Some Logical Aspects of Mathematical Reasoning

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Abstract. This article discusses the applications of logic structure in Mizar Language, and discusses mathematical principles and methods, such as concepts and definitions, judgments and propositions, reasoning and proof, etc. These principles and methods are strictly followed by mathematical axiomatic methods, so does Mizar language system.

On the 65th birthday of Professor Andrzej Trybulec, all the teachers and students in our Institute of Mathematics and Physics in Qingdao University of Science and Technology express our sincere wish to his health and happiness.

We want to show our admiration to professor Andrzej Trybulec for his outstanding work in the field of formalized mathematics especially in the field of Mizar, and we also want to thank him for his instructions to us for many years.

Xiquan Liang met Professor Andrzej Trybulec and Professor Yatsuka Nakamura (Shinshu University, Nagano) in 1997 when he was studying and working in the Tokyo Institute of Technology and Shinshu University. From then on, he began to learn Mizar Language and under the guidance of Professor Yatsuka, published his first Mizar article named "Solving Roots of Polynomial Equations of Degree 2 and 3 with Real Coefficients" [4].

During learning and using the Mizar Language, we generally realized the importance of formalized mathematics, and then took the further research of the theories of formalized mathematics. It is an important field to use programmed language to prove mathematical problems. The essential of formalized mathematics is to formalize the intelligent course of reasoning and deduction. Proving theorems by computer is one of the primary directions involving artificial intelligence. It is a great breakthrough of modern mathematics to prove theorems by machine automatically instead of by person themselves.

Xiquan Liang came back to China and worked in Qingdao University of Science and Technology in August, 2000. At the same time, he organized a scientific research group including professors, young teachers and postgraduates to continue the study in this field. On the base of discussing plenty of Mizar articles, we combined Mizar Language with concrete research directions, such as, mathematics morphology, matrix theory, fluxionary calculus theory, etc. The academic atmosphere was strong in our group, and we discussed with each other when we met problems, trying to

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improve our levels in this course. In order to improve our comprehension and applications of Mizar, we invited Professor Andrzej and Professor Yatsuka in 2004 and 2006. They explained the knowledge and applications of Mizar Language to our young teachers and postgraduates and meanwhile, they illustrated the confusions and difficulties of writing the papers vividly.

We have proved many theorems using Mizar Language from 2004, and accomplished some achievements. All the proved theorems were accepted by Mizar mathematical library. We plan to do the research and make some breakthroughs in the field of manifolds, algebra, number analysis, etc.

After 30 years of development, many scholars in Poland and Japan have made the great achievements in Mizar Language. But in Qingdao, it's a new field. We have constructed the good cooperative connection with Białystok University and Shinshu University, we hope to strengthen scientific collaboration of the three universities in the future, and promote the development of Mizar Language by our collective efforts.

1 Concept and Definition

First, we'll discuss the meaning, connotation and extension of concepts, and further discuss the methods and rules of definition with Mizar language.

1.1 Concept

Concept is our thinking form reflecting essential attributes of objective things. Like other concepts, mathematical concepts are also abstracted from the realistic world. For example, in geometry, point, line, plane, body are abstracted from the spacial form which objects occupy; in algebra, natural number, fraction, rational number, irrational number, etc., are abstracted from quantitative relationship between investigative objects.

The abstraction of the concept mainly expresses essential attributes and interior relations of investigative objects, instead of the phenomenon and external relations of investigative objects. These essential attributes of investigative objects and general comprehension are just the significate meaning of the scientific abstraction.

Attributes of objects can be divided into "essential attributes" (or "proper attributes") and "nonessential attributes". Essential attributes reflect characteristics of objects, namely they only belong to a certain type of objects, and don't belong to other types. Thus we can distinguish this type of objects from others, and infer other nonessential attributes of objects according to essential attributes. For example, the attribute that quadrangles are equilateral and equiangular is enough for us to distinguish square from all quadrangles, but other attributes, such as diagonals are equal; diagonals are vertical and halved each other; diagonals and sides are not commensurable; quadrangles can make the circumscribed circle and the inscribed circle, etc., are nonessential attributes.

