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THE GÖDELIAN SPEED-UP AND OTHER STRATEGIES TO ADDRESS DECIDABILITY AND TRACTABILITY

Editorial Comment on this Volume

Gödelian speed-up is a handy concept when dealing with decidability and tractability, even if it did not become common in logical literature yet.[1] Since the phrase "decidability and tractability", when uttered in one breath (as we shall often do), is a bit clumsy, let it be shortened for "D&T". In this pair, the theoretical (or, in principle) possibility of solving a problem is what one calls *decidability*, while the practical one is called *tractability* (more on this distinction – in item 1.3 below).

No wonder that it is Gödel whom we shall follow as the great master of D&T issues. To him we owe deepest insights into the feedback between human minds and computers in their common enterprise of problem solving.[2] His idea of speed-up (though not the word itself which appeared later) is to the effect that one can obtain ever shorter proofs, hence ever shorter times of problem solving, with engaging ever more abstract languages. Abstraction is to mean in such a sense in which talking about sets is more abstract than talking about individuals, while a talk about sets of sets is still more

[1] This text, when introducing to this volume, at the same time sums up the Research Project "Nierozstrzygalność i algorytmiczna niedostępność w naukach społecznych" [Undecidability and Intractability in Social Sciences] Grant No. 2 H01A 030 25, supported by the Polish Committee for R&D, next by Ministry of Science, in years 2003-2006. When mentioning the Polish title of the Project, a terminological note concerninig Polish terminology will be in order here. The phrase "niedostępność algorytmiczna" (that might be rendered as "algorithmic inaccesibility"), as used in the statement of the Project, is not the only one to be recommended as a counterpart of the English "intractability". There is also in the Polish dialect of computer scientists the term "nieobliczalność praktyczna" ("practical uncomputability") to express the same concept. May be, it is the latter which has more chances to become commonly accepted among Polish-speaking authors.

[2] There were no physical computers in the time of Gödel's revolutionary discoveries. However, the idea of mechanical problem-solving belongs to the very core of mathematical logic, especially in the inquiries of Hilbert, Turing and Gödel.

ISBN 978–83–7431–108–3 ISSN 0860-150X

abstract, and so on (a more systematic explanation is to be given below in item 1.3). It is that feedback which brings about the speed-up in problem solving, contributing to a rapid growth of computational powers, by which the range of what can be treated in an algorithmic way gets ever greater.

It is worth while to mention social implications of the increase of computational power. That constant and rapid increase is what constitutes the essence of modern information society. A popular definition (in Wikipedia) says that an information society is a society in which the creation, distribution and manipulation of information is a significant economic and cultural activity. The vagueness of the adjective 'significant' makes the whole definition too vague. This can be remedied if significant activity is to mean ever more massive information-processing due to the dramatic (in some segments even exponential) increase of computational power which, in turn, has the computational speed-up among its main sources. This is why a thorough inquiry into the nature of that speed-up is needed, to find an efficient strategy of further social development. Related claims appear in EU documents postulating a dramatic growth of information society in Europe. Some documents of the Lisbon Strategy emphatically call for improvement of the knowledge management procedures. Such a progress should heavily draw on the feedback between minds and computers, and this again would lead to the D&T issues.

In a previous stage of research on D&T these properties had been studied with regard to theories of deductive sciences. It was a novelty when Stephen Wolfram [1985] raised the D&T question in physics. There were authors who followed that train of thought with regard to economy, e.g. Latsch [2003], and other social sciences.

A relevant quotation from Latch's [2003] abstract runs as follows. "A highly complex computational economy can evolve and self-organize but it also displays computational universality that means that many problems are not decidable. The inherent limits of computability become evident. This paper proposes incorporating a particular (constructive) non-computability into our view of economic behavior and processes. The paper defines constructively non-computational behavior, discusses its origins in Roger Penrose's writings, and provides an application of this concept to the question of realistic counterfactuals in economic models."

The title and content of this volume – *Issues of Decidability and Tractability* – is to serve as a productive challenge to researchers ready to engage themselves in such a study with respect to social sciences. No new results on D&T are presented here but some material is provided as a background to put a new question: *what about D&T in social sciences?* An incentive to such a study is what is intended with this volume; it should offer a thought-provoking survey of some issues, statements and ideas being relevant to such a study.

1. D&T story in three chapters: Gödel, Boolos, a current research

1.1. The D&T story is foreworded with what historians call Hilbert Programme. It was Hilbert's famous project that involved the task of solving the problem of decidability (*Entscheidungsproblem*) of first-order predicate logic (called by him *der engere Funktionenkalkül*). In Hilbert and Ackermann [1928: 73] we read what follows (ad hoc translation by WM).

> "The decision problem gets solved if one knows a procedure which for a given logical expression allows to decide, with finitely many steps, about its validity or its satisfiability. The solution of the decision problem has a fundamental impact for all those theories whose statements are at all capable of being logically derived from finitely many axioms."

The negative solution of this problem came soon with the famous results by Turing [1936/37], Church [1936] and Post [1936] regarding undecidability of first-order logic (FOL, for short).

