Some Remarks on The Language of Mathematical Texts*

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We want to remind the readers of an article published nearly fifteen years ago in the joint issue 10/11 of "Studies in Logic, Grammar and Rhetoric". The decision about its reprint in the same periodical is not due to publishing of a special edition of "Studies..." devoted to the Mizar project and its author Andrzej Trybulec, even though dr Trybulec is the central figure of this text. It was him who initiated the subject and was the critical consultant of the whole article, which in spite of its limited influence due to a restricted circulation of the issue and its local character at that time, was quoted and referred to especially in texts of researchers working on formalization of the language of mathematics. Even if we treat this text as a yet historical contribution to the discussion on properties of the mathematical discourse, we are convinced that some of the ideas contained in it are still topical and can become a subject of a detailed analysis anew, particularly in the domain of studies concerned with standardization of the language of mathematics.

In mathematical literature, in popular science texts, in the teaching of mathematics, in the literature concerned with scientific methodology, and in handbooks of stylistics we encounter the concept of "the language of mathematics".

* The present paper is intended as a discussion of selected problems related to the language of mathematical texts and the ways of their organization. The complexity of mathematical statements – at both the language level and the text level – prevents us from dealing in this paper with all detailed issues related to the main subject matter. Hence we focus our attention on finding a proper definition of the language used to describe mathematical objects and their properties. We also try to answer the question about the universal nature of the language of mathematical texts by analysing its semantics and the properties of mathematical symbolism (mainly exemplified by variables).

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The conviction that mathematics has a language of its own is so common among mathematicians that the self-evident distinctive characteristics of speaking about mathematical objects absolves those who use that term from offering its precise definition. Some authors formulate an approximate meaning of that term, but their formulations are not in full agreement with one another and they partly obscure the picture of that self-evident reality. It is said that the language of mathematics is a set of formulas intermixed with sentences in natural language [6]. It is emphasized that the language of mathematics is a largely heterogeneous product, a set of languages, each of which refers to a different branch of mathematics [5]. It is also claimed that the language of mathematics is a set of symbols and the rules of using them, completed by the words and grammar drawn from natural language [3]. From some others, finally, it is merely a formalized language with a precisely defined vocabulary and syntax.

The distinctive manner of speaking about, or describing, mathematical objects accounts for the fact that the language of mathematics is often opposed to languages used in other disciplines, which – in the opinion of mathematicians – avail themselves of a freedom of expression which is denied to them. “The terms and expressions in those works (i.e., pertaining to domains not related to mathematics – Z.T. & H.Ś.) need not be precisely defined, and theorems should be only roughly true. Mathematicians suffer from their conviction that terms without precise definitions are meaningless, and statements which are not true are false” [11]. We shall not engage in a polemic with the opinion of the author quoted above, but we have to note that in works on stylistics the style of mathematical texts is singled out as something distinctive. On the other hand, no references are made to the languages of such disciplines as medicine, geology, or agronomy.

The concept of the language used in described mathematical reality is understood in various ways, according, among other things, to the purposes for which language is studied. Definitions of language usually, if not always, are constructed for natural language or for an artificial language as something which in some sense is opposed to ethnic languages.

If we want to speak about the language of mathematics, then we must be convinced about its systemic nature. If this is so, then it should be subsumed under the general definition of language used by linguists. According to one such definition, language is a system of signs used to convey thoughts, that is serving the purpose of communication among members of a given community. This definition is applied to natural language, but one could hardly claim that it suffices to describe the language of mathematics. First of all, even if we agree that by community we mean the community of mathematicians only, we face the fact that researchers who specialize in a single domain of mathematics need not be familiar with symbolism used in other branches of mathematics. This is so because it is claimed that in mathematics we do not have to do with any single language, and that in fact every branch of mathematics uses a language of its own and its specific symbolism. And even if one accepts the opinion that the language of mathematics is merely a specialized fragment of natural language, then that opinion is weakened by the statement that language is even less known than natural language and that the tools used in linguistics prove insufficient for the analysis of its properties [5].