1.2 Connotation and Extension of Concept

Concept is a basic form of thought and reasoning. Concept must be unambiguous; otherwise it can't correctly reflect the objective things and their characteristics. Therefore, we can't correctly judge, reason and prove in the thinking process. How on earth can the concept be unambiguous? This problem involves connotation and extension of the concept.

Connotation of the concept is the summations of all the essential attributes of the investigative objects.

For instance, connotations of the parallelogram are that it is a quadrilateral and its two sets of subtenses are parallel respectively. Connotations of the square are that it is a quadrilateral, its four sides are equal and its two adjacent angles are equal.

Extension of the concept is the summations of all investigative objects. In other words, it is inherent scope of the concept.

For example, triangles include many varieties. According to the angles, they are classified into acute triangles, obtuse triangles and right angled triangles. According to the sides, they are classified into non-equilateral triangles, isosceles triangles and equilateral triangles, etc. All the triangles in these varieties belong to the extension of the concept – triangle. In the same way, the extension of the quadrangle is all the quadrangles.

According to relationship between species and genus-difference, quadrangles can be divided as follows.

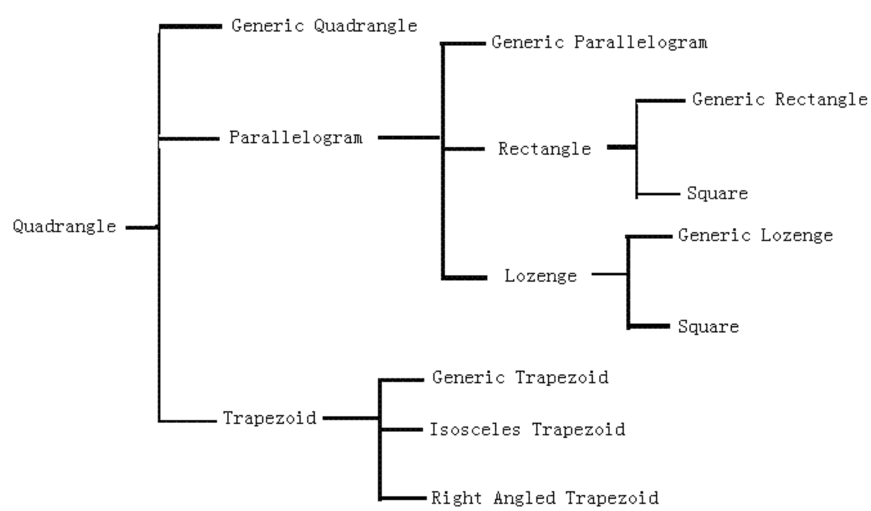

Fig. 1.

All quadrangles are extension of the concept – quadrangle. Generic parallelograms, rectangles, lozenges and squares constitute the extension of the parallelogram. Generic rectangles and squares constitute the extension of the rectangle. Generic lozenges and squares constitute the extension of the lozenge. In this way, we can see the rule that the extension of concept will decrease if its connotation increases. On the contrary, the extension of concept will increase if its connotation decreases.

Connotation and extension of the concept are two parts of the concept. A concept is unambiguous, which means its connotation and extension must be unambiguous and also means its essential attributes must be unambiguous. Therefore, we can use the concept correctly.

1.3 Species and Genus of Concept

If the extension of concept B belongs to the extension of concept A, the A is called species and B is called genus. The two kinds of concepts are constituted to have relationship between species and genus.

Species and genus are relative. The same concept is a species concept for one concept, but is possibly a genus concept for another concept. Every concept with bigger extension is a species concept for the concept with smaller extension. And every concept with smaller extension is a genus concept for the concept with bigger extension. For instance, in geometry, parallelogram is a genus concept for quadrangle, but it is a species concept for rectangle or square.

Circle and geometric figure have relationship between species and genus, but circle and quadrangle don't have relationship between species and genus. Both pentagon and hexagon are genus concepts for polygon, but they don't have relationship between species and genus.

