totalities are possible if we have no contradiction regarding their restrictions to finite sets. "*Mathematics is in its development entirely free and is only bound in the self-evident respect that its concepts must both be consistent with each other, and also stand in exact relationships, ordered by definitions, to those concepts which have previously been introduced and are already at hand and established. In particular, in the introduction of new numbers, it is only obligated to give definitions of them which will bestow such a determinacy and, in certain circumstances, such a relationship to the other numbers that they can in any given instance be precisely distinguished. As soon as a number satisfies all these conditions, it can and must be regarded in mathematics as existent and real*" [1].

Cantor distinguished between the improper infinite ("*Uneigentlich-Unendliches*") and the proper infinite ("*Eigentlich-Unendliches*"). He called the first kind of infinity the variable finite (*veränderliches Endliches*) and syncategorematic infinity ($\alpha\pi\varepsilon\iota\rho o\nu$, *synkategorematice infinitum*). The second kind of infinity is regarded by him as the actual infinite (*transfinitum*) and categorematic infinity ($\alpha\phi\omega\rho\iota\sigma\mu\varepsilon\nu o\nu$, *kategorematice infinitum*). According to Cantor, the set of natural numbers is improper infinite. He believed that there exists the proper infinite, which includes unattainable objects.

Modern mathematics and computer science are based, as a rule, on discrete objects, therefore they are constructed in the framework of finite foundations of mathematics. These foundations were formulated by Hilbert and they came to be known as *Hilbert's Program*, in which a formalization of all mathematics in axiomatic form, together with proof that this axiomatization of mathematics is consistent, is supposed. The consistency proof itself was to be carried out using only what Hilbert called "finitary" methods. Hilbert's Program was optimistic – it was assumed that the subject of mathematics consists only of the problems that can have a positive or negative solution by means of finitary methods. This optimistic standpoint allows us to set up 23 mathematical problems, which Hilbert addressed to the International Congress of Mathematicians in 1900 (see [5]). Hilbert thought that every well defined mathematical problem can have a solution: "*Take any definite unsolved problem, such as the question as to the irrationality of the Euler-Mascheroni constant C, or the existence of an infinite number of prime numbers of the form $2^n + 1$. However unapproachable these problems may seem to us and however helpless we stand before them, we have, nevertheless, the firm conviction that their solution must follow by a finite number of purely logical processes*" [5].

However there exist mathematical problems that are effectively insolvable, i.e. it is possible to set an effective proof of their insolvability. For example, Gödel and Cohen showed that the first of Hilbert's 23 problems does not have a solution (see [3], [2]). On the other hand, Matiyasevich proved the same result concerning the tenth of Hilbert's 23 problems (see [7]). These results show the limits of application of regarding the finite approach.

The tenth of Hilbert's problems was the following. "*Given a Diophantus equation with any number of unknowns and with rational integer coefficients: devise a process, which could determine by a finite number of operations whether the equation is solvable in rational integers*". Yuri Matiyasevich proved the insolvability of this problem using Cantor's diagonal construction (see [7]).

The first problem was the following: *The set of real numbers is well ordered*, or equivalently there is no a transfinite number between that of a denumerable set and the numbers of the continuum, i.e. $\mathfrak{c} = \aleph_1 = 2^{\aleph_0}$. This statement was proposed by Cantor and it is called *continuum hypothesis*. It is also connected to the axiom of choice. Since the continuum hypothesis is true, every object must be attainable – *for any two upper bounded subsets of an ordered set there exists an upper bound*.

Also, if we accept continuum hypothesis, then we set infinite objects as an infinite union of finite objects. For example, we can define the set ω of natural numbers as follows: $\omega = \sup\{n : n \text{ is a natural number and } n < \omega\}$. In Hilbert's opinion, we cannot avoid infinite constructions, therefore we must obtain infinite unions of finite objects using continuum hypothesis and the axiom of choice. For instance, a set of real numbers should be obtained by the union of rational number sets (denumerable sets): "*It should also be remarked that the process just described for obtaining an upper bound amounts to forming a* union set*. In fact every real number is defined by a partition of the rational numbers into larger and smaller ones or by the set of the smaller rational numbers. The given set of real numbers is therefore represented as a set* $\mathfrak{M}$ *of sets of rational numbers. And the upper bound of the set* $\mathfrak{M}$ *is formed from the set of those rational numbers which belong to at least one of the sets in* $\mathfrak{M}$*. The totality of these rational numbers is, however, exactly the union set of* $\mathfrak{M}$" [4]. On the other hand, "*it follows from the property of an upper bound that for every integer* n *there is a number* c_n *in the set such that* $a - \frac{1}{n} < c_n \leq a$ *and so* $|a - c_n| < \frac{1}{n}$*. The numbers* c_n *constitute therefore a sequence which converges toward* a*, and they all belong to the set under consideration.*

When we argue in this way our manner of expression hides a fundamental point in the proof. For when we use the notation c_n *we presuppose that* *for each number n among those real numbers c belonging to the set under consideration and satisfying the inequality* $a - \frac{1}{n} < c \leq a$*, a certain one is distinguished*." [4].