Undecidability of FOL is also supported by the fact of incompleteness of arithmetic of natural numbers (ANN, for short), the property demonstrated by Gödel [1931]. To see this, we need a method of expressing any ANN formula in the language of FOL; such a method has been elaborated by Hilbert and Ackermann [1928]. Then the problem whether a formula can be proved within ANN reduces to the question whether it logically follows from ANN axioms. Let the conjunction of ANN axioms, written in FOL notation, be denoted as κ. We ask whether a formula ϕ is provable from κ. Were FOL decidable, that is, were there a universal and mechanical decision procedure to recognize whether $\kappa \Rightarrow \phi$ is valid or not, then the recognition of its validity would make ϕ provable, while the recognition of non-validity would evidence non-provability, and so ANN would turn out to be complete.[3]

[3] Cp. Kneale and Kneale [1962: 736]. The transition from a valid formula of the form $\kappa \Rightarrow \phi$ to the inference from κ to ϕ is due the *converse deduction theorem* that holds for FOL. Cp. Surma [1981] and Marciszewski [1981: "Predicate Logic", 6.1].

1.2. Let us consider the message about FOL limitations vis-á-vis the main problem of information society, to wit the question of how to increase its computational power. This should be viewed in a larger perspective of the destiny of our civilization, and this, in turn, in a still larger cosmological perspective.

Our ancestors, looking for fundamental elements of the universe, found them naively in earth, fire, air and water. Nowadays, in the same role of the basic constituents of the universe, we entertain the triad of Matter, Energy and Information, each being capable of highly productive transformations.

Such a cosmological insight yields us the broadest frame for understanding the development of civilization. It should be seen as succession of three eons, each of them being defined with respect to the skill of transforming an element of that triad. First, people acquired the skill of transforming matter through agriculture and use of various tools and machines, as turning cereal grains into wheat, and that into bread, or a piece of clay into a jar, etc. The next era started with the invention of engines to produce energy through its transformations, say, mechanical energy from its other forms, as chemical or electric. And the third era came with inventing machines to process information. The third coincides with the beginning of the space age which is due to the united forces of all the three technologies: those of transforming matter, energy and information.

The first space flights, as those performed in the 20. century, should be seen as small preliminary steps towards the titanic work of colonizing the universe and of cosmic engineering. We, as the human race, are bound to be farsighted enough to engage ourselves into such enterprises, taking into account that warning by Stephen Hawking: "I don't think the human race will survive the next thousand years unless we spread into space." The accomplishment of such projects will require unimaginable energies and not less computational powers. The latter would depend on the dramatic development of both software and hardware.

When considering perspectives of developing software, we at last come to the essential role of FOL (First-Order Logic). This is so because each computer program is based on an algorithm, and each algorithm presupposes a mathematical proof. This relationship is obvious regarding programs of computing; say, a program to compute $y = \sqrt{x}$ owes its efficiency to the fact that the underlying formula is provable from the axioms of arithmetic. As to any other program, for example, one translating texts from English into Chinese, its functioning is due to the encoding of instructions into strings of ones and zeros, hence some arithmetic expressions. Such expressions are again formulae which are provable in arithmetic, while the range of what can be proved depends on the strength of logical rules of proof. The strength of FOL rules (let us recall) is limited – in that sense that in some cases they do not decide whether a formula does follow from relevant axioms.

Thus we come to realize the important fact: that reinforcing inference rules, be it possible, would extend the scope of problems being solvable through decision procedures, and thereby accordingly extend the scope of computing with the aid of software.

Is it in fact possible? It is Gödel to whom we are indebted an illuminating hint. Let us focus on his contribution and how it is continued nowadays, up to a recent mechanized deduction research.

1.3. On June 19, 1934, at the seminar run by Karl Menger in Wien, Kurt Gödel presented the communiqué entitled "Über die Länge von Beweisen" (on the length of proofs). The text appeared in 1936 in reports on the seminar (referred to as Gödel [1936: 23-24]). Its main point runs as follows (the numbering of items by WM).[4]

> Thus, passing to the logic of the next higher order has the effect, not only of (1) making provable certain propositions that were not provable before, but also of (2) making it possible to shorten, by an extraordinary amount, infinitely many of the proofs already available.

Thus we attain at what one may call *the Gödelian speed-up in problem-solving* (as in the title of this essay). The term 'speed-up' renders the acceleration of the processes of proving due to reducing the number of steps.

This assertion has not been either demonstrated or exemplified by Gödel himself (examples to test the assertion were to come later, produced by other authors – see Section 1.4 below). This may be a reason why this theorem had not been much referred to in the first years after its publishing; not as much as the theorems on the completeness of first-order logic, the incompletability of arithmetic, etc., although its import is comparable with that enjoyed by those famous results. However, one can find illuminating comments on its content, for instance in Kneale and Kneale [1962: 722], noticing the novelty of the approach suggested by Gödel in 1936. After discussing

[4] Translation in Gödel [1986: 397] under the title *On the length of proofs*. It is in order here to quote the original statement for its historical significance. *Der Übergang zur Logik der nächst höheren Stufe bewirkt also nicht bloß, daß gewisse früher unbeweisbare Sätze beweisbar zu werden, sondern auch daß unendlich viele der schon vorhandenen Beweise außerordentlich stark abgekürzt werden können.*

the unprovability of Gödel's formula, referred to as G, and the possibility of strengthening ANN (Arithmetic of Natural Numbers) by simply adding G to ANN axioms, the authors appreciate the new approach in the following passage.