An object possibly has many species concepts. The species that is the closest to it is called the closest species of it. For example, polygon, quadrangle and parallelogram are all species of rectangle. Here, the closest species of rectangle is parallelogram, but not the polygon or quadrangle. There are two closest species of square, which are rectangle and lozenge.

1.4 Definition

Defining a concept is to disclose the essential attributes of the concept, and also to disclose the connotation of this concept. When the concept has an explicit definition, we can distinguish it from other concepts in essence.

"Genus-difference" is the differences in the essential attributes between the concept being defined and other genus concepts paratactic with it. For example, generic quadrangle, parallelogram and trapezoid are paratactic genus concepts corresponding to the species concept quadrangle. But parallelogram has two sets of parallel subtenses, and trapezoid has only one set of parallel subtenses and another set of unparallel subtenses, and generic quadrangle has two sets of unparallel subtenses. These differences are the genus-difference among them. Another example: triangle,

quadrangle and pentagon are parallel genus concepts for polygon, and their genus-difference is that their numbers of sides are different.
We usually apply a method to disclose the essential attributes of the concept and correctly define it. This method is "the closest species" plus "genus-difference". Its formula is:

Defined concept = genus-difference + the closest species.

According to the above formula, trapezoid and parallelogram can be defined respectively as follows:

A parallelogram (the nearest species) is called trapezoid, if its one set of subtenses is parallel and another set of substenses is not parallel (the genus-difference).

A parallelogram (the nearest species) is called parallelogram, if its two sets of subtenses are parallel respectively (the genus-difference).

Definitions have to obey the following rules:

Rule 1. Definitions must be both corresponding and matching with, which means the extension of the defined concept must be equal with the extension of the definition.

Breaking this rule may lead us to make the mistakes that the scope of the definition is too wide or too narrow. For example, there is a definition as follows:

"Two polygons that their corresponding angles are equal respectively are called similar polygons."

In this definition, the extension is too wide. It includes not only all of similar polygons but also many nonsimilar polygons, because according to this definition, all rectangles and squares are similar with each other. There is another definition as follows:

"The polygons that their corresponding angles are equal respectively and the ratios of corresponding sides are equal with a certain positive integer, are called similar polygons."

In this definition, the extension is too narrow. Because the similarity ratio is limited in the scope of positive integers, this definition eliminates such similar polygons that their similarity ratio does not equal a positive integer (such as, positive fractions).

Rule 2. The definition can not be circulated.

This rule means to avoid two kinds of mistakes. One kind of mistakes is that concept A is defined by concept B, then concept A is used to define concept B again.

2 Judgment and Proposition

We discuss the meaning and common types of judgments first, and then discuss the mathematical judgment - propositions, including compositions, changes of a proposition and rules used to judge the equivalent relation of propositions, etc.

2.1 Judgment

Judgment is a thinking form expressing certain affirmation or negation of the objects.

Judgment is a thinking form understanding objective things on the foundation of concepts. Different from concept and other thinking forms, judgment is to disclose the relationship between objects and a certain special attribute by affirmation or negation. It isn't a judgment if it doesn't affirm anything or negate anything.

Concepts are realized by predicates. But judgments are realized by sentences. At the same time, the logical judgment has own structure.

Judgments can be divided into positive judgments and negative judgments by nature. They are expressed by "is" or "isn't".

There are some kinds of judgments.

2.2 Categorical Judgment

The categorical judgment reflects the simple contacts or differences between objects. It is the most common and the simplest and the most basic judgment. It can be divided as follows:

1. Single Judgment. It is the simplest form of the judgment. It affirms or negates a certain property of a single object. It includes as follows:
   - Single Positive Judgment. Its formula is "S is P".
   - Single Negative judgment. Its formula is "S isn't P".
2. Special Judgment. It affirms or negates properties of some objects in a certain class. It includes as follows:
   - Special Positive Judgment. Its formula is "some S are P".
   - Special Negative Judgment. Its formula is "some S aren't P".
3. Full Judgment. It affirms or negates properties of all objects in a certain class. It includes as follows:
   - Full Positive Judgment. Its formula is "all S are P".
   - Full Negative Judgment. Its formula is "all S aren't P".