Also, Hilbert's first problem is to constraint or formalize set theory and thereby prove continuum hypothesis and axiom of choice. Let us take Zermelo-Fraenkel's set theory **ZF**. Suppose M is a model for **ZF** and α is an ordinal. Define M_α by setting (1) $M_0 = \emptyset$, (2) if $\alpha \neq 0$, then $M_\alpha = \bigcup_{\beta<\alpha} M_\beta$. A set x is constructible if there exists an ordinal α such that $x \in M_\alpha$. Gödel showed that continuum hypothesis and the axiom of choice are provable in **ZF** if we take his universe of constructible sets.

At the same time, Cohen proved that we can refute continuum hypothesis and the axiom of choice in other models for **ZF**. Define a new model N that is a denumerable and transitive extension of M. Then we can prove in N that there is no universe of constructible sets. Let $\mathrm{Fn}(A, B) := \{p \subset A \times B$: p is finite function$\}$. Then continuum hypothesis is refuted on the base of $\mathrm{Fn}(\omega \times \lambda, 2)$, where λ is an ordinal, and on the basis of other assumptions, because already $\aleph_1 \neq 2^{\aleph_0}$. Cohen's idea is that there exist infinite unions of finite objects that are not attainable.

Thus, Gödel and Cohen proved that the solution to the first Hilbert's problem depends on the particular version of the set theory assumed, i.e. the acceptance or rejection of the continuum hypothesis and the axiom of choice is subjective and it is not connected to the axiomatic formalization of the set concept. Consequently, the setting of infinite sets as attainable infinite union (as upper bound) is no less subjective than its setting as unattainable infinite union (as actual infinity). The finite approach to foundations of mathematics is no less subjective than a transfinite one!

Consider some examples of sets that cannot be regarded as attainable infinite union (as upper bound).

1. **The set of incomputable functions**. Let $d(n) = U(n, n)$, where U is a two-place computable function that is universal for the class of one-place computable functions. Define the new function as follows

$$d''(x) = \begin{cases} 1 & \text{if } d(x) = 0, \\ 0 & \text{if } d(x) > 0. \end{cases}$$

Each completely defined extension of the function $d''(x)$ will be incomputable. This implies that the set of all completely defined extensions of $d''(x)$ is actual infinity, i.e. we cannot get this set as a minimal one in the given class (it has no an upper bound).

2. **The set of non-constructible real numbers**. Assume that any real numbers of $(0, 1)$ are contained in a sequence $x_1, x_2, \ldots, x_n, \ldots$ and

every number x_n can be represented as an infinite decimal fraction $x_n = 0, a_1^{(n)} a_2^{(n)} \ldots a_n^{(n)} \ldots$ that is not repeating with a repeater of 9. Take a number b_n for any $a_n^{(n)}$ such that $b_n \neq a_n^{(n)}$. Let us consider the infinite decimal fraction $0, b_1 b_2 \ldots b_n \ldots$ This number is called *non-constructible real number*. The set of such numbers is not attainable infinite union – it is the maximal set in the given class.

3. **The set of transcendental numbers**. According to Liouville's theorem, for any algebraic number α with degree $n > 1$, there exists positive λ such that $\left|\alpha - \frac{p}{q}\right| \geq \frac{\lambda}{q^n}$ for any rational number $\frac{p}{q}$. From this it follows that each transcendental number α satisfies the following inequality $\left|\alpha - \frac{p}{q}\right| < \frac{\lambda}{q^n}$, i.e., it has a stronger convergence than an algebraic number. This means that if we set the union of two transcendental numbers, then we obtain their maximum such that the law of strong convergence absorbs the law of weak convergence. Conversely, the law of weak convergence absorbs the law of strong convergence for the union of two algebraic numbers. In fact, if $\alpha_1 \geq \alpha_2$, the law of convergence for an algebraic number α_1 is weaker.

Also, in some cases we cannot use Hilbert's idea of an upper bound as an attainable infinite union for the transfinite setting of mathematical objects and we must postulate actual totalities in advance. We can propose the following understanding of actual infinity. *A set of mathematical objects of the same nature is called an actual infinite set if we are always able to obtain their infinite intersection, but cannot obtain their infinite union (as upper bound) in a general sense.* For instance, if A is the set of incomputable functions (resp. the set of non-constructible real numbers or the set of transcendental numbers) and $A \subset B$, then B is also the set of incomputable functions (resp. the set of non-constructible real numbers or the set of transcendental numbers). Therefore the union for these sets is not upper bound. In the same way, we can postulate the following principle.

Principle 1 (Principle of non-linearity)*An actual infinite totality cannot be represented as linear sequence of its objects.*

Suppose that we have built *a transfinite metalanguage of mathematics*, in which we define well formed formulas by setting some actual infinities. Suppose also that our formulas and proofs can be coded by some numbers (evidently, it is possible that they are actual infinite numbers). Then, our formulas can be seen as a number of transfinite programs to set operations on actual infinite numbers. In consequence we obtain the following new principle.

Principle 2 (Transcendental principle) *Transfinite logical language is infinite-order.*

According to this, transfinite logical language has no a metatheory in the sense that the truth concept is formalized in such metalanguage.

