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FROM THE IDEA OF DECIDABILITY TO THE NUMBER Ω^*

*It is not once nor twice but times without number that
the same ideas make their appearance in the world.*

(Aristotle, *On the Heavens*)

Abstract. Information science is based on the idea of computation. In the 20th century that idea was developed in connection with the problem of decidability. David Hilbert's formulation of the so-called *Entscheidungsproblem*, i.e. classical problem of decidability, produced a plethora of ideas that – in particular – gave rise to information science and is still abundant opening new horizons to philosophy, mathematics and computer science.

The paper will discuss methodological and cognitive premises of posing the question of decidability as well as the ideas that have been born since the appearance of the problem of decidability.

The last part of the paper will be devoted to speculations on a computer that could cross the barriers of the Turing machine. Nevertheless, even quantum and biological computers – if possible – would not be able to cross the barriers of the most random number Ω .

1. Introduction

Practical and theoretical limitations of computers pose a question of a possibility of overcoming them. Due to the progress in technology, intractability¹ could be lowering but new tasks and tendencies to make compu-

* The work has been supported by Grant 2 H01A 030 25: Nierozstrzygalność i algorytmiczna niedostępność w naukach społecznych (Undecidability and Intractability in Social Sciences).

¹ *Webster's Dictionary* defines:

Tractable

- (a) easily managed or controlled; docile; yielding.
- (b) easily worked, shaped, or otherwise handled; malleable.

Intractable

- (a) not docile or manageable; stubborn
- (b) hard to shape or work with: an intractable metal.
- (c) hard to treat, relieve, or cure.

Tractable derives from Latin *tractabilis*, from *tractare*, to handle, to manage, frequentative of *trahō*, to draw, to drag. **Intractable** is from Latin *intractabilis*, from *in-*, “not”

ters “more friendly” result in even more complex programs, consequently leading to its increasing. The progress in the performance of computers is lower than the growth of time and hardware resources need to execute ever more complicated tasks.

The variety of approaches to high performance computing may be divided into three main directions, cf. (Burgin 1999):

1. hardware-oriented (aimed mostly at the development of computational elements and their networks),²
2. software-oriented (focused on the advancement of computational methods and procedures),
3. pointed at the evolution of computed structures (such as data and knowledge).

There is a connection between these approaches, in particular, advances in software and structures depend on the hardware solutions.

The main topic of the author’s considerations will be the second approach. Universal computation is one of the foundation concepts in computer science. Any computation that can be carried out by one general-purpose computer can also be carried out on any other general-purpose computer. In order to avoid having to refer to different computers, it is defined as a model of computation that can simulate all computations. The most popular model of a universal computer is the Turing machine, cf. (Akl 2005).

The idea of computer is based on the concept of calculus, which has a very long history with its origins in antiquity. At the beginning of the 20th century the problem of decidability posed by Hilbert gave rise to the

+ *tractabilis*, “manageable”, from *trahere*, “to draw (along), to drag, to pull”. A programming task is considered as tractable if it can be accomplished in a reasonable period of time or with a reasonable supply of physical resources (usually space). Otherwise it is intractable. The study of tractability has a theoretical and a practical aspect, yielding theoretical and practical definitions of terms.

² In 1965 Gordon Moore, a co-founder of Intel and co-inventor of the integrated circuit, observed that the number of transistors per square inch on integrated circuits would double every year since the integrated circuit was invented (monthly “Electronics”, 19 April 1965). The current definition of Moore’s Law predicts that the data’s density doubles every 18 months. Most experts expect Moore’s Law to hold for at least another two decades. Moore’s Law is the key-defining trend of the technology age. In order to understand the future in the exponential world, Kurzweil in his famous book *The Age of Spiritual Machines: When Computers Exceed Human Intelligence* (1992) tells the story of the price that was promised to the inventor of the chess game by the emperor of China. The inventor asked the emperor to place a grain of rice on the first square of a chess board, and double the number of grains thereafter on each of 64 subsequent squares. To fill the entire board with rice it requires 18,446,744,073,709,551,615 grains. If a grain was counted out every second, it would take 584 billion years to count them all. The age of the planet Earth is only 4.5 billion years. The story goes that either the emperor lost his kingdom, or the inventor lost his head. Any way, the consequences were severe.

question of definition of calculus. The Turing-Church thesis (conjecture) claims that an intuitive notion of calculus is adequate to the notion of the Turing machine (or equivalent notions such as: recursive functions, the lambda-calculus by A. Church, the canonical systems by E. Post, the normal algorithms by A. A. Markov, the Minsky machines, the random access machines (RAM), the Kolmogorov algorithms, etc.). The Turing machine is a theoretical device that computes all the functions that are computable in any reasonable sense. Conversely, it is also believed that if a computation cannot be performed by the Turing machine, then it cannot be computed at all. In other words, the thesis states that all effective computational models are equivalent to, or weaker than, the Turing machine, cf. (Shoenfield 1991, p. 26). It has long been assumed that the Turing machine is able to do all that computers equipped with recursive algorithms do and all that can be done by the Turing machine may be executed by such computers (if they have enough time³). Thus the Turing-Church thesis states that any computation solvable by a precisely stated set of instruction (an algorithm) can be run on the Turing machine or a digital process computer. In recent years the number of people who maintain that the Turing machine cannot capture the entire spectrum of applications of computers has been growing. There are important constraints on the ability of the Turing machine. Moreover, some of these limitations do not concern physical restrictions. There are well-stated problems that are not computable by the Turing machine. Generally speaking, there are problems related to tractability, commensurability and computability. Thus the question arises if there are possible devices which can compute more than the Turing machine.

The attempts both to conceive a new notion of calculus and to abolish the Turing-Church thesis are still ineffectual. It would be a mistake to dismiss such theorizing as idle speculation. Moreover, the study of more powerful models of computation is of considerable importance, with many far-reaching implications in computer science, mathematics, physics and philosophy. The inability to develop ways in which we could build machines with more computational power than the Turing machine is not good evidence that this is impossible. Perhaps the path oriented on enhancing and enriching the already existing methods and procedures is deceptive. Maybe

³ The notion of the “speed of computation” makes little sense in the classical understanding of mathematics. Its importance has been recognized with the advent of modern computers and their applications, especially such that need to be accomplished in real time.

only inventing of extremely original computational structures will perform a real breakthrough. It can also be the case that the task exceeds the ability of our intuition. Perhaps we are not able to accomplish it only in our inner world. Thus we have to look for other ideas searching for them in the external world. What we require is a radically new technique and/or even new developments in physics and biology.

Computers are used to forecast natural phenomena. For Plato, the world has a soul and God speaks through mathematics. Leibniz's saying: *Cum Deus calculat et cogitationem excercet, fit mundus* (when God thinks things through and calculates, the world is made)⁴ could be, in this context, assumed to mean: computers' usage is effective because the nature calculates (Galileo's principle of natural computation), too. Today, we may omit "Deus" from this precept: via calculation we create everything from the neutron bomb to computers or to the human genome map project. There is a parallelism between a computer and the nature. If so, the question of calculus is no longer purely theoretical. We may ask about natural computers, i.e. about those phenomena that "calculate" (in nature). The nature has inspired Turing. The deliberation on brain and its way of operation provokes many ideas in computer science. According to Fredkin⁵ and Zuse,⁶ the universe is a kind of computational device – it is a cellular automaton.⁷ Per-

⁴ This sentence is on the margin of *Dialogus*, (1846–1863, pp. 190–193).