Difficulties emerge already when we try to fix the alphabet of that language. It may be said to consist of all letters of a given ethnic language plus special symbols and letters drawn from alphabets of other languages. eg., Greek, Latin, Hebrew, and Gothic letters, useful in denoting functions, variables, constants, etc. The vocabulary of the language of mathematics is a set of expressions which differ from both those occurring in the everyday language and those used in other disciplines. It includes strictly mathematical terms, terms drawn from everyday language but used in a new, specialized sense, and words drawn from everyday language useful in the construction of mathematical statements. Note in this connection the characteristic processes in the sphere of word-formation and the productivity of some formants, such as **sub-** in such terms as **subset**, **subgroup**, **subdomain**, **subbundle**. Peculiarities are also observed in expressions which are at variance with standard syntactic rules, such as "**f** is a mapping onto ...", "**f** is a mapping into ...", "**f** is a transformation onto ...". Such constructions, plus expressions such as "for almost all", "there are arbitrarily large natural numbers", "the function takes on the value zero almost everywhere", "if and only if", have a fixed meaning in the language of mathematics. Hence, it seems, it is not always correct to compare the vocabulary of the language of mathematics with units from the morphological level of any ethnic language. If we include grammar, then we may merely say that the so-called language of mathematics maps or assumes the grammatical system of natural language in those its fragments in which mathematical statements meet the criteria of linguistic correctness imposed upon it by the grammar of the ethnic language in which those statements are formulated.

It seems neither useful nor correct to treat the language of mathematics as natural language. Nor would it be right to consider the language of mathematical texts as a formalized language in the sense of a language with a precisely definite and explicitly described syntax, in spite of the endeavours to write mathematical texts using such a language. Most mathematical texts show a considerable freedom in expressing mathematical ideas, which is due to the fact that if we know well what are we talking about, then it is less important how we convey that.

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The term "language of mathematics", as has been shown above, has no sufficiently precise meaning. But we have at our disposal a wealth of empirical data in the form of mathematical texts written in various ethnic languages. It may be assumed that those texts, for all the differences of content, are marked by a certain similarity. But it would be rather risky to claim that mathematical texts comply with a linguistic system that might be termed the system of the language of mathematics. That would imply the necessity of describing that language in accordance with linguistic criteria or by methods adopted for the description of formal systems.

In view of definitional problems we want to study the language used in the formulation of mathematical statements. We accordingly suggest, for that purpose, to replace the term "the language of mathematics" by the term "the language of mathematical texts". Terminological decisions necessitate the explanation of the goal of the study of the subject matter so defined. Note that the study of texts in natural language may, on the one hand, serve the purpose of constructing a model of the linguistic system of a given language, but on the other, the data available may serve as the foundation for the formulation of a theory of text. If we assume that the language of mathematics does not exist in such a form as an arbitrary ethnic language, and that it is not legitimate to speak about it as a formalized language, we thereby preclude the possibility of constructing a system. In our opinion, mathematical texts are utterances which comply not so much with a certain linguistic system (even though it is obvious that they reproduce fragments of some systems) as with certain patterns of texts, a universal structure filled with content suitable for the described fragment of mathematical reality. Our task is to reveal that structure and to show features which are common to all kinds of mathematical statements. Hence the present paper has its place in that trend of study which is usually termed text theory.

The language of mathematical texts is a language in which people formulate statements pertaining to the various branches of mathematics. Such statements are sequences of signs, usually in a written form, although they may take on the form of, say, a spoken lecture. When referring to mathematical texts we shall hereafter mean their written form. Mathematical texts take on several stylistic variations, whose choice depends on the scope of the problem presented by a given author and the intended group of readers. These variations are: paper (article), monograph, and handbook. We want here explicitly to oppose strictly mathematical texts, that is those whose content refers to mathematical objects, to texts in which the mathematical apparatus is used to explain or to interpret non-mathematical domains.

Let us reflect here on the criteria which make it possible to distinguish mathematical texts from the remaining ones, drawn from other disciplines. It is common knowledge that in mathematical texts symbols play an essential role. The proper choice of such symbols is no less important, because an incorrect choice of symbols not only makes mathematical texts illegible, but makes them fail to perform their main function: it is not conductive to solving intricate problems. A contemporary mathematical text lacking symbolism altogether is unthinkable, either.

Hence it might seem that it is symbolism which is the main element (or even discriminant) of mathematical texts, and an average reader who looks on shelves for a non-mathematical text is guided above all by that criterion. He divides texts into those who lack symbolism and classes them as non-mathematical ones, and those which include it, and treats them as mathematical ones.