It is known that actual infinite numbers does not satisfy Archimedes' axiom. Let us remember that this axiom is the formula of infinite length which has one of the following notations:

- For any ε that belongs to the interval $[0, 1]$, we have

$$(\varepsilon > 0) \supset [(\varepsilon \geq 1) \vee (\varepsilon + \varepsilon \geq 1) \vee (\varepsilon + \varepsilon + \varepsilon \geq 1) \vee \ldots], \tag{1}$$

- For any positive integer ε, we have

$$[(1 \geq \varepsilon) \vee (1 + 1 \geq \varepsilon) \vee (1 + 1 + 1 \geq \varepsilon) \vee \ldots]. \tag{2}$$

Formulas (1) and (2) are valid in the field $\mathbf{Q}$ of rational numbers as well as in field $\mathbf{R}$ of real numbers. In the ring $\mathbf{Z}$ of integers, only formula (2) has a nontrivial sense, because $\mathbf{Z}$ does not contain numbers of the open interval $(0, 1)$. Also, Archimedes' axiom affirms the existence of an integer multiple of the smaller of the two numbers which exceeds the greater: for any positive real or rational number ε, there exists a positive integer n such that $\varepsilon \geq \frac{1}{n}$ or $n \cdot \varepsilon \geq 1$. The negation of Archimedes' axiom has one of the following forms:

- There exists ε which belongs to the interval $[0, 1]$ such that

$$(\varepsilon > 0) \wedge [(\varepsilon < 1) \wedge (\varepsilon + \varepsilon < 1) \wedge (\varepsilon + \varepsilon + \varepsilon < 1) \wedge \ldots], \tag{3}$$

- There exists a positive integer ε such that

$$[(1 < \varepsilon) \wedge (1 + 1 < \varepsilon) \wedge (1 + 1 + 1 < \varepsilon) \wedge \ldots]. \tag{4}$$

Notice that (3) is the negation of (1). It is obvious that formula (3) reveals there exist *infinitely small numbers* (or *infinitesimals*), i. e., numbers that are smaller than all real or rational numbers of the open interval $(0, 1)$. In other words, ε is said to be an infinitesimal if and only if, for all positive integers n, we have $|\varepsilon| < \frac{1}{n}$. Further, formula (4) reveals there exist *infinitely large integers* that are greater than all positive integers. Infinitesimals and infinitely large integers are called *nonstandard numbers* or *actual infinities.*

The field that satisfies all the properties of $\mathbf{R}$ without Archimedes' axiom is called the field of *hyperreal numbers* and it is denoted by ${}^*\mathbf{R}$. The field that satisfies all the properties of $\mathbf{Q}$ without Archimedes' axiom is called the field of hyperrational numbers and it is denoted by ${}^*\mathbf{Q}$. By definition of field, if $\varepsilon \in \mathbf{R}$ (respectively $\varepsilon \in \mathbf{Q}$), then $1/\varepsilon \in \mathbf{R}$ (respectively $1/\varepsilon \in \mathbf{Q}$). Therefore ${}^*\mathbf{R}$ and ${}^*\mathbf{Q}$ simultaneously contain infinitesimals and infinitely large integers: for an infinitesimal ε, we have $N = \frac{1}{\varepsilon}$, where N is an infinitely large integer.

The ring that satisfies all the properties of $\mathbf{Z}$ without Archimedes' axiom is called the ring of hyperintegers and it is denoted by ${}^*\mathbf{Z}$. This ring includes infinitely large integers. Notice that there exists a version of ${}^*\mathbf{Z}$ that is called the ring of p-adic integers and is denoted by $\mathbf{Z}_p$.

The main originality of non-Archimedean number systems consists in that the set of *hypernumbers* cannot be well ordered – e.g., there is no effective ordering relation on the set of infinitesimals. Therefore hypernumbers satisfy the principle of non-linearity.

Set up a problem to construct a metalanguage of mathematics in that well formed formulas have their truth values in the set of hypernumbers (namely, in ${}^*\mathbf{R}$ and ${}^*\mathbf{Q}$). I will show that this metalanguage is infinite-order, i.e., it satisfies the transcendental principle and it truly is a transfinite metalanguage.

2. Non-Archimedean Valued Matrix

Consider a set Θ. Let I be any infinite index set. Then, we can construct an indexed family Θ^I, i.e., we can obtain the set of all functions: $f\colon I \to \Theta$ such that $f(\alpha) \in \Theta$ for any $\alpha \in I$. The set of all complements for finite subsets of I is a filter and it is called a *Frechet filter*. A maximal filter (ultrafilter) that contains a Frechet filter is called a *Frechet ultrafilter* and is denoted by $\mathcal{U}$. Let $\mathcal{U}$ be a Frechet ultrafilter on I. Define a new relation $\approx$ on the set Θ^I by $f \approx g \equiv \{\alpha \in I\colon f(\alpha) = g(\alpha)\} \in \mathcal{U}$. It is easily proven that the relation $\approx$ is an equivalence. Notice that the aforementioned formula means that f and g are equivalent iff f and g are equal on an infinite index subset. For each $f \in \Theta^I$ let $[f]$ denote the equivalence class of f under $\approx$. The *ultrapower* $\Theta^I/\mathcal{U}$ is then defined to be the set of all equivalence classes $[f]$ as f ranges over Θ^I: $\Theta^I/\mathcal{U} := \{[f]\colon f \in \Theta^I\}$.