⁵ Edward Fredkin (1934–), one of the first modern computer programmers and hackers. Despite a quite non-traditional style of thinking and even without being a graduate student, in the sixties of the XX century he was invited to become a professor of MIT. For more about the life and ideas of Fredkin see (Wright 1988a, Wright 1988b). See also Fredkin's website <http://www.digitalphilosophy.org/>

⁶ Konrad Zuse (1910–1996), a German civil engineer and painter. He invented the first programmable computer (Z1) and the first high-level programming language. In 1936 Zuse applied for a patent, which he did not get. In spite of it, Zuse decided to build the machines.

⁷ Fredkin is its originator. Zuse (1967, 1969) arrived at that idea independently of Fredkin. "A cellular automaton is a collection of 'colored' cells on a grid of specified shape that evolves through a number of discrete time steps according to a set of rules based on the states of neighboring cells. The rules are then applied iteratively for as many time steps as desired. Von Neumann was one of the first people to consider such a model, and incorporated a cellular model into his 'universal constructor.' Cellular automata were studied in the early 1950s as a possible model for biological systems (Wolfram 2002, p. 48). Comprehensive studies of the cellular automata have been performed by S. Wolfram starting in the 1980s, and Wolfram's fundamental research in the field culminated in the publication of his book *A New Kind of Science* (2002) in which Wolfram presents a gigantic collection of results concerning automata, among which are a number of groundbreaking new discoveries." See (Weisstein 2005). Pythagoras, an ancient philosopher, maintained that "the whole thing is a number" and that "everything can be calculated". In the information era this thought has been repeated by Stephen Wolfram: the universe is an enormous computer.

haps we have to think about computers starting from the idea of “universe as a computer”.⁸ We may ask then if the nature-computer is more powerful than the Turing machine. It is rather an empirical than a theoretical problem. From a philosophical angle, there are good reasons to hope that the answer is positive and – if so – it is not excluded that the future computer programs will not be constrained by the Turing machine.

In the theory of computation,⁹ the Church-Turing thesis has a position comparable to that of the fifth of Euclid’s axioms in the 18th and 19th centuries. When in 1817 Carl Friedrich Gauss, the prince of mathematicians, began to study the consequences of dropping of the fifth axiom, he did not publish his results fearing the reactions of the mathematical community. Non-Euclidean geometries of János Bolyai and Nikolai Lobachevski were considered to be in obvious breach of the real-world geometry that the mathematics sought to model and thus an unnecessary and fanciful exercise. The new ideas and results were applied in Einstein’s general theory of relativity according to which the real geometry of our universe was non-Euclidean. Nowadays the idea of the non-Turing or super-Turing or hyper-Turing (a restricted form of super-Turing) machine seems to be a mere object of speculation. Nevertheless, there are some good reasons to believe in its applicability in quantum physics and molecular biology. If hypercomputation¹⁰ is possible, we may ask whether there are any limitations of (hyper)computability. The answer is positive. If the universe has a random structure, then no matter how efficient the future computer would be, there would remain some non-(hyper)computable numbers.

⁸ The idea of universe as a computer has many supporters. For them “the universe indeed may be some kind of universal computational device, or, to say the least, there may be some advantage to look at the Universe as if it is a computer”. See eg. (Petrov August 30, 2003, Petrov n.d.a, Petrov n.d.b).

⁹ The original name was *recursion theory*, since the mathematical concept claimed to cover exactly the computable functions called *recursive function*. That name was changed into *computability theory* during the last years. In many titles the term “recursion theory” still occurs.

¹⁰ Hypercomputation sometimes is defined as a method of computation of non-computable functions. Apparently, there is a *contradictio in adiecto*. Hypercomputation was first introduced by Turing (1939), which investigated systems with an oracle to compute a single arbitrary (non-recursive) function from naturals to naturals. Here we are not interested in a device that stops for an input for that the Turing machine does not, as e.g., the Turing machine in which an oracle is available. What we aim to discuss is the notion of computability and the possibility of abolishing the Church-Turing Thesis.

2. Hilbert's concept of mathematical knowledge

While searching for the origins of the problem of decidability, it is necessary to take into account the address *Mathematische Probleme* (1900) that David Hilbert (1862–1943)¹¹ delivered at the second International Congress of Mathematicians in Paris, 8th August 1900.¹² The questions of the lecture were consulted with Hilbert's friends and most distinguished mathematicians, Hermann Minkowski (1864–1909)¹³ and Adolf Hurwitz (1859–1919).¹⁴

¹¹ By 1900 Hilbert had emerged as a leading mathematician. He solved the major problems of invariant theory. He was famous for his great *Zahlbericht* (1897) that was written for the request of Deutsche Mathematiker-Vereinigung (the German Mathematical Society). In Hurwitz's opinion the implications of *The Foundations of Geometry* (1899) reached far beyond its immediate field. In a letter to Hilbert he wrote: "You have opened up an immeasurable field of mathematical investigation which can be called the "mathematics of axioms" and which goes far beyond the domain of geometry." Cf. (Gray 2000b).

For Otto Blumenthal, Hilbert's biographer and his first student, Hilbert was a man of problems (Hilbert 1932–1935, vol. 3, p. 405): "In the analysis of mathematical talent one has to differentiate between the ability to create new concepts that generate new types of thought structures and the gift for sensing deeper connections and underlying unity. In Hilbert's case, his greatness lies in an immensely powerful insight that penetrates into the depths of a question. All of his works contain examples from far-flung fields in which only he was able to discern an interrelatedness and connection with the problem at hand. From these, the synthesis, his work of art, was ultimately created. Insofar as the creation of new ideas is concerned, I would place Minkowski higher, and of the classical great ones, Gauss, Galois, and Riemann. But when it comes to penetrating insight, only a few of the very greatest were the equal of Hilbert." For Hermann Weyl (1944, p. 612) "No mathematician of equal stature has risen from our generation."

¹² At the Congress because of time constraints Hilbert presented only ten of the twenty-three problems. For more details and about the twenty fourth problem (it asks for the simplest proof of any theorem), see (Thiele 2003). Thiele (2003, p. 1) writes: "for a century now, the twenty-fourth problem has been a Sleeping Beauty." Let us remark that Polish logicians obtained many results concerning economy and simplicity in expressing (e.g., a language with only one functor – Leśniewski, Łukasiewicz) and formulation of formal theory (e.g., with only one axiom – Łukasiewicz).