> If, instead of merely adding G as a new axiom, we enlarge our formalism by adding some new general apparatus of proof which enables us to obtain [...] G as a theorem, we have an instrument which is more powerful for the ordinary purposes of arithmetic. The novelty here is the use of axioms containing variables of higher orders than those occurring in the number theory; and Gödel has shown that it not only enables us to prove formulae that were hitherto unprovable, but allows very much shorter proofs for many of the previously obtainable formulae. Within this new system it is possible, however, to construct a new undecidable formula, and so the whole process can be repeated *ad infinitum.*

The time in which this comment was made (1962) was not ripe enough to fully appreciate the import of the second part of Gödel's assertion, that concerning the shortening of proofs already available. Only just with the rise of the theory of computational complexity of algorithms (cp. Hartmanis and Stearns [1965]) people started to realize the significance of this part of assertion. The greater was becoming the computational power of hardware (increasing exponentially according to Moore's law), the more was growing the awareness of the import of the computational power of software. When we now are able to estimate the speed of processors available in the coming time, we can also estimate how fast algorithms we need in order to solve (with the united forces of hardware and software) the problem having a definite complexity. And so we learn messages like that: there are problems so hard that even most speedy computers, programmed with the fastest available algorithms, do not give us a hope of solving them even in some millions or more years.

What should reasonable people do in such a drawback? They should look for methods of *making it possible to shorten, by an extraordinary amount, infinitely many of the proofs already available* – as reads item 2 of Gödel's assertion (instead of 'proofs', now read 'algorithms'). When enjoying such realization due to Gödel, what we need bad is a paradigm of attaining such methods.

1.4. Before I report on such a serious argument to show the enormous difference between the efficiency of first-order and second-order logic, let mi give just a rough idea of that difference. Consider the following reasoning expressed in that part of ordinary English which corresponds to some part of second-order logic.

> In a village there are three parental couples, each having two little children, and there are no more parental couples in that village. Hence there are exactly six little children in that village.

Even in such a childish problem, one resorts to the second-order ideas because of talking of sets (couples) as existing. If a radical nominalist like, say, Tadeusz Kotarbiński regards second-order logic as an unscientific metaphysics, he should express such a reasoning in the first-order language. Then in its conclusion he has to use six individual variables (say, $x_1, \ldots, x_6$) in order to state about each individual the fact of being a little child, and then to say about any other individual denoted by a variable, say y, that if y were a little child, then it would be identical either with x_1 or [...] with x_6. This would require using more than hundred symbols. (And what if one is to speak about millions of individuals?). The formalism needed to express the premiss would be even more cumbersome, involving binary relation of being, so to speak, parentally coupled, and ternary relation of being a child of each member of the couple in question.

The comparison of the above reasoning in the first-order logic and that in the second-order logic suggests that some mental mechanisms of reasoning in humans happen to function according to second-order pattern. Is it possible for computers to simulate and match humans also in this respect? This is the question. The one we shall address with tracing the way from Gödel's [1936] idea to recent research in mechanized deduction.

A milestone in this way is found in George Boolos' [1987] seminal paper "A Curious Inference".[5]

Let the inference in question be called BP (for Boolos' Proof). I postpone presenting the content of BP (to be roughly presented below, item 1.5), and focus on hinting at the scale of difference when one compares the lengths of first-order and second-order proofs. The latter, as performed by Boolos, is a short derivation taking no more than one page, while the number of symbols used in the FOL derivation is represented by the exponential stack of as many 2's as 64536. It is larger than any integer that might appear in science.

An important source of this difference is found in the fact that BP reduces to so short derivation owing to the use of comprehension axioms

[5] There were earler studies which resorted to Gödel's idea of shortening proofs, e.g. Parikh [1973], but none of them has made a comparable impact on claryfying the issue.

(typical second-order means). These are propositions being subsumed under the following schema:[6]

$$\exists_X \forall_y (y \in X \Leftrightarrow \phi(y)).$$

1.5. This fact is crucial for the next step of our inquiry into the feedback of minds and computers. This is the step of entering the realm of mechanized deduction. The most expert guide to that realm I could find so far is the survey by Christoph Benzmüller and Manfred Kerber [2001] entitled "A Challenge for Mechanized Deduction".[7] The authors start from the message that the art of general purpose automated theorem proving has been best developed for first-order logic. This was convincingly highlighted by the success of first-order theorem provers in producing the machine-generated proof of the Robbins problem.[8] Their argument nicely fits into the idea I express with the phrase *the feedback between minds and computers.* They mean something like that when considering *facilities to combine interaction with automation.* Let the following quotation render such a course of thought.