Also, we can say that each nonempty set Θ has an ultrapower with respect to a Frechet filter/ultrafilter $\mathcal{U}$. (Notice that if $\mathcal{U}$ is a Frechet filter, we have no well-ordering relation. On the other hand, suppose that $\mathcal{U}$ is a Frechet ultrafilter, in this case we obtain an ineffective well-ordering relation. In the sequel we propose that $\mathcal{U}$ is a Frechet filter.) This ultrapower $\Theta^I/\mathcal{U}$ is said to be a *proper nonstandard extension* of Θ and it is denoted by ${}^*\Theta$. There exist two groups of members of ${}^*\Theta$: (1) functions that are constant, e.g., $f(\alpha) = m \in \Theta$ for an infinite index subset $\{\alpha \in I\}$ (a constant function $[f = m]$ is denoted by ${}^*m$), (2) functions that are not constant. The set of all constant functions of ${}^*\Theta$ is called a *standard set* and is denoted by ${}^\sigma\Theta$.

The members of ${}^{\sigma}\Theta$ are called *standard*. It is readily seen that ${}^{\sigma}\Theta$ and Θ are isomorphic: ${}^{\sigma}\Theta \simeq \Theta$. If Θ was a number system, then members of ${}^{*}\Theta$ are called *hypernumbers*.

Assume that ${}^{*}\mathbf{Q}_{[0,1]} = \mathbf{Q}_{[0,1]}^{\mathbf{N}}/\mathcal{U}$ is a nonstandard extension of the subset $\mathbf{Q}_{[0,1]} = \mathbf{Q} \cap [0,1]$ of rational numbers and ${}^{\sigma}\mathbf{Q}_{[0,1]} \subset {}^{*}\mathbf{Q}_{[0,1]}$ is the subset of standard members. We can extend the usual order structure on $\mathbf{Q}_{[0,1]}$ to a partial order structure on ${}^{*}\mathbf{Q}_{[0,1]}$: (1) for rational numbers x, $y \in \mathbf{Q}_{[0,1]}$ we have $x \leq y$ in $\mathbf{Q}_{[0,1]}$ iff $[f] \leq [g]$ in ${}^{*}\mathbf{Q}_{[0,1]}$, where $\{\alpha \in \mathbf{N}\colon f(\alpha) = x\} \in \mathcal{U}$ and $\{\alpha \in \mathbf{N}\colon g(\alpha) = y\} \in \mathcal{U}$, i.e., f and g are constant functions such that $[f] = {}^{*}x$ and $[g] = {}^{*}y$, (2) each positive rational number ${}^{*}x \in {}^{\sigma}\mathbf{Q}_{[0,1]}$ is greater than any number $[f] \in {}^{*}\mathbf{Q}_{[0,1]} \backslash {}^{\sigma}\mathbf{Q}_{[0,1]}$, i.e., ${}^{*}x > [f]$ for any positive $x \in \mathbf{Q}_{[0,1]}$ and $[f] \in {}^{*}\mathbf{Q}_{[0,1]}$, where $[f]$ is not constant function.

These conditions have the following informal sense: (1) The sets ${}^{\sigma}\mathbf{Q}_{[0,1]}$ and $\mathbf{Q}_{[0,1]}$ have an isomorphic order structure. (2) The set ${}^{*}\mathbf{Q}_{[0,1]}$ contains actual infinities that are less than any positive rational number of ${}^{\sigma}\mathbf{Q}_{[0,1]}$.

Define this partial order structure on ${}^{*}\mathbf{Q}_{[0,1]}$ as follows:

$\mathcal{O}_{{}^{*}\mathbf{Q}}$ For any hyperrational numbers $[f], [g] \in {}^{*}\mathbf{Q}_{[0,1]}$, we set $[f] \leq [g]$ if $\{\alpha \in \mathbf{N}\colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$. For any hyperrational numbers $[f], [g] \in {}^{*}\mathbf{Q}_{[0,1]}$, we set $[f] < [g]$ if $\{\alpha \in \mathbf{N}\colon f(\alpha) \leq g(\alpha)\} \in \mathcal{U}$ and $[f] \neq [g]$, i.e., $\{\alpha \in \mathbf{N}\colon f(\alpha) \neq g(\alpha)\} \in \mathcal{U}$. For any hyperrational numbers $[f], [g] \in {}^{*}\mathbf{Q}_{[0,1]}$, we set $[f] = [g]$ if $f \in [g]$.

Introduce two operations max, min in the partial order structure $\mathcal{O}_{{}^{*}\mathbf{Q}}$: (1) $\min([f], [g]) = [h]$ iff there exists $[h] \in {}^{*}\mathbf{Q}_{[0,1]}$ such that $\{\alpha \in \mathbf{N}\colon \min(f(\alpha),\ g(\alpha)) = h(\alpha)\} \in \mathcal{U}$; (2) $\max([f], [g]) = [h]$ iff there exists $[h] \in {}^{*}\mathbf{Q}_{[0,1]}$ such that $\{\alpha \in \mathbf{N}\colon \max(f(\alpha), g(\alpha)) = h(\alpha)\} \in \mathcal{U}$.

Note there exist the maximal number ${}^{*}1 \in {}^{*}\mathbf{Q}_{[0,1]}$ and the minimal number ${}^{*}0 \in {}^{*}\mathbf{Q}_{[0,1]}$ under condition $\mathcal{O}_{{}^{*}\mathbf{Q}}$.