¹³ About his friendship with two-year-younger Minkowski, Hilbert wrote: "Seit meiner Studienzeit war mir Minkowski der beste und zuverlässigste Freund, der an mir hing mit der ganzen ihm eigenen Tiefe und Treue. Unsere Wissenschaft, die uns das liebste war, hatte uns zusammengeführt; sie erschien uns wie ein blühender Garten. [...] Gern suchten wir dort auch verborgene Pfade auf und entdeckten manche neue, uns schön dünkende Aussicht, und wenn der eine dem anderen sie zeigte und wir sie gemeinsam bewunderten, war unsere Freude vollkommen. [...] Er war mir ein Geschenk des Himmels, wie es nur selten jemand zuteil wird, und ich muss dankbar sein, dass ich es so lange besaß. [...] Jäh hat ihn der Tod von unserer Seite gerissen. Was uns aber der Tod nicht nehmen kann, das ist sein edles Bild in unserem Herzen und das Bewusstsein, dass sein Geist in uns fortwirkt." About Minkowski himself he wrote, (1971): "Er war mir ein Geschenk des Himmels, wie es nur selten jemand zuteil wird, und ich muß dankbar sein, daß ich es so lange besaß. Jeder, der ihm näher stand, empfand die Harmonie seiner Persönlichkeit und den Zauber seiner Genialität; sein Wesen war wie der Klang einer Glocke, so hell in dem Glück bei der Arbeit und der Heiterkeit seines Gemütes, so voll in der Beständigkeit und Zuverlässigkeit, so rein in seinem idealen Streben und seiner Lebensauffassung. Wie er gelebt hat, so starb er – als Philosoph."

¹⁴ Hurwitz was three years older than Hilbert. Hilbert remembered him: "Damals noch

Weyl (1944, p. 613) wrote about the role that friendship had played in Hilbert's intellectual development:

More decisive than any other influence for the young Hilbert at Königsberg was his friendship with Adolf Hurwitz and Minkowski. He got his thorough mathematical training less from lectures, teachers or books, than from conversation.

In a letter of 5th January 1900 Minkowski (1973)¹⁵ advised Hilbert to seize the moment, writing:

Most alluring would be the attempt to look into the future and compile a list of problems on which mathematicians should test themselves during the coming century. With such a subject you could have people talking about your lecture decades later.

Mathematical research has been influenced by that address, which continues to have a tremendous importance for all mathematicians.¹⁶ Already by September 1900, George Bruce Halsted had written that Hilbert's beautiful paper on the problems of mathematics "is epoch-making for the history of mathematics" (Halsted 1900, p. 188). A substantial part of Hilbert's fame rests on it. Hilbert spoke on the problems of mathematics to such effect that it has been remarked by Hermann Weyl (1951, p. 525):

We mathematicians have often measured our progress by checking which of Hilbert's questions had been settled in the meantime.

Hilbert and his twenty-three problems have become proverbial. Throughout the 20th century the solution to the problem was the occasion for praise and celebration. The Hilbert problems have made their impact, just as Minkowski predicted. It would take more than a single book to describe the results

Student, wurde ich bald von Hurwitz zu wissenschaftlichem Verkehr herangezogen und hatte das Glück, in der mühelosesten und interessantesten Art die geometrische Schule von Klein und die algebraisch-analytische Berliner Schule kennenzulernen. Dieser Verkehr wurde um so anregender, als auch der geniale Hermann Minkowski zu unserem Freundschaftsbund hinzutrat. Auf zahllosen, zeitweise Tag für Tag unternommenen Spaziergängen haben wir damals während acht Jahren wohl alle Winkel mathematischen Wissens durchstöbert, und Hurwitz mit seinen ebenso ausgedehnten und vielseitigen wie festbegründeten und wohlgeordneten Kenntnissen war uns dabei immer der Führer."

¹⁵ See also (Reid 1970, p. 69).

¹⁶ During the Millennium Meeting in Paris in May 2000, the Clay Mathematics Institute of Cambridge, Massachusetts, identified seven Millennium Prize Problems, for each of which it has put up a one million dollar prize for a solution. It has been declared that the problems "are not intended to shape the direction of mathematics in the next century" (The Clay Mathematics Institute http://www.claymath.org/prize_problems/html).

produced in this mushrooming field in the 20th century, cf. (Gray 2000b). Hilbert's impact on the world of mathematicians is summed up by Weyl (1944, p. 132):

I seem to hear in them from afar the sweet flute of the Pied Piper that Hilbert was, seducing so many rats to follow him into the deep river of mathematics.

According to Hilbert,

... jedes Zeitalter eigene Probleme hat, die das kommende Zeitalter löst oder als unfruchtbar zur Seite schiebt und durch neue Probleme ersetzt.
... every age has its own problems, which the following age either solves or casts aside as profitless and replaces by new ones.¹⁷

In the introduction to the Paris lecture the future problems that are expected for the new generation of mathematicians are characterized first of all on the ground of methodological and epistemological premises.

Science needs unsolved problems. Due to them it is alive and develops. Hilbert says:

Solange ein Wissenszweig Überfluß an Problemen bietet, ist er lebenskräftig; Mangel an Problemen bedeutet Absterben oder Aufhören der selbstständigen Entwicklung.

As long as a branch of science offers an abundance of problems, so long it is alive; a lack of problems foreshadows extinction or the cessation of independent development.¹⁸

A good mathematical problem has to be clear and understandable. What it means was explained by Hilbert as follows:

Ein alter französischer Mathematiker hat gesagt: Eine mathematische Theorie ist nicht eher als vollkommen anzusehen, als bis du sie so klar gemacht hast,

¹⁷ All fragments quoted here are from the lecture translated into English by David Joyce, Clark University.

¹⁸ In 1947 that Hilbert's thought was repeated by André Weil. He said: "Great problems furnish the daily bread on which the mathematician thrives." And turning to Hilbert's famous list of problems he singled out the 5th problem on Lie groups, the Riemann hypothesis, and the problem of generalizing the theorems of Kronecker's Jugendtraum which "still escapes us, in spite of the conjectures of Hilbert himself and the efforts of his pupils". Bourbaki, the group of French mathematicians (André Weil, Jean Dieudonné), shared Hilbert's view of mathematics. Till now the supply of problems in mathematics is inexhaustible, and as soon as one problem is solved numerous others come forth in its place. Contemporary mathematicians produce approximately two hundred thousand theorems a year (multiplying the number of journals by the number of yearly issues by the number of papers per issue by the average number of theorems per paper).

daß du sie dem ersten Manne erklären könntest, den du auf der Straße triffst. Diese Klarheit und leichte Faßlichkeit, wie sie hier so drastisch für eine mathematische Theorie verlangt wird, möchte ich viel mehr von einem mathematischen Problem fordern, wenn dasselbe vollkommen sein soll; denn das Klare und leicht Faßliche zieht uns an, das Verwickelte schreckt uns ab.

An old French mathematician said: A mathematical theory is not to be considered complete until you have made it so clear that you can explain it to the first man whom you meet on the street. This clearness and ease of comprehension, here insisted on for a mathematical theory, I should still more demand for a mathematical problem if it is to be perfect; for what is clear and easily comprehended attracts, the complicated repels us.

The “clearness and ease of comprehension” is the first feature of a good mathematical problem. The second feature of it – that also attracts – is its difficulty. Hilbert wrote:

Ein mathematisches Problem sei ferner schwierig, damit es uns reizt, und dennoch nicht völlig unzugänglich, damit es unserer Anstrengung nicht spotte; es sei uns ein Wahrzeichen auf den verschlungenen Pfaden zu verborgenen Wahrheiten – uns hernach lohnend mit der Freude über die gelungene Lösung. Moreover a mathematical problem should be difficult in order to entice us, yet not completely inaccessible, lest it mock at our efforts. It should be to us a guide post on the mazy paths to hidden truths, and ultimately a reminder of our pleasure in the successful solution.