2.3 Proposition

The proposition is a sentence describing a certain judgment in mathematics. Both theorems and axioms are propositions.
Condition proposition is the manifestation of mathematical proposition.

Condition proposition reflects more complicate relationship or causality between objects. Theorems are condition propositions generally. Their formula is "if S is P, then R is Q". For example, if point P is in the exterior of a circle O, then $OP<r$ (O is a center of the circle O, r is the radius).

2.4 Compositions of Proposition

Many mathematical propositions are condition propositions. They are constituted by two or more than two judgments. If the former judgment "S is P" of the condition proposition is written as A, the latter one "R is Q" is written as B, then this kind of propositions is written as "if A, then B". Here A is called the premise, B is called the conclusion. In other words, a mathematical proposition is constituted by premise and conclusion, and its standard format is "if A, then B". It is written as "→" or "if A, then B" or "if A, prove B".

2.5 Four Kinds of Forms of Proposition

A proposition has four kinds of changes, then it has four kinds of forms.

We can get the converse proposition of an original proposition if we exchange its condition and conclusion. We can get the negative proposition of an original proposition if we negate its condition and conclusion at the same time. We can get the converse-negative proposition of an original proposition if we negate its negative proposition again or exchange premise and conclusion of its negative proposition. Its standard form is as follows:

Original proposition: if A, then B.
Converse proposition: if B, then A.
Negative proposition: if not A, then not B.
Converse-negative proposition: if not B, then not A.

Four kinds of forms of a proposition are expressed as follows:

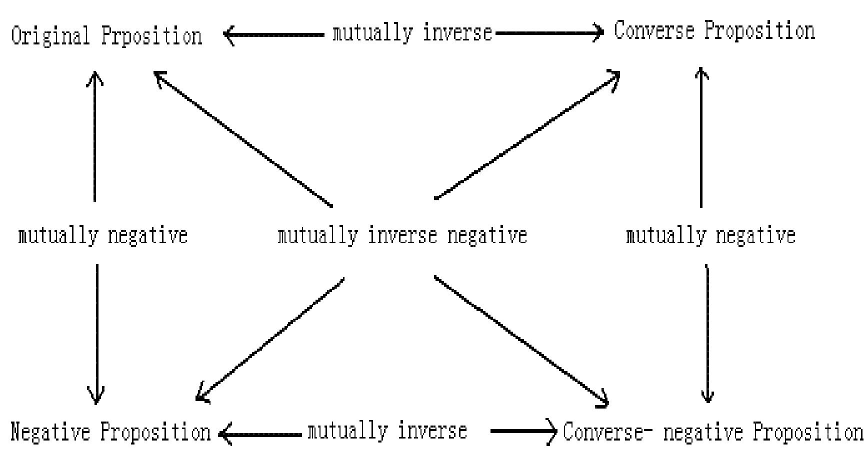

Fig. 2.

2.6 Equivalence in Four Kinds of Forms of Proposition

Proposition α and proposition β in the four kinds of forms of a proposition are called equivalent propositions, if α is true, then β is true; on the contrary, if β is true, then α is true. Equivalent propositions must be true or false simultaneously.

Propositions are not always true. So four kinds of forms of a proposition are not all true.

It is important to study the equivalent relation of transformations of propositions. It helps us to understand the effects of propositions and prove propositions. There are some rules to judge equivalent propositions:

1. Rule of Converse-negative Proposition

From 3.5, we can see that original proposition and converse-negative proposition, converse proposition and negative proposition are mutually inverse negative respectively. They are always true or false simultaneously. So two mutually inverse negative propositions are equivalent. This rule is called "Rule of converse-negative proposition". It is the theoretical foundation of the reduction to absurdity (it is also called "Contradictory Method") which is a indirect proof method.