Now define *hyperrational valued matrix logic* $\mathbf{M}_{{}^{*}\mathbf{Q}}$ as the ordered system $\langle V_{{}^{*}\mathbf{Q}}, \neg, \supset, \vee, \wedge, \widetilde{\exists}, \widetilde{\forall}, \{{}^{*}1\}\rangle$, where (1) $V_{{}^{*}\mathbf{Q}} = {}^{*}\mathbf{Q}_{[0,1]}$ is the subset of hyperrational numbers, (2) for all $x \in V_{{}^{*}\mathbf{Q}}$, $\neg x = {}^{*}1 - x$, (3) for all $x, y \in V_{{}^{*}\mathbf{Q}}$, $x \supset y = {}^{*}1 - \max(x, y) + y$, (4) for all $x, y \in V_{{}^{*}\mathbf{Q}}$, $x \vee y = (x \supset y) \supset y = \max(x, y)$, (5) for all $x, y \in V_{{}^{*}\mathbf{Q}}$, $x \wedge y = \neg(\neg x \vee \neg y) = \min(x, y)$, (6) for a subset $M \subseteq V_{{}^{*}\mathbf{Q}}$, $\widetilde{\exists}(M) = \max(M)$, where $\max(M)$ is a maximal element of M, (7) for a subset $M \subseteq V_{{}^{*}\mathbf{Q}}$, $\widetilde{\forall}(M) = \min(M)$, where $\min(M)$ is a minimal element of M, (8) $\{{}^{*}1\}$ is the set of designated truth values.

The truth value ${}^{*}0 \in V_{{}^{*}\mathbf{Q}}$ is false, the truth value ${}^{*}1 \in V_{{}^{*}\mathbf{Q}}$ is true, and other truth values $x \in V_{{}^{*}\mathbf{Q}}$ are neutral.

If we replace the set $\mathbf{Q}_{[0,1]}$ by $\mathbf{R}_{[0,1]}$ and the set ${}^{*}\mathbf{Q}_{[0,1]}$ by ${}^{*}\mathbf{R}_{[0,1]}$ in all above definitions, then we obtain *hyperreal valued matrix logic* $\mathbf{M}_{{}^{*}\mathbf{R}}$.

3. Non-Archimedean Valued Propositional Logical Language

An infinite-order propositional logical language $\mathcal{L}_V^\infty$ consists of the following symbols: (1) first-order propositional formulas: φ, ϕ, ψ, ... of $\mathbf{n}$-valued Łukasiewicz's logic $Ł_V$, where $|V| = \mathbf{n}$; (2) logical symbols: (i) various order propositional connectives of arity n_j: $F_0^{n_0}$, $F_1^{n_1}$, ..., $F_r^{n_r}$, which are built by a superposition of negation $\neg$ and implication $\supset$, (ii) vertical quantifiers of various order Q_1, Q_2,...,Q_{i-1},... such that an i-order quantifier has the lower index $i-1$; (3) auxiliary symbols: (,), and , (comma).

If $V = \mathbf{Q}_{[0,1]}$, then we have $\aleph_0$ universal vertical quantifiers at the level i and $\aleph_0$ existential vertical quantifiers at the level i: $\forall_i^{y \in \mathbf{Q}_{[0,1]}}$, ..., $\forall_i^{y' \in \mathbf{Q}_{[0,1]}}$, ..., $\exists_i^{y \in \mathbf{Q}_{[0,1]}}$, ..., $\exists_i^{y' \in \mathbf{Q}_{[0,1]}}$, ... If $V = \mathbf{R}_{[0,1]}$, then we have $2^{\aleph_0}$ universal vertical quantifiers at the level i and $2^{\aleph_0}$ existential vertical quantifiers at the level i: $\forall_i^{y \in \mathbf{R}_{[0,1]}}$, ..., $\forall_i^{y' \in \mathbf{R}_{[0,1]}}$, ..., $\exists_i^{y \in \mathbf{R}_{[0,1]}}$, ..., $\exists_i^{y' \in \mathbf{R}_{[0,1]}}$, ...

Well-formed formulas of $\mathcal{L}_V^\infty$ are inductively defined as follows: (1) If φ is a first-order propositional formula of $\mathbf{n}$-valued Łukasiewicz's logic $Ł_V$, where $|V| = \mathbf{n}$, and $\mathrm{Q}_1, \mathrm{Q}_2, \ldots, \mathrm{Q}_{i-1}$ are a finite sequence of vertical quantifiers, then

$$\mathrm{Q}_{i-1}(\ldots(\mathrm{Q}_1\varphi)\ldots)\ldots\mathrm{Q}_1\varphi(\varphi)$$

is an i-order formula denoted sometimes by φ_i or by $\mathrm{Q}_{i-1}\varphi(\varphi_{i-1})$ to emphasize what is the least quantifier in φ_i. It is called atomic or an atom. Its outermost logical symbols are $\mathrm{Q}_1, \mathrm{Q}_2, \ldots, \mathrm{Q}_{i-1}$. If the first of these quantifiers is $\forall_1^{y_1 \in V}$, then the other are also universal with the upper indices that are equal to y_1. If the first of these quantifiers is $\exists_1^{y_1 \in V}$, then the other are also existential with the upper indices that are not possibly equal to y_1. (2) If $\varphi_i, \ldots, \psi_i$ are formulas of i-order and F^n is a propositional connective of arity n, then $F^n(\varphi_i, \ldots, \psi_i)$ is an i-order formula with outermost logical symbol F^n. (3) If φ is a first-order propositional formula and $\mathrm{Q}_1, \mathrm{Q}_2, \ldots, \mathrm{Q}_{i-1}, \ldots$ are an infinite sequence of vertical quantifiers, then