Where is the source of mathematical problems? It is the next question posed by Hilbert. First of all, mathematical problems may have been found in the physical world and empirical experience. They are in mechanics, astronomy and physics. By the end of the address the question of unity of mathematics was discussed. According to Hilbert:

Der einheitliche Charakter der Mathematik liegt im inneren Wesen dieser Wissenschaft begründet; denn die Mathematik ist die Grundlage alles exacten naturwissenschaftlichen Erkennens. Damit sie diese hohe Bestimmung vollkommen erfülle, mögen ihr im neuen Jahrhundert geniale Meister erstehen und zahlreiche in edlem Eifer erglühende Jünger!

The organic unity of mathematics is inherent in the nature of this science, for mathematics is the foundation of all exact knowledge of natural phenomena. That it may completely fulfil this high mission, may the new century bring it gifted masters and many zealous and enthusiastic disciples!

The source of problems is also in human mind. In some cases, even without any influence of external world, mathematical problems result from logical operations such as comparison, generalization, specification, analysis and

grouping of notions. There is a continuous interchange between thinking and experience.

According to Hilbert, to solve a mathematical problem, methods of deduction and strictness should be used. A solution for a problem has to be based on a finite number of premises and a finite number of inferences. The premises should be precisely formulated. This is the requirement of strictness of proving, which corresponds to general philosophical expectations of human mind. Only in such a way the fullness of meaning of a problem and its fruitfulness are revealed. Mathematics and any other science needs inventing sharper tools and simpler methods.

The strictness of proof is not an enemy of its simplicity. On the contrary, many examples show that problems which demand strictness of proof can be solved unexpectedly easily or without much effort. Such a case may be true about all sciences. Due to sharper tools and simpler methods a unity of mathematics would be preserved.

Symbols have to associate with their objects of denotation. Arithmetical symbols are geometrical figures and geometrical figures are drawn formulas. Geometrical representation is indispensable for mathematicians. Hilbert shares Leibniz's view,

Drawing is a very useful tool against the uncertainty of words.

A similar notion of intuition and geometrical knowledge could be found in Kant, to whom Hilbert appealed explicitly. For Kant the intuition directly produces knowledge of a general geometric theorem: a requisite geometrical proof is one diagram, and constructing the diagram in intuition provides knowledge of the theorem, cf. (Zach 2001, p. 153).

An application of geometrical figures as means of strictly proving assumes the exact knowledge and complete mastery of axioms that are fundamental to the theory of these figures. For these reasons their intuitively perceived content has to be strictly axiomatized.¹⁹ For Hilbert (1922a):

Diese Zahlzeichen, die Zahlen sind und die Zahlen vollständig ausmachen, sind selbst Gegenstand unserer Betrachtung, haben aber sonst keinerlei Bedeutung.

¹⁹ Axiomatization and its importance is the topic of *Axiomatisches Denken*. Hilbert believes that any possible subject of scientific knowledge, if it is mature for theoretical study, is subjected to axiomatic method and indirectly to mathematics. The unity of science is based on a leading role of mathematics, (1918, p. 415). For more see (Hilbert n.d., p. 93), cf (Thiele 2003, p. 19):

Axiomatics is the rhythm that makes music of the method, the magic wand that directs all individual efforts to a common goal.

These digits (signs of numbers), that the numbers are and the numbers completely express, themselves are a subject of our perception, do not have any meaning.

By the end of December 1899, in a letter to Frege, Hilbert wrote (Frege 1976, p. 66):

Wenn sich die willkürlich gesetzten Axiome nicht einander widersprechen mit sämtlichen Folgen, so sind sie wahr, so existieren die durch die Axiome definierten Dinge. Das ist für mich das Criterium der Wahrheit und Existenz. When arbitrary chosen axioms do not contradict to the totality of consequences, so they are true, so the things defined by the axioms exist. It is for me the criterion of truth and existence.²⁰

The axiomatic method – an area of interest and a research program, which Hilbert pursued till the end of his activity – began with his work on axiomatic geometry and the publication of *Grundlagen der Geometrie*.²¹ Axioms provide implicit definitions of non-logical terms.²² In a famous remark:

Man muß jederzeit an Stelle von ‘Punkte, Geraden, Ebenen’ ‘Tische, Stühle, Bierseidel’ sagen können.

It must be possible to replace the words ‘point, line, plane’ with ‘table, chair, beer mug’.

already in 1891 the main idea of *Grundlagen der Geometrie* was expressed.²³ It was in contrast with Euclid’s view of axioms. Euclid conceived

²⁰ Later a quite similar thought was expressed by Henri Poincaré (1914, p. 137):

In der Mathematik kann das Wort: ‘existieren’ nur einen Sinn haben: es bedeutet ‘widerspruchsfrei sein’.

Frege asks Hilbert to consider the following example: Suppose we know that the propositions

1. A is an intelligent being
2. A is omnipresent
3. A is omnipotent

together with their consequences did not contradict one another; could we infer from this that there was an omnipotent, omnipresent, intelligent being? (Frege to Hilbert, 6/1/1900 (Frege 1980b)).

²¹ Hilbert’s work in geometry has the greatest influence in that area after Euclid. First published in 1899 *Grundlagen der Geometrie* has appeared in many new editions. Its impact on the axiomatic approach to mathematics throughout the 20th century is hard to overestimate.

²² Logical terms are topic-neutral. Learning a system of logic involves learning how to use these terms correctly.

²³ See (Blumenthal 1935, p. 402f), (Toepell 1986, p. 42).

axioms as propositions that embody intuitive truth.²⁴ Later, in 1921, in a paper of a programmatic character “The new grounding of mathematics” Hilbert (1922*b*) wrote:

If logical inference is to be certain, then these objects [certain extra-logical discrete objects, which exist intuitively as immediate experience before all thought] must be capable of being completely surveyed in all their parts, and their presentation, their difference, their succession (like the objects themselves) must exist for us immediately, intuitively, as something that cannot be reduced to something else. [...] The solid philosophical attitude that I think is required for the grounding of pure mathematics – as well as for all scientific thought, understanding, and communication – is this: In the beginning was the sign.²⁵

The idea that (in mathematics) there is nothing that is not given explicitly even logic which should be reduced to operations on symbols (things)²⁶ is significant for the development of information science and especially for artificial intelligence (AI). The process of computation by a computer is an abstract principle and it is independent of a physical form or mechanism used (i.e., symbol manipulation is as accurately accomplished with tables, chairs, beer mugs as with CPUs and memories). Computers process signals, thus for computers mathematics is a game played according to certain simple rules. Formal theories are physical objects of a specific kind: they all can be implemented as programs of digital computers. The concept of mechanization (in Turing’s sense) is equivalent as to its scope with that of formalization (in Hilbert’s sense). Mechanization and formalization differ

²⁴ Frege argued with Hilbert. In a letter to him, Frege wrote (Frege to Hilbert, 6/1/1900, (Frege 1980*b*)):

Given your definitions, I do not know how to decide the question whether my pocket watch is a point.