> Automating proof search in higher-order logic is a very challenging enterprise, such that the above systems all provide facilities to combine interaction with automation. The idea is that the interactive human provides the crucial proof steps while simple subgoals are handled automatically by the prover. Of course, many non-trivial proofs can be already automated in higher-order logic. [...] A well known example illustrating the expressiveness and elegance of automated higher order theorem proving is Cantor's theorem, where the diagonalisation argument, in form of a lambda-term, is synthesised by higher-order unification.
>
> However, Boolos' example perspicuously demonstrates the limitations of current first-order and higher-order theorem proving technology. With current technology it is not possible to find his proof automatically, even worse, automation seems very far out of reach. Let's first give a high-level description why this is so. Firstly, Boolos' proofs need comprehension principles to be available and it employs different complex instances of them. [...] Secondly, the particular instances of the comprehension axioms cannot be determined

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[6] In the mentioned 'naive' form, this schema is exposed to danger of antinomies, but its sole purpose in this context is to suggest, in a possibly simplest form (without refinements), the very idea of comprehension through hinting at its second-order character.

[7] A considerable number of other studies on this subject can be found with the search: citeseer.ist.psu.edu/. There may be of special interest the paper by Natarajan Shanker "Using Decision Procedures with a Higher-Order Logic" which refers to excellent surveys of higher-order logics as offered by S. Feferman, J. van Benthem, etc.

[8] A report on that event, compiled by the present author, can be read in Mathesis Universalis, No. 4, Autumn 1997. See www.calculemus.org/MathUniversalis/4/6com-rob.html.

> by higher-order unification but are so-called Heurika-steps which have to be guessed. However, the required instantiations here are so complex that it is unrealistic to assume that they can be guessed. [...] Here it is where human intuition and creativity comes into play, and *the question arises how this kind of creativity can be realised and mirrored in a theorem prover.* [From Section 1; italics – WM.]

The question raised in the last sentence is fundamental. To give it a case study exemplification, the authors analyze difficulties involved in Boolos' 'curious inference'. It contains the following premises (1-5) and concludes with line 6. This conclusion may be reached in a short (one-page size) derivation in the second-order logic, being practically unattainable in the first-order logic (the notation as below is adjusted to a programming idiom).

1. FORALL n. f(n,1)=s(1)
2. FORALL x. f(1,s(x))=s(s(f(1,x)))
3. FORALL n. FORALL x. f(s(n),s(x))=f(n,f(s(n),x))
4. D(1)
5. FORALL x. (D(x) –> D(s(x)))

hence

6. D(f(s(s(s(s(1)))),s(s(s(s(1))))))

This case yields opportunity to observe the importance of critical reflexion on the strategy in planning a proof. As the authors assert (in Section 5.1), it depends on the reasoner's experience. The case in question requires enormous expertise, hence the following challenge for mechanized deduction: how to impart such a superior skill to a mechanical prover. This would mean constructing provers which together with planning proofs at object level would in parallel critically reflect on that planning, thus having a kind of meta-level understanding of their object-level activity. This challenge might be expressed using Emil Post's maxim that "Symbolic Logic may be said Mathematics self-conscious".[9] The endavour to make a mechanical prover critical about its own doings is something like to simulate a logical self-consciousness. This seems to be not only technical but also philosophical challenge.

An equally great challenge consists in simulating the process of concept-forming. It is specially perspicuous in Boolos' inference in which so great a role is played by searching for comprehension axioms. Selecting

[9] This saying is taken from Post's Diary entitled "Time Accounts", being the Appendix to Post's paper "Absolutely Unsolvable Problems and Relatively Undecidable Propositions – Account of an Anticipation" published by Davis [ed. 1994].

useful comprehension axioms is the same question (the authors notice) as to form interesting concepts. Another task consists in teaching a prover to resort to a model-based reasoning (this would require, if I properly quess, equipping the system with a large data-base concerning the domain which a given inference is concerned with).[10]

1.6. All that is highly interesting and amenable for the discourse on the development of information society. Mechanized deduction is essentially involved in knowledge management and expert systems as well as artificial society projects, the latter using the computer simulation of social phenomena cellular automata, agent programs, etc. All the listed enterprises, when undertaken at a large scale, are characteristic of information society. The input of mechanized deduction is in them obvious, as each of them includes reasonings.

Do researchers in the listed areas happen to recognize some encountered problems as likely to be either undecidable or intractable? Certainly, such awareness did appear in economics in the famous Hayek-Lange debate on socialist calculation (cp. Marciszewski [2002]). It was Oskar Lange who claimed all economic problems be tractable in the system of socialist central planning, thus endorsing the strong AI with regard to economic issues. It was Friedrich Hayek who claimed the intractability of central planning problems and advised therefore to resort to free market mechanisms as using better methods of data-processing, to wit taking advantage of paralell and interactive processing.

However, what would be the most interesting in that debate, has not been explicitly identified. It is the problem of the kind of logic as used by market agents. Let me direct our attention to a deep observation in the paper by Boolos [1987: 380]. It runs as follows.