$$\ldots\mathrm{Q}_{i-1}(\ldots(\mathrm{Q}_1\varphi)\ldots)\ldots\mathrm{Q}_1\varphi(\varphi)$$

is an infinite-order formula denoted sometimes by φ_∞. It is called atomic or an atom. Its outermost logical symbols are $\mathrm{Q}_1, \mathrm{Q}_2, \ldots, \mathrm{Q}_{i-1}, \ldots$ If the first of these quantifiers is $\forall_1^{y_1 \in V}$, then the other are also universal with the upper indices that are equal to y_1. If the first of these quantifiers is $\exists_1^{y_1 \in V}$, then the other are also existential with the upper indices that are not possibly equal to y_1. (4) If $\varphi_\infty, \ldots, \psi_\infty$ are formulas of infinite order and F^n is a propositional connective of arity n, then $F^n(\varphi_\infty, \ldots, \psi_\infty)$ is a formula with outermost logical symbol F^n and this formula is an infinite-order formula.

Consider the set ${}^*V$ of all equivalence classes $[f]$ under a Frechet ultrafilter $\mathcal{U}$ such that $f\colon \mathbf{N} \to V$. Recall that for each $i \in V$, ${}^*i = [f = i]$, i.e., it is a constant function. Every element of ${}^*V$ has the form of an infinite tuple $[f] = \langle y_0, y_1, \ldots \rangle$.

Let 1 be the designated truth value of $\mathbf{n}$-valued Łukasiewicz's logic Ł$_V$, where $|V| = \mathbf{n}$. An i-order truth assignment is a function $v_i(\cdot)$ whose domain is the set of all i-order formulas of $\mathcal{L}_V^\infty$ and whose range is the set ${}^*V$ of truth values such that:

1. For any first-order propositional formula φ_1, $v_1(\varphi_1)$ is a truth assignment of $\mathbf{n}$-valued Łukasiewicz's logic.
2. For any first-order propositional formula φ_1, $v_i(\varphi_1) = \langle \underbrace{y_1, y_1, \ldots y_1}_{i} \rangle$ iff $v_1(\varphi_1) = y_1$.
3. For any i-order atomic propositional formula $\forall_{i-1}^{y_{i-1} \in V} \varphi(\varphi_{i-1})$, (1) if $v_{i-1}(\forall_{i-2}^{y_{i-2} \in V} \varphi(\varphi_{i-2})) = \langle \underbrace{y_{i-1}, \ldots, y_{i-1}}_{i-1} \rangle$ for all valuations, then
$$v_i(\forall_{i-1}^{y_{i-1} \in V} \varphi(\varphi_{i-1})) = \langle \underbrace{y_{i-1}, \ldots, y_{i-1}}_{i} \rangle;$$
(2) if (i) $v_{i-1}(\forall_{i-2}^{y_{i-2} \in V} \varphi(\varphi_{i-2})) \neq \langle \underbrace{y_{i-1}, \ldots, y_{i-1}}_{i-1} \rangle$ for some valuations, (ii) $v_{i-1}(\forall_{i-2}^{y_{i-2} \in V} \varphi(\varphi_{i-2})) = \langle y'_1, \ldots, y'_{i-1} \rangle$ and $v_1(\varphi) = y'_0$ for some valuations, then $v_i(\forall_{i-1}^{y_{i-1} \in V} \varphi(\varphi_{i-1})) = \langle y'_0, y'_1 \ldots, y'_{i-1} \rangle$.
4. For any i-order atomic propositional formula
$$\varphi_i = \exists_{i-1}^{y_{i-1} \in V} (\ldots (\exists_1^{y_1 \in V} \varphi) \ldots) \ldots \exists_1^{y_1 \in V} \varphi(\varphi),$$
(1) if (i) $v_{i-1}(\varphi_{i-1}) = \langle y_1, \ldots, y_{i-1} \rangle$ for some valuations and (ii) $v_1(\varphi_1) = y_0$ for some valuations, then $v_i(\varphi_i) = \langle y_0, y_1, \ldots, y_{i-1} \rangle$; (2) if (i) $v_{i-1}(\varphi_{i-1}) \neq \langle y_1, \ldots, y_{i-1} \rangle$ for all valuations, (ii) $v_{i-1}(\varphi_{i-1}) = \langle y'_1, \ldots, y'_{i-1} \rangle$ and $v_1(\varphi) = y'_0$ for some valuations, then $v_i(\varphi_i) = \langle y'_0, y'_1 \ldots, y'_{i-1} \rangle$.
5. For any formula φ_i, $v_i(\neg\varphi_i) = \langle 1 - y_0, 1 - y_1, \ldots, 1 - y_{i-1} \rangle$, where $v_i(\varphi_i) = \langle y_0, y_1, \ldots, y_{i-1} \rangle$.
6. For any formulas φ_i and ψ_i, $v_i(\varphi_i \supset \psi_i) = \langle (1 - \max(x_0, y_0) + y_0), (1 - \max(x_1, y_1) + y_1), \ldots, (1 - \max(x_{i-1}, y_{i-1}) + y_{i-1}) \rangle$, where $v_i(\varphi_i) = \langle x_0, x_1, \ldots, x_{i-1} \rangle$ and $v_i(\psi_i) = \langle y_0, y_1, \ldots, y_{i-1} \rangle$.