In debate with Hilbert, Frege held that axioms express determinate truths, fundamental facts of intuition. See (Kambartel 1975, Resnik 1974, Resnik 1980). If so, then they must be facts which follow from concepts which we already have. The logical concepts are the only ultimate intuitive concepts. Thus mathematical concepts should be reduced to purely logical ones. See Frege (1884).

²⁵ See (Hilbert 1922*b*, p. 202); repeated almost verbatim (Hilbert 1925), (Heijenoort 1967, p. 376). This is the text of a talk given in Hamburg, July 25–27, 1921. Cf. (Zach 2001, pp. 8, 115–116). Discussing with Hilbert, Oskar Becker (1927) maintained that signs that did not have meaning were not signs.

²⁶ Hilbert’s *Grundlagen der Geometrie* (1899) is the first work where (at least implicitly) logic in a model theoretic conception is involved. Frege for whom logic was a language (not as for Hilbert a calculus) criticized Hilbert from that universalistic position. Hilbert partly took it into account in the second edition of *Grundlagen der Geometrie* in 1903. Cf. (Müller 2001, p. 19).

in their pragmatic roles. The Turing machine is a mathematical model of a mathematician who acts according to Hilbert's formalistic programme, cf. (Marciszewski 2003, p. 79). Computers are "engines" that by means of a program generate theorems without involving human skills, intuitions, etc. Turing maintained that (Hodges 1992, p. 361):

... if a machine is expected to be infallible, it cannot also be intelligent.

Hence, as it is said in a contemporary handbook of mathematics:

The ultimate goal of mathematics is to eliminate all need for intelligent thought, (Graham, Knuth & Patashnik 1989, p. 56).

By the end of the introductory part of his lecture, Hilbert expressed a very characteristic view that (according to his opinion that view was shared by all mathematicians):

daß ein jedes bestimmte mathematische Problem einer strengen Erledigung notwendig fähig sein müsse ...
that every definite mathematical problem must necessarily be susceptible of an exact settlement ...²⁷

The belief in the solvability of every problem is not a peculiarity of mathematics. It is a general law inherent in the nature of the mind. The similar thought could be found in Wittgenstein's *Tractatus Logico-Philosophicus*, 6.5:

For an answer which cannot be expressed the question too cannot be expressed.
The *riddle* does not exist.
If a question can be put at all, then it *can* also be answered.

To solve a problem, we have to know it. Thus, for us there is a difference between the conviction of solvability of every problem and the conviction that we are able to achieve complete knowledge.²⁸ This difference is not clear when Hilbert writes:

²⁷ Let us remark that there is a difference between the thesis that all mathematical problems are solvable and the thesis of decidability. It may be assumed that this difference was not clear for Hilbert. The belief that every mathematical question is solvable in principle has been called the Hilbert axiom. See (Thiele 2003).

²⁸ The difference is clearer if we consider decidability and omniscience. For an omniscient being, an undecidable problem remains undecidable. We can also imagine (it is not self-contradictory) that everything is decidable but there are truths that we are not able to know. If 'to know' means 'to be axiomatizable', then, according to Gödel (1906–1978), our knowledge is convicted to be incomplete. Some of true propositions (e.g., Ramsey theorem) that are independent of the Peano Arithmetics may be proved in ZFC. But what

Diese Überzeugung von der Lösbarkeit eines jeden mathematischen Problems ist uns ein kräftiger Ansporn während der Arbeit; wir hören in uns den steten Zuruf: *Da ist das Problem, suche die Lösung. Du kannst sie durch reines Denken finden; denn in der Mathematik giebt es, kein Ignorabimus!*

This conviction of the solvability of every mathematical problem is a powerful incentive to the worker. We hear within us the perpetual call: *There is the problem. Seek its solution. You can find it by pure reason, for in mathematics there is no ignorabimus.*

The “ignorabimus” of the Paris address referred to famous agnostic catchwords “Ignoramus et ignorabimus” [We are ignorant and we shall (always) be ignorant]²⁹ of Emil du Bois-Reymond’s fashionable academic pessimism. Hilbert insisted that in mathematics:

Wir müssen wissen. Wir werden wissen.
We can know, and we shall know.³⁰

Hilbert even believed:

That there is no *ignorabimus* in mathematics can probably be proved by my theory of logical arithmetic. See (Thiele 2003).

This epistemological optimism was not restricted to mathematics. Hilbert (1932–1935, p. 292) in his Paris report speaks about “pre-established

about propositions independent of ZFC? The continuum hypothesis is (for Gödel and other platonists) true or false, but – as it has been proved by Cohen in 1963 – it is independent of ZFC. By the end of life Gödel (1970) was searching for the answer. If we agree with the thesis that the limits of axiomatization coincide with the limits of mathematics itself, then the Gödel’s incompleteness theorem states that the (mathematical) world is incomplete. In that interpretation the Gödel’s theorem is an argument against platonism. As an argument against the possibility of complete knowledge, the Skolem-Löwenheim theorem can also be used. The first-order (countable) language is too weak to describe models that are uncountable.

²⁹ It was given currently by Emil du Bois-Reymond (1831–1889), a German physiologist, in his *Über die Grenzen des Naturerkennens* [On the limitations of knowledge in the natural sciences] of 1872, (Du Bois-Reymond 1912). His brother Paul Du Bois-Reymond was a mathematician. Problems of natures of matter and the origin of motion were examples of unsolvable questions.

³⁰ These famous six words ended the address that Hilbert gave at the meeting of the Society of German Scientists and Physicians in Königsberg in 1930, on the occasion of presentation to him of an honorary citizenship of the town. *Wir müssen wissen. Wir werden wissen* is engraved on his tombstone in Göttingen. See <http://www.psych.uni-goettingen.de/home/ertel/ertel-dir/morehome/4gallerypast/4.04gravesofrenownscientists.html> “Reide-meister and Szego made arrangements for Hilbert to repeat the last part of his speech over the local radio station; a record of this talk pronounced at the broadcasting studio exists and was recently acquired by Victor Katsnelson. He kindly supplied his colleague Victor Vinnikov with the transcript of Hilbert’s address, which was then diligently translated to English by Amelia and Joe Ball.” See <http://www.math.ucsd.edu/williams/motiv/hilbert.html>; <http://math.sfsu.edu/smith/Documents/HilbertRadio/HilbertRadio.pdf>; (Vinnikov 1999).

harmony” between mathematics and physics. After 30 years, enriched by the experience of the participation in elaboration of the general theory of relativity and quantum mechanics, he speaks again about this “pre-established harmony”, the most magnificent and most wonderful example of “which is the general theory of relativity and quantum mechanics” (Hilbert 1930, p. 961). In the Königsberg address Hilbert stresses the special role of mathematics in science. It is as Galileo said:

Die Natur kann nur der verstehen der ihre Sprache und die Zeichen kennengelernt hat, in der sie zu uns redet.

Only one who has learned the language and signs in which nature speaks to us can understand nature.

After Kant, Hilbert repeated:

Ich behaupte, daß in jeder besonderen Naturwissenschaft nur so viel eigentliche Wissenschaft angetroffen werden kann, als darin Mathematik enthalten ist.

I maintain that, in any particular natural science, genuine scientific content can be found only in so far as mathematics is contained therein.