The above criterion, simple in application as it is, does not, however, seem satisfactory. It is true that mathematical texts include symbolism, but not every text which includes it is a mathematical one. For instance, chemical texts abound in symbolism, as also do those concerned with medicine, physics, and biology, and the same even applies to linguistic texts, seemingly so free from mathematical elements. It suffices to inspect some philosophical texts to treat them as mathematical one on the strength of that criterion, and yet we would protest against consider chemical and philosophical texts as mathematical ones.

An even cursory inspection of mathematical texts allows us to conclude that in addition to symbolism to be found in them, they are formulated in some their parts in a language which is a variety of a given ethnic language. Even such a superficial contact with the language of mathematical texts shows that those parts which are recorded in symbols and those recorded in an ethnic language are in various proportions. One can encounter texts written almost like texts concerned with other disciplines, also completely deprived of symbolism or including a very small amount of symbols, as well as other texts abounding in symbolism so much that they are outright illegible to a person not already familiar with the domain to which they pertain. Mathematical texts, next to symbols and parts recorded in an ethnic language, may include diagrams and drawings, which must also be recognized as elements of the language of mathematics.

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In this paper by a mathematical text we mean, following L. A. Kaluzhnin [4], a sequence of symbols and verbal text in a definite ethnic language used in contemporary mathematical literature. Unlike scientific texts in other disciplines, a mathematical text has certain specific features. Note, for instance, that regardless of the ethnic language in which it is formulated, a reader who has at least an elementary mathematical training will recognize its organization, its structure, and a mathematician, even if he does not know the ethnic language in question, can also recognize its content.

As follows from the above, every mathematical text can be split into two parts. One of them is formal (representable in a formalized form), and the other, informal. The formal text, called the proper text by some authors, is usually presented in a language which is a mixture of a formalized language and an ethnic one. This is how a large number of theorems are recorded. The recording of the text proper in a formalized language alone makes that text difficult to read, because a large amount of information is concentrated in its small fragments. That is why the proper text is usually written in a language which might be termed hybrid. Definitions, theorems and proofs are such a formal text. Next to it in mathematical texts we find fragments of text written usually in an ethnic language; they convey the author's intuitions, interpretations of some fact, and serve as a *sui generis* connective tissue of the proper text.

Thus in the structure of a mathematical text we shall single out two strata: the formal (consisting of the formalized text) and the informal (intuitive). Following A. Trybulec ([12], [13]) we shall term the former the objective stratum, and the latter, the subjective one.

The objective stratum in a mathematical text differs quite visibly from the subjective one. It begins where it has its own name (definition, theorem, lemma, conclusion, postulate, proof) or is printed in a different type. The end of a proof is usually indicated by "q.e.d." or by some other sign. Some authors do not do that and confine themselves to stating in the paragraph that follows the proof that the proof has been carried out. Very often both definitions and theorems are numbered to make references to them easier in further parts of the text. For instance, in A. Grzegorczyk's *Outline of Theoretical Arithmetic* (in Polish), [2] definitions are preceded by abbreviations D1, D2, ...; axioms, by A1, A2, ...; theorems, by T1, T2, ...; rules, by R1, R2, ...; lemmas, by L1, L2 ... The beginning of the proof is marked "dowód", which in Polish means 'proof'.

The objective stratum of a mathematical text, and its subjective stratum, are – as in the case of symbolism and natural language – in various proportions. Some mathematical texts have the former stratum very expanded, as the just quoted text by A. Grzegorczyk (this usually applies to monographs and papers); others have a rich subjective stratum (handbook, teaching aids, popular papers).

Note that the primary goal of every mathematical text is to present its formal stratum, the secondary goal consisting in offering the reader a method which would allow him to incorporate that formal stratum with the knowledge he has already had and to keep it there as part of his working intellectual endowment. That secondary goal is attained by those parts of the text which we have termed the informal stratum.