7. For any formulas φ_i and ψ_i, $v_i(\varphi_i \vee \psi_i) = \langle \max(x_0, y_0), \max(x_1, y_1), \ldots, \max(x_{i-1}, y_{i-1})\rangle$, where $v_i(\varphi_i) = \langle x_0, x_1, \ldots, x_{i-1}\rangle$ and $v_i(\psi_i) = \langle y_0, y_1, \ldots, y_{i-1}\rangle$.
8. For any formulas φ_i and ψ_i, $v_i(\varphi_i \wedge \psi_i) = \langle \min(x_0, y_0), \min(x_1, y_1), \ldots, \min(x_{i-1}, y_{i-1})\rangle$, where $v_i(\varphi_i) = \langle x_0, x_1, \ldots, x_{i-1}\rangle$ and $v_i(\psi_i) = \langle y_0, y_1, \ldots, y_{i-1}\rangle$.

An infinite-order truth assignment is a function $v_\infty[\cdot]$ whose domain is the set of all infinite-order formulas of $\mathcal{L}_V^\infty$ and whose range is the set ${}^*V$ of truth values such that:

1. For any first-order propositional formula φ_1, $v_1(\varphi_1)$ is a truth assignment of **n**-valued Łukasiewicz's logic.
2. For any first-order propositional formula φ_1, $v_\infty[\varphi_1] = {}^*y_1 = \langle y_1, y_1, \ldots\rangle$ iff $v_1(\varphi_1) = y_1$.
3. For any infinite-order atomic propositional formula

$$\varphi_\infty = \ldots \forall_{i-1}^{y_{i-1} \in V}(\ldots(\forall_1^{y_1 \in V} \varphi)\ldots)\ldots \forall_1^{y_1 \in V} \varphi(\varphi),$$

(1) if $v_\infty[\varphi_\infty] = {}^*y_1 = \langle y_1, \ldots, y_1, \ldots\rangle$ for all valuations, then $v_\infty[\varphi_\infty] = {}^*y_1$; (2) if (i) $v_\infty[\varphi_\infty] \neq {}^*y_1$ for some valuations and (ii) $v_\infty[\varphi_\infty] = [f]$ for some valuations, then $v_\infty[\varphi_\infty] = [f]$.

4. For any infinite-order atomic propositional formula

$$\varphi_\infty = \ldots \exists_{i-1}^{y_{i-1} \in V}(\ldots(\exists_1^{y_1 \in V} \varphi)\ldots)\ldots \exists_1^{y_1 \in V} \varphi(\varphi),$$

(1) if $v_\infty[\varphi_\infty] = [f] = \langle y_0, y_1, \ldots, y_{i-1}, \ldots\rangle$ for some valuations, then $v_\infty[\varphi_\infty] = [f]$; (2) if $v_\infty[\varphi_\infty] \neq [f]$ for all valuations and $v_\infty[\varphi_\infty] = [f']$ for some valuations, then $v_\infty[\varphi_\infty] = [f']$.

5. For any formula φ_∞, $v_\infty[\neg\varphi_\infty] = {}^*1 - v_\infty[\varphi_\infty]$.
6. For any formulas φ_∞ and ψ_∞, $v_\infty[\varphi_\infty \supset \psi_\infty] = {}^*1 - \max(v_\infty[\varphi_\infty], v_\infty[\psi_\infty]) + v_\infty[\psi_\infty]$.
7. For any formulas φ_∞ and ψ_∞, $v_\infty[\varphi_\infty \vee \psi_\infty] = \max(v_\infty[\varphi_\infty], v_\infty[\psi_\infty])$.
8. For any formulas φ_∞ and ψ_∞, $v_\infty[\varphi_\infty \wedge \psi_\infty] = \min(v_\infty[\varphi_\infty], v_\infty[\psi_\infty])$.

Note that the function $v_\infty[\cdot]$ is an infinite sequence of functions $v_i(\cdot)$.

Suppose that **n**-valued Łukasiewicz's logic $Ł_V$, where $|V| = \mathbf{n}$, is truth-functionally complete thanks to Słupecki's operators $\mathrm{T}_k(\varphi) : v(\varphi) \mapsto k \in V/\{0, 1\}$, where $v(\varphi)$ is any truth valuation of $\varphi \in Ł_V$. In this case some formulas of $\mathcal{L}_V^\infty$ have truth values of the form ${}^*k$. Without Słupecki's operators, only ^{*0}, ^{*1} are constant functions that can be truth values for formulas of $\mathcal{L}_V^\infty$.