2. Rule of Identity

Negative propositions and original propositions are not always true or false simultaneously. If premise and conclusion of a proposition exist uniquely, its original proposition and converse proposition must be equivalent. So we call this proposition satisfies the rule of identity. For example, the proposition that the bisector of vertex angle of an isosceles triangle is the middle line of the base obviously satisfies the rule of identity. For a certain isosceles triangle, its bisector of vertex angle and the middle line of the base are existential and unique, and its converse proposition, namely, the middle line of the base of an isosceles triangle is the bisector of vertex angle, is apparently true. There is another example. The proposition that if a straight line goes across a point and is vertical with the radius across this point, this straight line is tangent with the circle at this point also satisfies the rule of identity. Because the vertical line and the tangent across the endpoint of a radius are existential and unique, its converse proposition, if a straight line is tangent with a circle at a point, this line is vertical with the radius across this point, is also true.

3. Segmented Proposition

A proposition is called a segmented proposition, if it is constituted by several propositions, and the contents that are expressed by condition and conclusion of these propositions are elaborate in every aspect (all probable situations among objects are listed exhaustively), but not consistent. For example,

The theorem that in a triangle, if two angles are equal, their corresponding sides are also equal; if two angles are not equal, their corresponding sides are not equal either, and the big angle corresponds the big side and the small angle corresponds the small side.

This proposition is constituted by three propositions in fact, and in the triangle ABC, can be expressed by the symbols as follows:

(i) If $\angle A = \angle B$, then BC=CA;
(ii) If $\angle A > \angle B$, then BC>CA;
(iii) $\angle A < \angle B$, then BC<CA;

Conditions of the above three propositions include all situations of the relations of sizes of two angles. These situations include "equal", "greater than" and "lessen than". At the same time, conclusions of three propositions also include all situations of the relations of the corresponding sides of these two angles. That is to say, conditions and conclusions of propositions are elaborate.

2.7 Condition of Proposition is As Follow:

In mathematics whether many propositions are true or not has something to do with the condition of it. This kind of condition is the hypothesis of the proposition. According to the function of condition A to conclusion B, the condition of proposition has two characters and three forms.

Two characters:

Sufficient character – if we can get conclusion B from condition A, then the condition A is called the sufficient condition of the conclusion B.

Necessary character – if we can get condition A from conclusion B, then the condition A is called the necessary condition of the conclusion B.

Three instances:

1. Sufficient but not necessary condition. This condition has the character of sufficient but not the character of necessary.
2. Necessary but not sufficient. This condition has the character of necessary but not the character of sufficient.
3. Sufficient and necessary. This condition has not only the character of sufficient but also the character of necessary.

3 Inference and Proof

3.1 Inference

Inference is a thinking form of getting a new judgment from one or several known judgments. It's a significant thinking form for people to know the objects. But where does the judgment come from? Some are got from direct observations and experimentations; some are got from inference, which is a more advanced thinking form than judgment.

Inference is generally constituted by two parts: the premise and the conclusion. In the process of inference, the premise is the known judgments from which we can get a new one. And the new judgment which is got from the premise is called the conclusion. The special relation among the judgments which constitute inference is also the relation of premise and conclusion. The inferences which are frequently used are inductive inference, deductive inference and analogy inference etc.

1. Inductive Inference

Inductive inference is a thinking method from special to general, it is also the inference method to get general conclusion from the premise of the individual judgment or special judgment.

Usually, there are two kinds of inductive inference being used in mathematics: complete inductive inference and incomplete inductive inference.

Complete inductive inference: The amounts of same kind of objects which are investigated is finite, every object will be investigated, generalized and then concluded.

The conclusion which comes from true premise is reliable by complete inductive inference.

Incomplete inductive inference: The amounts of the same objects which are investigated are infinite. Even they are finite, people can't be investigate them one by one. So we can only investigate part of them to speculate more general conclusion.

The conclusion got from incomplete inductive inference may be wrong, and it also may be correct, but it can help people to get the new regular patterns. Because when people are trying to get the new principle of facts, first they'll recognize some individual facts of practice which may be the clue for them, then people will get some general conclusions and at last, they'll get the final confirmation by inference or other method.