Hilbert (n.d., p. 95), (1931, p. 485) believed in parallels between nature and thought:³¹

Between thought [Denken] and event [Geschehen] there is no fundamental and no quantitative difference. This explains the pre-established harmony [between thought and reality] and the fact that simple experimental laws generate ever simpler theories.³²

³¹ See (Thiele 2003, p. 18).

³² The question of simple principles that generate theory is discussed by Leibniz. In his *Discourse on Metaphysics* (2005, sections 5–6) we read:

When the simplicity of God’s way is spoken of, reference is specially made to the means which he employs, and on the other hand when the variety, richness and abundance are referred to, the ends or effects are had in mind. Thus one ought to be proportioned to the other, just as the cost of a building should balance the beauty and grandeur which is expected. It is true that nothing costs God anything, just as there is no cost for a philosopher who makes hypotheses in constructing his imaginary world, because God has only to make decrees in order that a real world come into being; but in matters of wisdom the decrees or hypotheses meet the expenditure in proportion as they are more independent of one another. The reason wishes to avoid multiplicity in hypotheses or principles very much as the simplest system is preferred in Astronomy. [...] Thus we may say that in whatever manner God might have created the world, it would always have been regular and in a certain order. God, however, has chosen the most perfect, that is to say the one which is at the same time the simplest in hypotheses and the richest in phenomena, as might be the case with a geometric line, whose construction was easy, but whose properties and effects were extremely remarkable and of great significance. I use these comparisons to picture a certain imperfect resemblance to the divine wisdom, and to point out that which may at least raise our minds to conceive in some sort what cannot otherwise be expressed. I do not pretend at all to explain thus the great mystery upon which depends the whole universe.

Mathematics is a tool that joins thoughts and practice. All our culture is based on the intellectual exploration of the nature and the exploitation of the nature has its fundamentals in mathematics, cf. (Hilbert 1930).

It is clear that Hilbert was in duty to justify his epistemological optimism. This optimism was grounded in his concept of mathematics. At the International Congress of Mathematicians in Bologna in 1928 Hilbert (1929) added a new problem of completeness³³ to the old problem of consistency.³⁴ Hans Hahn (1879–1934) communicated Hilbert’s program to the Vienna Circle. In 1929 the problem of the completeness of the first order logic was presented. It also provided the young Kurt Gödel with his dissertation topic. Gödel (1930a) in his Ph.D. dissertation demonstrated that the first-order logic is complete, i.e. every valid formula can be derived from the axioms based on the *Principia Mathematica* (1910, 1912 and 1913) by Whitehead and Russell.³⁵ In the same year the problem of decidability (Entscheidungsproblem) was also raised in the famous textbook of logic *Grundzüge der theoretischen Logik* (1928) by Hilbert and Ackerman that proved to be extremely influential.

The questions of completeness, consistency and decidability may be formulated as follows:

1. is mathematics complete, in the sense that every truth about a given subject matter is inferable by means of a finite number of well-definite steps from a well-definite (recursive) set of axioms,³⁶

Hilbert also was convinced that: there is a realm beyond phenomena, and the universe is governed in such a way that a maximum of simplicity and perfection is realized, cf. (Thiele 2003, p. 18).

The idea of simplicity governed Einstein’s scientific work and his searching for unitary field theory, see (Norton 2000).

³³ The problem was already mentioned in (Hilbert 1899). The question was also a subject of academic lectures (Hilbert 1905). In 1929, Mojżesz Presburger provided a partial solution to Hilbert’s problems. He proved that natural number arithmetic with only addition and no multiplication is consistent (void of contradiction) and complete (capable of proving all valid statements).

³⁴ This problem was formulated for the first time in Hilbert’s lecture at the Congress in Paris as the second problem of proving the consistency of axioms of arithmetic (number theory and analysis): “Any contradiction in the deductions from the geometrical axioms must thereupon be recognizable in the arithmetic of this field of numbers. In this way the desired proof for the compatibility of the geometrical axioms is made to depend upon the theorem of the compatibility of the arithmetical axioms.

On the other hand, a direct method is needed for the proof of the compatibility of the arithmetical axioms. The axioms of arithmetic are essentially nothing else than the known rules of calculation, with the addition of the axiom of continuity.” (Hilbert 1900)

³⁵ The completeness theorem was a step towards the resolution of Hilbert’s Entscheidungsproblem.

³⁶ It means that we can mechanically check for any given statement if it is an axiom

2. is mathematics consistent, in the sense that only valid formulas may be proved,
- and
3. is mathematics decidable, i.e. there exists a definite procedure which, in principle, be applied to any formula and which guarantees producing a correct decision as to whether that formula is valid or not?

It is believed that Hilbert expected the answer to each question to be “yes”.

If an axiomatic system is complete and consistent, i.e. all valid and only valid formulas are its theorems, as is the case of the first-order logical calculus, 3 may be replaced by:

4. there is a method that in the case of any formula in a finite number of well-definite steps allows to decide whether this formula is a theorem or not.

If an infinite set of axioms is allowed, then the condition 3 is equivalent to the conjunction of 1, 2 and 4: all valid and only valid formulas can be taken as the axioms. An inconsistent system is decidable in the sense 4 though not necessarily in the sense 3. The difference between 3 and 4 is apparent in the case of an incomplete system.

On 7 September 1930 (one day before Hilbert’s address) at the second Meeting of Epistemology, September 5–7, 1930 (Zweiten Tagung für Erkenntnislehre der exakten Wissenschaften, a meeting of members of Wiener Kreis and Berlin Society for Empirical Sciences that also took place in Königsberg) Gödel lectured on his incompleteness theorem (the First Incompleteness Theorem).³⁷ In the proof³⁸ Gödel combined two insightful ideas. First, formulas were associated with natural numbers, to allow statements in arithmetic to refer to other statements in arith-

as well as the validity of proofs so that there can be no doubt that a theorem follows from the starting list of axioms. In theory, such a proof can be checked by a computer.

³⁷ Gödel spoke on the third, last day of the conference. Perhaps it was not tiredness that was the reason for the lack of interest in Gödel’s lecture (being 25 years old, he was not known yet). Even in the conference materials there is a lack of information about it. John von Neumann (He met David Hilbert on a visit to Göttingen in 1926, after which he was offered a position of a Privatdozent, an unsalaried lecturer, at the University of Berlin and then at the University of Hamburg. In 1930 he visited the United States, accepting a salaried lecturership at Princeton University, a move which would shape the rest of his life) was the only participant that immediately understood and conceived the significance of Gödel’s result: *genius recognized genius*. After Gödel’s talk he had a long discussion with him asking about details of the proof. See (Hahn et al. 1930, Gödel 1930b).

³⁸ (Gödel 1931) is arguably the most important mathematical paper of the 20th century and one of the greatest and most surprising results in the whole history of mathematics. The results of Gödel, however, have been also achieved – but not published – by Zermelo and Post.

metic.³⁹ Second, the predicate 'is provable in *Principia Mathematica*' was translated into a statement in arithmetic. Thus it was allowed to construct a statement that defined itself as not provable within *Principia Mathematica*. He proved that the leading formalization of mathematics, *Principia Mathematica*, was either an incomplete or inconsistent theory of the natural numbers. In other words, he stated that there are propositions in the first-order language of arithmetic that are either true but unprovable within *Principia Mathematica* or false but provable. Since consistency is required of all serious proof systems, Gödel's result is considered a proof of the incompleteness of *Principia Mathematica*. The completeness of the axioms of arithmetic can never be proved. If a system of sufficient power to express arithmetic is consistent, then it must be incomplete.⁴⁰ More generally, the first theorem can be paraphrased as:

there is no all-encompassing axiomatic system which is able to prove all mathematical truths, but no falsehoods.