The informal stratum of a mathematical text consists of those fragments of that text which include motivations, analogies, examples, and non-mathematical explanations. N. R. Steenrod [11], by analysing L. C. Young's work *Lectures on the Calculus of Variations* and the book by Hurewicz and Wailman, *Dimension Theory*, drew a list of the informal material which can be found in mathematical works. That list covers the introductory material, which should include a brief review of the basic material which forms the content of the formal structure. The intuitive stratum should also present motivations and an analysis of examples which suggest hypotheses. There is also a place in it for a cursory description of the results that will be arrived at, and the methods that will be used. The informal material should also include a review of the content of the book. It is worth noting, too, those fragments of the intuitive stratum which usually follow the formal material and discuss connections with other subjects or alternative approaches to the same problem. The formal material may be presented in various ways. The author is not required formally to expound the alternative solutions, but it is assumed that in such cases a brief description of the idea of the alternative solution should be included. If the work is intended as a handbook, then it usually has at the end material for exercises.

The intuitive stratum of the text is written in a language which relatively less often includes formulas and symbols used in the formal part. The principle of not using formulas and symbolism in the informal material is observed in particular in mathematical handbooks. It is claimed outright that the best way of expounding those fragments is to avoid all symbols [3]. A good example of the application of those principles can be seen in the handbook by J. Słupecki, H. Hałkowska and K. Piróg–Rzepecka, *Elements of Theoretical Arithmetic* (in Polish), Warsaw 1980.

It must be emphasized that some fragments of the intuitive stratum, especially examples and analogies, may be presented in a formalized form. This is due to the fact that we have there, for instance, to do with the substitution of a theorem proved earlier, or an equivalent, though merely outlined, way of presenting the formal material. The formalization of the entire intuitive stratum is usually difficult. On the other hand, the possibility of paraphrasing fragments of the formalized text in natural language is a certain regularity.

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Let us reflect on what accounts for the fact that the language of mathematical texts is also comprehensible to mathematicians who only poorly, or not at all, know that ethnic language in which a given text has been written. Why is it so that since the early years of the development of science it has been accepted that precisely that language should be the universal language of science, and why no ethnic language can become such a language (on which there is a general consent)? Where are we to look for the essential difference between the language of mathematical texts and ethnic languages? Has the language in which mathematical texts are written some peculiar properties of which other languages are deprived, and if so, then what these properties are? And what role in that language is played by symbolism, without which we can hardly imagine a contemporary mathematical text? Our working hypothesis is: the semantics of the language of mathematical texts accounts for its universal comprehension. That semantics differs essentially from the semantics of any ethnic language. One might also risk the statement that ethnic languages determine various perspectives of the world, whereas the language of mathematics has only one and the same such perspective[1].

The above hypothesis of course requires an at least superficial verification and argumentation. We assume that natural language makes use of continuous (analog) semantics, opposed to discrete (digital) semantics. Terms used in ethnic languages usually have a fuzzy (blurred) meaning (e.g., vague terms). This can be illustrated by the well-known fact that we are unable to indicate a precise boundary showing the use of colours in ethnic languages. We usually resort to gradation to indicate differences between colours that do not differ much from one another (for instance, *blue*, *dark blue*, *light blue*). Moreover, indicating such a boundary is impossible because there is a margin in which the boundaries would differ from one another even if they were described as precisely as possible. It may be said without exaggeration that, for instance, not to every Pole the blue colour is the same blue colour. Hence, it may be said in most general terms that the continuity of semantics finds reflection in non-eliminable vagueness. When speaking about the continuity of semantics we do not mean ambiguity, so characteristic of everyday language, because ambiguity does not cause fuzziness of meanings of words. Ambiguity occurs systematically in both natural language and mathematics. But that does not deprive mathematical texts of their semantic clarity. Ambiguity does not make semantics discrete, and the type of the semantics with which we have to do is determined by whether the boundaries between meanings of words are fuzzy or not.

That, for all ambiguities, the semantics of the language of mathematics remains discrete is illustrated, for instance, by the concept "kernel of transformation", which

[1] When speaking about the perspectives of the world determined by ethnic languages we refer to the hypothesis of linguistic relativity, formulated by Sapir and Whorf. That hypothesis includes the supposition that the ways of classifying objects in extralinguistic reality may be different in different languages. The Sapir-Whorf hypothesis states that languages as a social product shape our way of perceiving the world around us, and in view of the differences among linguistic systems people who think in those languages perceive that world in different ways. That hypothesis has not, it is true, turned into a theory in view of lack of convincing experiments that would confirm it, but has served as a stimulus for experimental research in field previously not covered by linguists.

in mathematics occurs in many meanings. We shall consider here several examples of its use.