> The fact that we so readily recognize the validity of I [so is referred to by Boolos his famous inference] would seem to provide as strong a proof as could be asked for that no standard first-order logical system can be taken to be a satisfactory idealization of the psychological mechanisms or processes, whatever they might be, whereby we recognize logical consequences.

Thus it does not seem unlikely that free market agents in their reasonings use higher-order logics without any computer assistance, so taking advan-

[10] These are just some examples from among a larger set of problems which the authors consider under the heading “How to find a solution?”. I discuss this particular report in such details, paying so much attention to it (as compared with other publications I mention here), for I see in it an illuminating example of how much advantage may take social researchers and philosophers of mind from expert reports on mechanized deduction.

tage of the Gödelian speed-up in their intuitive reasonings (like Boolos did in his short inference conducted in the second-order logic). On the other hand, a socialist planner (unlike a free market agent dealing with a highly restricted set of data) has to deal with so astronomically huge data amount that he is helpless without computer's assistance. However, that assistant is often doomed to fail in solving some problems at the second-order level – for the reasons discussed by Benzmüller and Kerber in their survey of various difficulties.

Let us look at the computational speed-up and tractability in the broad perspective of civilizational progress. The stage of information society we enter nowadays is but a preparatory step towards the age of biological engineering and cosmic engineering. Obviously, there will be enormous risks in such a course of events. However, let us assume that the human race will succeed (may be, in a distant future) in gaining the necessary moral potential to face the challenges. Then the rest would depend on a sufficiently large cognitive potential. Its core lies in mathematics as an indispensable tool of science and technology.

Now, let us look at the growing interaction, in the form of positive feedback, between the development of mathematics and the increase of computational power. That increase, apart from being due to ever greater hardware perfection (as claimed in Moore's law), haevily draws on software perfectioning. The kind of software most in this case relevant consists of programs for mechanized deduction. It is in order here to recall the lesson due to Gödel, Boolos and, lastly, such reserchers as Benzmüller and Kerber. To wit, if we like getting a necessary speed-up in mechanized deduction, we should resort to higher-order logics. Then some important problems, which were not tractable so far, become solvable in real practice.

A bit of reflection is due to European R&D policies regarding the challenges of information society, as considered specially in Lisbon Strategy. Among documents concerning that strategy there is one that reads as follows.

"The Union has today set itself a new strategic goal for the next decade: to become the most competitive and dynamic knowledge-based economy in te world. [...] Achieving this goal requires an overall strategy aimed at preparing the transition to a knowledge-base economy and society by better policies for the information society and R&D."

So reads a passage in the document, dated 2004.01.05, *Lisbon Strategy – the European Agenda for competitiveness, employment and social cohesion* by Professor Mario João Rodrigues being in charge of the preparation of the Lisbon Summit as special advisor to the Prime Minister. (Cp. www.bsssc.com/section.asp?id=217&pid=79.)

There is no need to exaggerate the role of logic and tractability issues in such enterprises. Nevertheless, there is a place for them in that R&D segment in which knowledge management, expert systems, and the like, require mechanized deduction. And this in turn – to make problems tractable – needs enhancing with more sophisticated logical tools like those from the height of higher-order logics.

2. On how does this volume deal with D&T issues

2.1. There are two contributions which offer a general historical account in the role of introduction: one by Roman Murawski and one by Kazimierz Trzęsicki. It is the former which opens this volume as the Editor decided to emphasize what is expressed by its title: *The present state of mechanized deduction, and the present knowledge of its limitations.* It is a revised and updated version of the closing chapter in the book by Witold Marciszewski and Roman Murawski *Mechanization of Reasoning in a Historical Perspective* [1995]. This is in accordance with the narrative of the previous Section where mechanized deduction is shown as an important factor of information society: the one likely to enrich the body of mathematical results (being in turn applied in high-tech), and profiting from inquiries into logics of higher orders.

Murawski's contribution explains how mechanized deduction techniques derive from theoretical achievements in logic. Namely, the results of Skolem and Herbrand demonstrate that if a theorem is true, this can be proved with an algorithm, hence in a finite number of steps. However, this does not hold if the theorem in question lacks truth. Then either one can prove in some cases the falsity of the given statement or the process does not stop.[11] These results are discusssed in an earlier chapter of the book mentioned to which the author makes references.

After the introductory remarks, the paper tells the story of early attempts of applying computers to prove theorems, in particular the results of Davis, Newell-Shaw-Simon, Gilmore, Gelernter, Hao Wang and Davis-Putnam. The next sections deal with resolution and unification algorithms of Prawitz and Robinson, and with their modifications. They proved crucial for the further research in mechanization and automatization of reasonings.

[11] Examples can be found in Jerzy Pogonowski's on-line draft of a textbook on logic, Section III.1, page 36, referred to from the page www.logic.amu.edu.pl/posluga_pogon.html.

The author does not continue his narration up to the point marked by Boolos' problem with its implication for automated theorem-proving. To complete the story, so relatively much attention is paid to that problem in the preceding section of this text. Another line of progress is discussed in the next paper.