Thus the condition 1 cannot be fulfilled. It was the first nail to coffin of Hilbert's program.⁴¹ While the proof of the incompleteness of *Principia Mathematica* was finished and unquestionable, the incompleteness

³⁹ The idea of arithmetization of theorems as well as rules applied in the proof by Gödel is the base of von Neumann architecture of computers (instructions are stored as binary values).

⁴⁰ Gödel remembered: "In summer 1930 I began to study the consistency problem of classical analysis [...] I reached the conclusion that in any reasonable formal system in which provability in it can be expressed as a property of certain sentences, there must be propositions which are undecidable in it." (Wang 1996)

⁴¹ In the 1920s, David Hilbert proposed a research program with the aim of providing mathematics with a secure foundation. It was one of the grandest research projects in the philosophy of mathematics. A number of mathematical principles, such as impredicative definitions, the axiom of choice, and the law of excluded middle for infinite totalities, were charged contradictory, false, or at least were unfounded assumptions. Hilbert distinguished between the unproblematic, finitistic part of mathematics and the infinitistic (ideal) part that needed justification. Infinitistic mathematics should be justified by finitistic methods. Only they are trustful. The ideal part of mathematics has to be formalized. In the proof theory only the finitistic methods could be used. Hilbert wanted to reduce all of mathematics to finite reasoning from a set of self-evident axioms. The purpose of Hilbert's program was to show that mathematics when the actual infinity is allowed is safe and free of any inconsistencies: *No one shall expel us from the paradise that Cantor has created for us* (Aus dem Paradies, das Cantor uns geschaffen, soll uns niemand vertreiben können), see (Hilbert 1925). In pursuing this objective Hilbert finally approached Kronecker's finitism. To a certain extent his metamathematics closely corresponded to finitary mathematics 'à la Kronecker. Due to Cantor the concept of the actual infinity for the first time became available for strict formal-logical (certainly, in the sense of classical Aristotle's logic) and mathematical analysis. In the field of mathematics Hilbert tried to answer the old question of philosophy: whether and how the limited human brain is able to handle the infinite?

of all formal systems was dependent upon an interpretation of ‘finite means’.⁴²

Though Gödel’s theorems are not the main subject of interest here, it is worth saying something more about them for the better understanding of further considerations. Roughly speaking, in the first theorem Gödel has proved that for any consistent first-order formal system that is sufficiently strong to axiomatize the arithmetic of natural numbers there are true sentences, moreover one can construct such a formula, that can neither be proven nor disproven within the system itself. There is not a complete list of axioms. Each time a statement is added as an axiom, if the set of axioms is recursive, there will always be another statement (an undecidable statement) out of reach. Mathematical knowledge is always imperfect; starting from a recursive consistent set of axioms we cannot prove everything that is to be known. In the axiomatic approach, the “tree” of all formulas growing from axioms is not only extremely expansive but even disconnected. In the theorem on the unprovability of consistency (today called Gödel’s Second Incompleteness Theorem) it is stated that such a formal system, if used as its own proof theory, cannot be used to prove both its own consistency and completeness.⁴³ It does not depend on the power of induction that is allowed within the system. It means that the condition 2 of page 89 is not feasible even if non-finitistic induction is allowed. It is worth mentioning that Church (1936*a*) showed that Gödel’s results cannot be obtained by finite means. To prove the incompleteness of a system, we have to prove its consistency. However, according to the second incompleteness theorem, the proof of consistency of a system containing arithmetics is not possible without using stronger logical means that are available in the system itself. In such a system one can construct a formula that:

1. can be neither proven nor disproven within that system;

or

2. can be both proven and disproven within that system.

⁴² The goal of Hilbert’s problem progressed into an effort to show that mathematics is reducible to logic. The school of thought that assumes that is referred to, alternatively, as positivism, logical positivism or logicism, see (Corbeil 1997). Bertrand Russell and Alfred North Whitehead using a particular set of axioms of arithmetic that seemed promising, commonly labeled as Peano’s axioms of arithmetic, proceeded to construct a set theory and a number theory. The result was *Principia Mathematica*, a monumental three-volume tome spanning thousands of pages of small print. When Gödel published his incompleteness theorems, Russell and Whitehead were working on the fourth part of *Principia Mathematica* (on geometry). The volume was never completed.

⁴³ For technical as well as philosophical and historical information on Gödel’s theorems see e.g., (Murawski 1999).

The Gödel theorems do not state that a theory of arithmetic is inconsistent or incomplete, only that it cannot be proven under given conditions. Gerhard Gentzen (1936, 1938, 1974) has showed that it is possible to prove both the consistency and completeness of a formal system, only if induction of strictly greater order is used.⁴⁴ Thus such a proof with induction of an equivalent order is not possible. It is proven that the arithmetic with finitistic induction is both consistent and complete by using non-finitistic induction to ϵ_0 .⁴⁵ Gödel's theorems are consequences of Gentzen's theorem.

Gödel's results confirmed Cantor's belief that there are no foundations of mathematics without metaphysics, i.e. without infinite methods.

Gödel (1931, p. 197) openly referred to Hilbert's programme:

It must be expressly noted that [these theorems...] represent no contradiction of the formalistic standpoint of Hilbert. For this standpoint presupposes only the existence of a consistency proof effected by finite means, and there might conceivably be finite proofs which **cannot** be stated in [Peano arithmetic].

The thought is repeated in a letter to Constance Reid:

I would like to call your attention to a frequently neglected point, namely the fact that Hilbert's scheme for the foundation of mathematics remains highly interesting and important in spite of my negative results. What has been proved is only that the specific epistemological objective which Hilbert had in mind cannot be obtained [...] As far as my negative results are concerned, apart from the philosophical consequences mentioned before, I would see their importance primarily in the fact that in many cases they make it possible to judge, or to guess, whether some specific part of Hilbert's program can be carried through on the basis of given metamathematical presuppositions. Cf. (Thiele 2003).

The incompleteness theorems do not only affect the philosophy of mathematics. They are pertained to knowledge in general. J. I. Austin, being informed about Gödel results that not all truth can be proved, asked:

Who would have ever thought otherwise?

While they are only a limitation for the formalization process, they were taken by many to argue that there are statements in arithmetic whose truth is unknowable. Gödel's theorems are thus usually considered to be a major limitation on the power of reasoning. For some philosophers Gödel's theorems

⁴⁴ Independently of Gentzen the consistency of arithmetic was proved by Ackerman (1940).