(i) *The Encyclopaedia of Physics* [1] states: "KERNEL, in mathematics the function $K(x,y)$ in the following integral transformation

$$\varphi(x) = \int_a^b K(x,y)f(y)dy,$$

which defines the relationship between the function $f(y)$ and the function $\varphi(x)$". Compare the entry *TRANSFORMATION* in the *Comprehensive Universal Encyclopaedia* [14].

(ii) In L. S. Pontriagin's *Topological Groups* [8] on p. 187 we find the information that "in the integral equation

$$\varphi(x) = \lambda \int k(x,y)(y)dy$$

the function $k(x,y)$ is termed the kernel of that equation".

(iii) Z. Semadeni and A. Wiweger in their *Introduction to the Theory of Categories and Functors* [10] state that "the counterimage

$$Ker\,\alpha = \alpha^{\leftarrow}(0) = ((\alpha \in A : \alpha(a) = 0))$$

of the natural (zero) element 0 of a group is the kernel of the morphism α".

(iv) In *Linear Algebra* by A. Mostowski and M. Stark [7] the definition of kernel is: "A set of vectors α, for which $f\alpha = 0$, is a linear space $\Sigma(f)$ termed the kernel or the zero space of the transformation f".

Obviously, in (i) and (ii) on the one hand, and also in (iii) and (iv) on the other, we have to do practically with one and the same concept. In (i) and (ii) it is interpreted as a function which determines an integral transformation, whereas in (iii) and (iv), as the counterimage of the zero element. Formally, however, all the four meanings are different. That ambiguity does not cause any misunderstandings in the interpretation of the texts quoted above. It is to be noted that in mathematical texts such ambiguity is encountered as frequently as in natural language.

The semantics of the language of mathematical texts is discrete. That is secured above all by definitions used in those texts, which, unlike in natural language, are never definitions by examples. This guarantees a lack of vagueness of the terms used and consistence in their use. Before any change in the meaning of a term used earlier the author ought to indicate the new meaning of that term by stating it explicitly. The semantic of the language of mathematical texts is usually too intricate for being expressed by the means at the disposal of natural language. That necessitates the use of stronger tools: that role is taken over by mathematical symbolism.

It is true that contemporary mathematics is unthinkable without the use of symbols, but these appeared in mathematical texts relatively late. It is legitimate to assume, for instance, that the use of letters (as variables) introduced in mathematics in the early 17th century made the birth and development of algebra possible. That led in turn to the emergence of analytic geometry, and the application of adequate symbolism in solutions of problems in the differential calculus turned the latter into an independent branch of mathematics.

The introduction and consistent use of the system of symbols in mathematics were in a close connection with works on the systems of symbols used in logical calculi. It is assumed that the rapprochement between logics and mathematics was bilateral. On the one hand, logic was coming closer to mathematics by a more and more general interpretation of logical relationships and operations, and on the other, mathematicians, following the development of their discipline, were forced to become interested in its logical structure. The development of studies in the logical foundations of mathematics must have accordingly had important consequences for logic as a whole. On the other hand, the development of mathematical content induced people to develop symbolism and brought out the usefulness of certain standard formulations serving the precise expression of thoughts. Verbal expressions were being replaced by a non-ambiguous and transparent system of symbols, free from the chance and obscurity of everyday language.

The task of constructing a theory of mathematical reasonings, to be presented in a symbolic form, was set themselves at the close of the 19th century by G. Peano in Italy and G. Frege in Germany. The thesis on the reducibility of mathematics to logic was demonstrated by A. N. Whitehead and B. Russell in their *Principia Mathematica*.