2.2. Roman Matuszewski's contribution "On computer-assisted approach to formalized reasoning" is also concerned with the both constituents of this volume's title – decidability and tractability. As for the latter, we find the message that, for instance, the problem of deciding validity of propositional sentences (in conjunctive normal form) belongs to the co-NP class where we consider a sequence of finite ever larger search spaces to eventually establish validity.

Even more interesting is to hear that a major difficulty in formalizing mathematics turns out to lie in its practical unfeasibility, that is, intractability, rather than the impossibility of formalizing all mathematical proofs. It is believed by most, if not all, mathematicians – the author says – that one is able to formalize most of the present day mathematics using a sufficiently strong system of set theory, say ZFC, while valid theorems which cannot be derived in such a strong system are supposedly uninteresting statements which would not occur in the mathematical literature. Let me comment, we ecounter here a philosophically intriguing question of what is involved in the realm of indecidability. If there is so as is claimed by the mentioned mathematicians, then a practical impact of limitative theorems, like those of Gödel, turns out to be relatively small; undecidable statements do exist, but they would not belong to what is interesting in mathematics. This issue seems to require a careful study.

The utility of the paper consists in detailed information on some research projects which endavour to make the enterprise of mechanized deduction as tractable as possible by doing this in two ways: with enhancing inferential mechanism and by widening the base of mathematical knowledge. This is mostly exemplified through experiences of the Polish project MIZAR (in which the author takes part). As to the former, the system of natural deduction created in 1934 by Stanisław Jaśkowski is discussed as a theoretical inspiration for MIZAR. As to the latter, the role of sufficiently strong systems of set theory is emphasized.

The author inserts some historical remarks which should be treated as means to demonstrate the import of the problem as having so significant historical antecedents. With such an intention, it is usual that some authors see rather similarities than differences between the past and the present. However, arguments of Euclid are not formalized, likewise Aristotle's logic is not one to suit Hilbert's project. It was not until Leibniz that the idea of formalized and mechanized proof emerged, but with Leibniz it was just the idea, not its accomplishment (thus there is bit of exaggeration in saying that "Leibniz developed a calculus of reasoning"). Anyway, such references make sense to the effect that they hint at the main evolutionary line. When sketching that line, the author rightly stresses the role of Gentzen's *Hauptsatz* as a milestone in the way towards mechanized deduction.

2.3. Kazimierz Trzęsicki contributed the extensive essay: *From the idea of decidability to the number* Ω. Among its merits there is the due appreciation of the role of David Hilbert in the history of computation, the role closely tied with his *Entscheidungsproblem.* The author gives a sympathetic account of Hilbert's ideas and activities, starting from his Paris address of 1900. The story includes many crucial ideas in the form of literal quotations, and numerous historical details, as much picturesque as instructive, concerning Hilbert and his contemporaries. Special attention is paid to the Turing-Church Thesis and the problem of a chance of going beyond the limitations it claims. Not less interesting are footnotes remembering the past times of Leibniz's projects compared with current state of issues. The vivid narration combined with precise historical workshop, e.g. in quotations (preserving also the original language) and references, make the reading both pleasantly entertaining and instructive.

The next part of the essay that encourages reading is that dealing with Gödel's results and its relation to Hilbert Programme. A point which seems worth special recommending yields evidence against the often repeated view that Gödel's so-called First and Second theorems totally ruined Hilbert Programme. One after another authors repeat that opinion, while any diligent reader of *Grundlagen der Mathematik* by Hilbert and Bernays, a monumental work written after Gödel's discoveries, can ascertain what follows: that study involves a quiet analysis and continuation of Gödel's results, without any feeling of dissonance with Gödel. Trzęsicki documents that relationship from Gödel's point of view. A quotation from Gödel found by Trzęsicki deserves to be re-quoted here. Here are Gödel's thoughts from his letter to Constance Reid.

> "I would like to call your attention to a frequently neglected point, namely the fact that Hilbert's scheme for the foundation of mathematics remains highly interesting and important in spite of my negative results. [...] As far my negative results are concerned, I would see their importance primarily in the fact

> that in many cases they make it possible to judge, or to guess, whether some specific part of Hilbert's program can be carried through on the basis of given metamathematical presuppositions."

Let this comment be read in the light of Gödel's communication on the length of proofs (as discussed in the Section 1 above) and Hilbert's program be construed as a demand that for every proof there should be a formalism to make it amenable to mechanized deduction, not necessarily a formalism within the first-order arithmetic. Such a claim may be supported by the fact that Hilbert and Ackermann [1928] convincingly encourage to adopt the second-order logic for mathematical purposes. The last chapter of the said book starts with the section entitled "On the indispensability of a higher-order logic" (*Notwendigkeit einer Erweiterung des Kalküls*). Hence the historical evidence mentioned by Trzęsicki appears to be in a nice accord with the contention of Section 1 of this text.