2. Deductive Inference

Deductive inference is a kind of thinking method from general to special. What we frequently meet in deductive inference is syllogism. Syllogism is a kind of inferable method that is to get a new judgment from two categorical judgments; moreover one of the categorical judgments must be full name judgment.

Syllogism is made up of three parts: assumption (full name judgment), minor premise (special name judgment), and conclusion (final judgment).

For example the diagonals of rectangle are equal (assumption).

Square is rectangle (minor premise).

So the diagonals of square are equal (conclusion).

3. Analogical Inference

Analogical inference is such a kind of inference as follows: We can infer that two objects may be same in other attributes from the fact that the two objects are same in some attributes. Such as the object A has the attributes of a, b, c, dthe object B has the attributes of a, b, c, then we can imagine that object B also has the attribute of d.

Analogical inference is a sort of probable reasoning, it can only provide clues to people, inspire people to consider and find problem; but whether the conclusion is right or not must be validated by other methods.

There are some common characters and differences between analogical inference and incomplete inductive inference. The common character is that the conclusion inferred by them is presumption; the difference is that the former is from special to special, the latter is from special to general.

3.2 Proof

Proof is a thinking form which states the judgment is actual and correct. That is a process to make use of some actual and reliable judgments as the gist to illuminate the authenticity of some judgments by one or several inference. There are logicality and continuity in these illations, which means that the conclusion of the former illation is usually the premise of the next illation, until getting the correctness of special judgment to be proved by inference.

The difference between proof and inference is: in the inference process, the procedure is from premise to conclusion. The premise is the judgment which is known beforehand, and then reason to get a new conclusion. The procedure of proof is reverse to the inference, firstly given a judgment, people will try to find the sufficient excuses which state that the judgment is actual. Only based on these excuses, can we can state the authenticity of the judgment which is provided beforehand by several reasoning. Proof is a logic form and method which is more complex and abstruse than inference.

The proof in mathematics can help us confirm the authenticity of some propositions or theories, and it is an important method to institute mathematics theory like inference.

3.3 Constitution of Proof

Three parts constitute the proof:

1. Thesis is the judgment that is needed to prove its authenticity, it's the thesis needed to be proved in mathematics.
2. Arguments are those judgments which are used to prove the authenticity of thesis. These arguments are definitions which are known beforehand and axioms and theorems which are put forward before the thesis needing to be proved.
3. Reasoning is the inference process from argument to conclusion.

3.4 Several Rules of Proof

A correct proof must obey the following rules:

1. The thesis must be correct and the change is forbidden, which means we can not change the content of hypothesis and conclusion.
2. The argument must be real and reliable. That is to say every argument we use must be the adequately confirmed fact.
3. The argument can not be inferred from thesis in one proof. Otherwise to get thesis through argument by reasoning and to get argument through thesis by reasoning, this will lead to the error of vicious circle.
4. The argument must reason the thesis. If the arguments have no relation with the thesis or the arguments are not enough, the whole proof have no effect.

3.5 Methods of Proof

Owing to different aspects, there are several methods of proof in mathematics.

It can be divided into the proof of deductive method and inductive method by the adoption of the different inference.

It also can be divided into the proof of analyze method and synthesize method because of the adoption of different thinking of "from unknown to known" and "from known to unknown".

It can be divided into the direct proof if we begin our thesis directly and indirect proof if we begin our thesis indirectly.

These methods of are classified according to the different views. Classifying in this way is in order to master the characters of every method of proof conveniently. By the reason that every proof is completed by inference, the deductive inference and the inductive inference are the base. By the reason that every proof can not be separated from the thinking process of analysis and synthesize, the analysis and synthesize are the key points to solve the problems; while whether starting the thesis directly or indirectly is the starting point of the whole proof, in fact every proof is the result of the associated effect of using several methods of proof. We should not only master the characters of every method but also apply them unitedly to complete each proof. All these proof methods have extensive applications in Mizar language.

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