Arriving at mathematical theories which would comply with the logical requirements of formal correctness became the goal of the formalist programme of D. Hilbert and his school. That programme, striving for the axiomatization of the whole of mathematics, when verified by the works of Gödel, Church and others, and also those of Tarski, proved unattainable in its philosophical aspect. But it marked a certain step forward towards the construction of a standard language of mathematics when it comes to certain techniques used in that discipline. The concept of the language of mathematics was, in the context of metalogical and metamathematical studies, identified with that of the formalized language. From the point of view of the language of mathematics, both the works of Hilbert and the earlier studies of Peano, Whitehead, Russell, and others show that any mathematical theory with its primitive concepts and proofs, can be recorded in a formal manner. It has also been demonstrated that the exposition of mathematics by totally or almost totally abstaining from the use of a verbal text is possible when one resorts to a symbolism specially worked out for that purpose.

It turned out in this connection, which is stressed by some authors, that as in the 16th and the 17th century, when algebra was being counched in symbols, two hundred years later the formalization of the mathematical language was not due to any new principles, but to the precision and standardization of the ordinary lexical and syntactical means of language: logical relationships replaced conjunctions, function symbols were used in describing the relations between the subject and the predicate, and the discovery of the usefulness of quantifying expressions, as a natural consequence of the development of the symbolism of the values of variables in mathematical analysis and geometry, preceded the introduction of quantifiers.

It seems that the idea of the formalization of mathematics is being achieved by the imposition upon natural language of limitations enforced by classical logic. One could, by the way, hardly imagine another sequence of events. Everyday language is that language in which all scientific theories, in particular mathematical ones, were formulated in a natural manner. The limitations imposed upon everyday language were, it seems, mainly due to the nature of the semantics of that language. But it should be assumed that the shortcomings of natural language resulting from a lack of precision in the expression of certain ideas, were the inspiration for adjusting that language to the needs of science, mathematics in the first place. It seems characteristic of mathematics to used bivalued logic, which entailed the adoption of the extensional nature of conjunctions (in technical language renamed "sentential connectives"), and also the fixing of the meanings of quantifying expressions, which, by the way, came rather late. The formal interpretation of the latter was essential in the sense that in natural language those expressions were never used in the meaning adopted in formal logic. The formal recording of logical reasonings was the next stage in formalization, which essentially influenced the formalization of mathematics.

How far recourse to symbolism in definitions facilitates their formulation and use is shown by the example drawn from Webster's dictionary, where under the entry metric we find: "a mathematical function that associates with each pair of elements of a set a real non-negative number constituting their distance and satisfying the conditions that the number is zero only if the two elements are identical, the number is the same regardless of the order in which the two elements are taken, and the number associated with one pair of elements plus that associated with one member of the pair and a third element is equal to or greater than the number associated with the other member of the pair and the third element" (*Webster Dictionary* 1977, p. 724–725). It is to be doubted whether that text is comprehensible to a non-mathematician, and even a mathematician will find it difficult to identify it as the clearly and precisely formulated definition of distance, commonly used in mathematical texts. That definition reads:

> the function $D(x, y) \geq 0$ is termed distance if and only if it satisfies the following conditions:
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> (1) $D(x, y) = 0 \leftrightarrow x = y$,
> (2) $D(x, y) = D(y, x)$,
> (3) $D(x, y) + D(y, z) \leq D(x, z)$.

The above example finely illustrates another element of the language of mathematics, very important in our opinion, namely the common occurrence of variables. Variables appeared in the language of mathematics more or less simultaneously with the first attempts to formalize mathematics, and their role cannot be overestimated. It seems that the "invention" of variables was decisive in the process of formalization, for in natural languages variables do not occur in such a form as in mathematical texts, and their place is occupied by other natural language expressions, most frequently pronouns[2], but not only them, as the example drawn from Webster has shown. There the role of variables is played by numerals. While the definition of distance formulated in natural language can be interpreted correctly with some difficulty, and possible to apply, many other definitions cannot in

[2] The opinion that pronouns in natural language may be treated as variables was voiced, among others, by W V. O. Quine in his paper "Logic and the reification of universals", (in: [9]).

any way be sensibly formulated in natural language. Here is one of the axioms of Moebius' geometry (where *circumference* stands for *circumference of a circle*:

Given eight different points A, B, C, D, E, F, G, H, then
if there is a circumference which passes through A, B, C, D, and
if there is a circumference which passes through E, D, C, H, and
if there is a circumference which passes through E, D, A, F, and
if there is a circumference which passes through F, A, B, G, and
if there is a circumference which passes through G, B, C, H, then
there is also a circumference which passes through E, F, G,H.