Unfortunately, this accord gets spoiled through the Author's opinion (shared by him with some other writers) that Hilbert believed in the following thesis: any human reasoning may be expressed in the first-order logic. In a previous page the Author is a bit more cautious as he adds in parentheses that this so-called Hilbert Thesis was not formulated by him explicitly. This is to mean the thesis be formulated implicitly. But the sole implicit evidence as given by Trzęsicki is limited to an exemplification which consists in Hilbert's formalizing of a geometrical argument, due to Pascal, with the means of the first-order logic. But it was due to the very content of this argument that the first-order logic proved sufficient to formalize it.

Hence the reasoning of those who attribute to Hilbert the said Thesis looks like a naive generalization: in at least *one* case Hilbert used FOL as a suitable tool of formalization, hence Hilbert regarded that in *any* case FOL were a suitable tool of formalization. Apart of that example, if one carefully reads the passages (in [1928] book) referred to, one does not find any formulation, even implicit, like the so-called Hilbert Thesis. On the contrary, the mentioned content of the last chapter of [1928] book provides the explicit Hilbert's statement that mathematics needs higher-order logics.

The continuation of the issue of decidability, entitled *Entscheidungsproblem* is again reach in descriptions of historical setting. To give an example, the Author fittingly applies Rudolf Carnap's concept of explication (as a concept-forming procedure) to Turing's analysis of computation.

A subject still more capturing our interest is treated in the section entitled *Beyond the Church-Turing Thesis.* After a thorough discussion of the meaning of the Church-Turing Thesis, we get an exhausting and vividly narrated survey of various chances of going beyond the said Thesis. When appreciating the Author's approach, I see a point for a bit of polemics, namely the following statement by Trzęsicki.

> The Church-Turing Thesis tells about the procedure of calculation carried out by a human being. It does not say anything about the "calculation" realized in the nature by physical or biological processes.

Assuming that human beings belong to the physical and biological world (a point by no means controversial), one has to infer from the first sentence that the Thesis tells about procedures realized in the biological world, contrary to what the second sentence says. A serious problem is whether humans may surpass the Universal Turing Machine owing to their biological nature. Something like that was suggested by John von Neumann by the end of his booklet *The Computer and the Brain* [1958] where he opposed the historically formed logic and mathematics to the logic and mathematics of the brain as two highly different systems.

At last the reader arrives at the ending section which tells about the number Omega mentioned in the title of the essay. In a natural way this account crowns the discussion on the idea of decidability for it is concerned, so to speak, with the furthest limits of undecidability. The Letter Ω represents a real number denoting halting probability, that is, the chance that a program running at Turing Machine will eventually stop. The digits of this number are distributed in such a random way that any attempt to find a rule for predicting them is doomed to failure. Thus, this number, consisting of infinitely many 0's and 1's, has no recognizable pattern. Such is the message carried and commented in the final passages. The comment made at the very end incites a philosophical question about decidability of empirical problems, that is, those forming a set which subsumes the class of social research issues. Chaitin's results on Omega are by Trzęsicki applied to the physical world in order to conclude that the physical world, like mathematics, has a random structure. However, one may philosophically believe that all the physical quantities we encounter in the world are measurable with computable numbers. Such a belief seems to be entertained by some physicists. Some other people may regard that such a world would be too miraculous to be real. Anyway, the question seems to remain open.

2.4. The title of Witold Marciszewski's contribution resembles that of Stephen Wolfram [1985]: "Undecidability and intractability in theoretical physics". This is not to mean that this paper parallels Wolfram's in listing and commenting results concerning either undecidability or intractability in scientific theories, in this case teories in social research. When imitating

Wolfram's title, with exchanging "theoretical physics" for "social sciences", the author intends to encourage social theorists to ask themselves: do they in their work encounter analogous metatheoretical problems?

The answer should be in the affirmative, though the cases of such metatheoretical awareness are not frequent among social scientists. Such a case occurs, although not quite explicitly, in the famous debate on socialist economic calculation between the Austrian School (Ludwig von Mises, Friedrich Hayek and others) and some socialist economists, especially Oskar Lange (see Section 2.2, Example 2). The Austrians are convinced that the enormous complexity of economic problems makes them intractable at the scale of a state economy. On the other hand, the free market, seen as information-processing system, provides us with such efficient methods of handling data as is computing being both parallel and interactive. These features exist solely in a free-market setting, where many independent units parallely process only those, relatively small, data amounts which they need for their individual purposes (Hayek's idea of *dispersed knowledge*). Moreover, otherwise than in central planning, their information processing activity in thoroughly interactive, so they can learn from experiences, and adjust their strategies to a changing environment. Oskar Lange was certain that the rapid development of computational power will enable handling even most complex problems of central planning. As he died in 1965, he could not check his views against the background of the theory of computational complexity which did not appear until about 1970. It is a challenge for his present followers (there are such ones, though not much numerous) to confront the socialist project with the current state of complexity research in order to estimate chances of this project. In the paper some rudiments of complexity theory are reported as a background to put the problem of decidability and tractability in economics with respect to the current state of complexity research.