4. Non-Archimedean Valued Logic

A hyperrational valued logic denoted by $Ł_{{}^*\mathbf{Q}_{[0,1]}}$ (resp. a hyperreal valued logic denoted by $Ł_{{}^*\mathbf{R}_{[0,1]}}$) is built on the basis of language $\mathcal{L}^\infty_{\mathbf{Q}_{[0,1]}}$ (resp. $\mathcal{L}^\infty_{\mathbf{R}_{[0,1]}}$).

The following properties of higher-order formulas are evident without proofs:

- $v_i(\forall_{i-1}^{y_{i-1}\in V}\varphi(\varphi_{i-1})) = \langle y_{i-1}, \ldots, y_{i-1}\rangle$:= "a formula φ has the truth value y_{i-1} for all truth valuations, a formula φ_1 has the truth value $\langle y_{i-1}, y_{i-1}\rangle$ for all truth valuations, etc.";
- $v_i(\exists_{i-1}^{y_{i-1}\in V}(\ldots(\exists_1^{y_1\in V}\varphi)\ldots)\ldots\exists_1^{y_1\in V}\varphi(\varphi)) = \langle y_0, \ldots, y_{i-1}\rangle$:= "a formula φ has the truth value y_0 for some truth valuations, a formula φ_1 has the truth value $\langle y_0, y_1\rangle$ for some truth valuations, etc.";
- a formula $\forall_{i-1}^{y_{i-1}\in V}\varphi(\varphi_{i-1})$ or $\exists_{i-1}^{y_{i-1}\in V}(\ldots(\exists_1^{y_1\in V}\varphi)\ldots)\ldots\exists_1^{y_1\in V}\varphi(\varphi)$ is a tautology iff a formula φ is a tautology;
- a formula $\forall_{i-1}^{y_{i-1}\in V}\varphi(\varphi_{i-1})$ or $\exists_{i-1}^{y_{i-1}\in V}(\ldots(\exists_1^{y_1\in V}\varphi)\ldots)\ldots\exists_1^{y_1\in V}\varphi(\varphi)$ is satisfiable iff a formula φ is satisfiable.

These properties allow setting a non-Archimedean calculus. Consequently, we can extend the Hilbert's type calculus of infinite-valued Łukasiewicz's logic $Ł_\infty$ for the non-Archimedean case. *The axioms of the Hilbert's type calculus for* $Ł_{{}^*\mathbf{Q}_{[0,1]}}$ (resp. for $Ł_{{}^*\mathbf{R}_{[0,1]}}$) are as follows.

$$(\varphi_i \supset \phi_i) \supset ((\phi_i \supset \psi_i) \supset (\varphi_i \supset \psi_i)), \tag{5}$$

$$\varphi_i \supset (\phi_i \supset \varphi_i), \tag{6}$$

$$((\varphi_i \supset \phi_i) \supset \phi_i) \supset ((\phi_i \supset \varphi_i) \supset \varphi_i), \tag{7}$$

$$(\neg\varphi_i \supset \neg\phi_i) \supset (\phi_i \supset \varphi_i), \tag{8}$$

$$(\varphi_\infty \supset \phi_\infty) \supset ((\phi_\infty \supset \psi_\infty) \supset (\varphi_\infty \supset \psi_\infty)), \tag{9}$$

$$\varphi_\infty \supset (\phi_\infty \supset \varphi_\infty), \tag{10}$$

$$((\varphi_\infty \supset \phi_\infty) \supset \phi_\infty) \supset ((\phi_\infty \supset \varphi_\infty) \supset \varphi_\infty), \tag{11}$$

$$(\neg\varphi_\infty \supset \neg\phi_\infty) \supset (\phi_\infty \supset \varphi_\infty), \tag{12}$$

$$(\neg(\psi_1 \equiv \psi_\infty) \wedge \neg(\phi_1 \equiv \bot)) \supset (\psi_\infty \supset \phi_1), \tag{13}$$

where $\bot$ is a contradiction.

Axioms (5)–(8) are called *horizontal.* Axiom (13) is called *vertical.* Axioms (9)–(12) are called *axioms of infinite length.* Notice that the upper indices of vertical quantifiers belong to the countable set $\mathbf{Q}_{[0,1]}$ in $Ł_{*\mathbf{Q}_{[0,1]}}$ and belong to the uncountable set $\mathbf{R}_{[0,1]}$ in $Ł_{*\mathbf{R}_{[0,1]}}$.

Inference rules are as follows: (1) *modus ponens* (if two formulas φ_i (resp. φ_∞) and $\varphi_i \supset \psi_i$ (resp. $\varphi_\infty \supset \psi_\infty$) hold, then we deduce a formula ψ_i (resp. ψ_∞)); (2) *substitution rule*: we can substitute a formula of the same order for an atomic formula.

The non-Archimedean valued Hilbert's type calculus has all the deductive and semantic properties of the Hilbert's type calculus of infinite-valued Łukasiewicz's logic $Ł_\infty$. At the same time the truth concept of $Ł_\infty$ can be syntactically expressed by means of non-Archimedean valued logic. Thus, non-Archimedean valued logic is more formally expressive than $Ł_\infty$.

5. Conclusions

In this paper I considered some perspectives of transfinite foundations of mathematics, namely I built the non-Archimedean valued propositional logic. This new metalanguage also has a lot of practical applications, in particular it can be applied in non-Kolmogorovian probability theory and in soft computing (see [6] and [8]).

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