In the formulation of the above axiom there are eight objects of the same type, namely points. There are too many of them to be sensibly distinguished from one another using means of natural language. It is true that this objects can be distinguished in the way suggested by Webster, namely by replacing letters with ordinal numerals, e.g., instead of *point C* to say *the first* point, instead of *point H*, *the second point*, and so on. The numerals in this case allow us to distinguish the points in question, but that is an inconvenient way of replacing letter variables by "descriptive" ones. It seems that recourse to letter variables is the only practical way of recording the axiom under consideration. Note that the variables referred to also occur in the same for instance in chemical texts. But similarity is merely apparent. Chemistry has a strongly expanded system of nouns used as technical terms, and letter symbols do not play there the role of variables but function rather as definite descriptions. The problem of variables does not arise even when very many objects are involved if they are of different kinds. In such a case in natural language it is just nouns which are used to describe them. The objection might be raised that it is not only in mathematics that we have to do with a rich terminology, and yet that does not require the use of symbolism abounding in variables. It is true that other disciplines have an expanded terminology and yet do not use variables in the manner typical of mathematics, but there is no such need because terminology, for instance in biology, serves different purposes than it does in mathematics. We have there to do with taxonomy, and hence with a classification of objects of various types. It is only in mathematics that so many objects of the same type are used, which requires a way of speaking about them without obscuring their picture. It is legitimate to think that the underestimated fact – the introduction of variable in mathematical texts – has been one of the greatest achievements of the symbolism used in the language of mathematical texts, and at the same time the element which marks an essential difference between the language of mathematics, on the one hand, and not only everyday language but the languages of other, non-mathematical, disciplines as well.

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The possibility of making mathematical statements uniform in the sense of a system of symbols created specially for that purpose does not imply its common acceptance. When one reads the majority of mathematical texts one finds that they make use to a little extent or not at all of the symbolism applied in mathematical logic, even though its use together with a verbal text increases the precision of exposition of the subject matter. Moreover, some authors are convinced that the symbolism of formal logic, while indispensable in the discussion of mathematical logic, becomes an intricate code when used as a means of conveying ideas (Steenrod 1978, Shiffer 1978). At the same time attention is drawn to the fact that formalized languages are too poor to cover large mathematical theories. For instance, logical and mathematical symbolism does not suffice in expounding the theory of differential equations. Verbal additions are necessary and, moreover, the need of availing one self of verbal text manifests itself in nearly all mathematical works. Without it many of them prove too difficult for the readers or even outright incomprehensible.

Note that a large part of the verbal text tends to be standardized: the vocabulary of the mathematical language (except for specific mathematical terms) is poor. There is a fixed number of operators which are used as formulas ("let us consider", "to do so it suffices to demonstrate" etc.).

Since the language of mathematical logic, combined with the language presently used in mathematics, does not suffice for a standard recording of mathematical contents, the need emerges of constructing a new language, which would meet the requirements of simplicity and adequacy and thus materialize the idea of a standard language of mathematics. The suggestions in that matter, known in the contemporary literature of the subject, are as follows:

(1) A precise statistical and logical analysis of the language of contemporary mathematical texts that would bring out its tendencies for standardization.

(2) Construction on that basis of a specialized and simplified mathematical language which, combined with mathematical symbolism and the symbolic means of mathematical logic, would suffice for the description of the results of contemporary mathematical research. According to L. A. Kaluzhnin, the simplification of that language in the first phase should consist in the selection and fixing of a not too rich vocabulary; the fixing of precise rules of the introduction of specialized terms; the simplification of grammar (e.g., in the sense of eliminating grammatical exceptions). This is based on the fact that even a superficial knowledge of the grammar of a given foreign language suffices one for understanding a mathematical text written in such a language.

The idea of creating a standard mathematical language is not new and emerges systematically in the literature of the subject, and the advances in information science and the resulting new prospects have stimulated those ideas a new. Theories of the construction of such a language and the properties which it should have are known in international literature, but in practice they have proved to be a difficult undertaking. The only standard mathematical language we know, used in recording mathematical texts and in automatically checking the correctness of mathematical proofs, namely Mizar, meets in practice the requirements set to a standard language. Its author is A. Trybulec, and it has already been successfully applied in practice both in Poland and abroad.

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