Other prerequisities to ask about the said properties of economic and social theories are due to the theory of games, devised by von Neumann and Morgenstern [1944]. It provides social researchers with a fitting model of human interactions motivated by quest for gains, while its mathematical formalism makes sense of asking computability questions. More recently, such inquiries profit much from addressing two computational studies, those on agent programs and those on cellular automata. It is natural to treat players as agents behaving according to rules for cellular automata, adjusted to problems and strategies of a game (the theory of such automata was devised by Stanisław Ulam and John von Neumann). A fovourite case discussed in the theory of games is called "prisoner's dilemma" for its anecdotic plot being concerned with the issue of solidarity of two accused in their defence in a trial. This is to exemplify the core of the question of how in a game to reasonably compare profits of loyal collaboration with a partner with those which one may win being selfish.

The paper, following recent literature, offers some examples of tractability results for the theory of games combined with the computational model of cellular automata. The data offered are not much abundant, but they should help to draw the following methodological moral.

To start, let us realize that any talking of tractability makes sense just with respect to those theories which have mathematical models. Such are some economic and social theories using, for instance, either a model provided by the theory of games or that of cellular automata (or both); this is why such theories alone are considered in the paper. Notwithstanding, sociologists, social psychologists, etc. used to report on their experimental results in the tone of full certainty, as if they relied on a perfect algorithm. It is up to philosophers to make their learned collegaues aware of two things.

First, if one uses a mathematical model, there may be problems which are in that model intractable or, even, undecidable. Like the question of whether a given game strategy will gain a permanent predominance over the rest (cf. item 1.3). A researcher, when observing a finite sequence of moves may be satisfied with perceiving that in this set such a predominance was the case, and thus declare his observation as a final result; however, such a success would be due rather to his ignorance than to experimental skill.

Second, if no mathematical model comes into play, then any empirical proceeding consists in guesses, or in the method of error and trial. Once upon a time, for example in the Vienna Circle, people believed in the so-called logic of induction which should have yielded algorithms to infer scientific laws from observational data. However that dream has gone with wind, especially with the wind of Popperian criticism. This is not to mean that all guesses are equally devoid of reliability. There is a fact significant for scientific method, to wit that there are varying degrees of relability, depending, e.g., on preciseness of operational definitions of theoretical concepts involved.

2.5. A concrete and significant piece of own research in modelling social interactions is reported in the paper by Anna Gomolinska "Rough information granules in social agent system modelling". The paper nicely fits into the frame of tractability issues and exemplifies a high degree of methodological awareness. The computational status of some problems of social interaction and konwledge engineering is diagnosed as their being computationally hard (exactly, NP-hard). Hence, because of such intractability, the researchers look for heuristics which are computationally more attractive though they provide us with suboptimal and approximate solutions alone. These are the other strategies, alternative to the speed-up phenomenon, as mentioned in the title of this essay.

The paper deals with the modelling of social, cognitive, and communicating agents, either natural or artificial. Agents and their systems are dynamic complex objects which can be represented by rough information granules. Information granules, mentioned in the title as this research subject, are defined as follows. An agent – an autonomous entity able to act on behalf of others or for itself – is viewed as a complex structure built of rough information granules; the same refers to multiagent systems. "Rough" is taken in the sense of rough sets theory as developed by Z. Pawlak; in the paper the classical Pawlak model of rough sets and its two extensions are recalled. Rules of some knowledge representation language are used as labels for information granules and, on the other hand, they are themselves objects to form information granules. An agent's knowledge and belief, value and norm systems, judgment systems, classification procedures, and (inter)action modules can be modelled as information granules consisting of rules. Patterns of interaction and systems of agents can be viewed as such granules as well.

A considerable merit of the theory, as developed by Gomolińska and the international team she belongs to, lies in the fact that it considers a wide spectrum of agents' utilities including moral values, cooperative attitudes etc. This constitutes a reasonable and desirable extension as compared with the standard approach of game theorists.

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Let me put in a nutshell how the content of this volume relates to its title. The preceding part of this introductory comment recalls Gödel's claim that decidability and tractability are system-relative: in the sense that problems lacking these properties in a particular system can obtain them in a stronger one. Trzęsicki's paper discusses a high and inconquerable treshhold of undecidability constituted by the number Omega. Both Trzęsicki and Murawski much contribute to the picture of how logic has developed towards the issue of decidability.

Once the issue is settled in the negative, logicians focus on partial solutions to find means of mechanical decision procedures to such an extent as it could be done. This story is told by Murawski and by Matuszewski, the first narrated in a wider historical perspective, the second focussing on most recent developments and exemplifying issues of tractability.

A similar relation of completing general considerations with a piece of concrete research holds between the contributions by Marciszewski and by Gomolińska, respectively, each applying to social sciences. The former states the question and offers a set of concepts to address it, the latter shows some means to devise a working model of game interactions, a model which proves efficient in spite of hard intractability, due to the art of obtaining some suboptimal and approximate solutions.

Clearly, this survey of problems, ideas and results is far from being either complete or duly detailed. However, it seems to shed some light on crucial issues of current logic and informatics and, thereby, issues of our civilization.

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