⁴⁵ ϵ_0 is a transfinite (though countable) ordinal. ϵ_0 is defined as the smallest ordinal ϵ such that $\omega^\epsilon = \epsilon$ or as the limit of the sequence $\omega, \omega^\omega, \omega^{\omega^\omega} \dots$

are sufficient reasons for the conclusion that human thought is uncomputable. This claim is addressed by Lucas (1961) and his argument is already classical. It has been reported by Marvin Minsky that Kurt Gödel told him personally that he believed that human beings had an intuitive, not just computational, way of arriving at truth and that therefore his theorem did not limit what can be known to be true by humans.⁴⁶ Gödel himself believed that it was not a limitation for human reasoning, see (Dawson 1997). Arguments based on Gödel's theorems and their implications are indispensable in contemporary considerations of the concept of mind and questions of human knowledge. These arguments are the main subject of many publications.⁴⁷ One of the most discussed is Roger Penrose's *The Emperor's New Mind* (1989) and *Shadows of the Mind* (1994), where he claims that minds depend on quantum mechanical phenomena that cannot be reproduced by computation. If human mind is able to transcend limitations proved by Gödel but computers are not able to do that, then the application of computers is limited and the artificial intelligence is fundamentally weaker than a human mind.

Gödel proved that the idea of consistent and complete axiom system is not feasible, i.e. to both the questions 1 of page 88 and 2 of page 89 the answer is negative. Nevertheless the question 3 of page 89 still remained unanswered.⁴⁸ However, Gödel's theorems say nothing about decidability. Gödel assumed that the system was consistent but incomplete. With a decision method, we can assume a consistent system that is incomplete, but still attempts to use the decision method to check for theorems. That aspect was addressed by Turing and Church.

3. The idea of decidability

The idea of mechanical method of solving problems is old.⁴⁹ Usually Lullus⁵⁰ is pointed out as its originator. The founder of contemporary logic,

⁴⁶ See http://www.absoluteastronomy.com/encyclopedia/G/G/Gödel's_incompleteness_theorem.htm.

⁴⁷ For more see (Krajewski 2003).

⁴⁸ If we ask about 4, a procedure to decide whether a sentence was or was not a theorem, we might still hope that there would be possible to produce a general algorithm that for a given statement determines whether it is undecidable or not.

⁴⁹ The problem of mechanization of reasoning in a historical perspective is a subject of (Marciszewski & Murawski 1995).

⁵⁰ Raimundus Lullus (1232–1316), a theologian, philosopher and logician. He was influenced by Christian, Muslim and Jewish cultures. Lullus wrote over 280 works in Latin and Arabic. In *Ars Magna* (1305–08) he presented the idea of mechanization of reason-

Leibniz,⁵¹ was inspired by *Ars Magna* of Ramon Lullus, although he criticized the author because of the arbitrariness of his categories and his indexing. Leibniz distinguished between two different versions of *Ars Magna*. The first version, *ars inveniendi*, finds all true scientific statements. The other, *ars iudicandi*, allows one to decide whether any given scientific statement is true or not. In *Dissertatio de arte combinatoria* (1923) Leibniz cites the idea of Hobbes that all reasoning is just a computation: *cogitatio est computatio*.

William Stanley Jevons was next after Lullus to build a logical machine,⁵² which he did in 1869.

In the seventeenth century Gottfried Wilhelm von Leibniz, after having constructed a successful mechanical calculating machine, dreamt of building a machine that could manipulate symbols in order to determine the truth values of disputed statements.⁵³ He dreamt about times when political and economic questions could be settled, not by disputes, but by a sort of reckoning (*calculemus*) through which it would be possible for all people concerned to agree at least in principle about the issues at stake. *Calculemus!*⁵⁴ (Let's calculate!) reflects Leibniz's conviction that all human

ning. He constructed a logical machine: on seven wheels different notions were written, rotation of the wheels resulted in different associations of the notions. Lullus had the proper intention to demonstrate all the truths of the Catholic church. The machine was used in his missionary work in North Africa and Asia Minor where Lullus taught the pacifist doctrines of St Francis of Assisi. Lullus used logic and mechanical methods involving symbolic notation and combinatorial diagrams to relate all forms of knowledge. The underlying goal of the Lullus text machine was the formulation of universal rules, which could cross boundaries and cultures. His great invention was the idea of producing the totality of human wisdom by a mechanical method of combining a strictly limited quantity of signs. Lullisten, the followers of Lullus, studied and copied his works though they were forbidden for many centuries. (In 1376 Pope Gregory XI charged Lullus with confusing faith with reason and condemned his teachings.) In the 17th century the Lullus's idea of an universal language had a special attention. Cf. (O'Connor & Robertson 2005), http://de.wikipedia.org/wiki/Raimundus_Lullus. Werner Künzel, with his colleague Heiko Cornelius (1991) translated Lullus's *Ars Magna* into the 'higher language' of Cobol.

⁵¹ Leibniz is among the authors who are often thought to have done more than they actually did. In Peckhaus (1999, s. 436) opinion:

The development of the new logic started in 1847, completely independent of earlier anticipations, e.g., by the German rationalist Gottfried Wilhelm Leibniz (1646–1716) and his followers; (Peckhaus 1994), (Peckhaus 1997, ch. 5). In that year the British mathematician George Boole (1815–1864) published his pamphlet *The Mathematical Analysis of Logic* (1847).

See also (Marciszewski & Murawski 1995).

⁵² In the case of a logical machine the output is YES or NO. In the case of calculating machine the output is – roughly speaking – a number.

⁵³ Leibnizian schemes are satirized in Jonathan Swift's *Voyage to Balnibarbi*.

⁵⁴ CALCULEMUS is an international research network. The goal of the group is the development of a new generation of mathematical assistance systems based on the integration of the deduction power of deduction systems and the computational power of computer algebra system. See <http://www.calculemus.net/>

reasoning may be turned into an object of mathematical demonstration and in such a way, any controversial truth can obtain the evidence of $2 + 2 = 4$.

... quando orientur controversiae, non magis disputatione opus erit inter duos philosophos, quam inter duos Computistas. Sufficiet enim calamos in manus sumere sedereque ad abacos, et sibi mutuo (accito si placet amico) dicere: c a l c u l e m u s.⁵⁵ (Leibniz 1890, vol. 7, p. 200)

Actually, when controversies arise, the necessity of disputation between two philosophers would not be bigger than that between two accountants. It would be enough for them to take the quills in their hands, to sit down at their abaci, and to say (as if inviting each other in a friendly manner): Let's calculate! (Calculemus!)

Leibniz was the first to realize that a comprehensive and precise symbolic language *characteristica universalis* (the perfect language which would provide a direct representation of ideas along with a calculus for the reasoning) is a prerequisite for any general problem solving method and much of his subsequent work was directed towards that goal. With the help of the language and a formal calculus (*calculus ratiocinator*) it would be possible to verify human thoughts like it is possible to verify an arithmetic calculation.⁵⁶ Leibniz's ideas were taken up again by Frege who defined the first

⁵⁵ Similar expressions occur in other texts of the same volume, e.g. p. 64–65, 125.

⁵⁶ The impact of Leibniz's ideas on computer science is discussed in *The Universal Computer: The Road from Leibniz to Turing* (2000) by Martin Davis. Leibniz's contributions to computer science are many-sided. E.g., he is the inventor of the binary system: ENIAC (**E**lectronic **N**umerical **I**ntegrator **A**nd **C**alculator) based on the decimal system (Stern 1981, p. 133); EDVAC (**E**lectronic **D**iscrete **V**ariable **A**utomatic **C**omputer) due to von Neumann was based on binary system (Goldstine 1972, p. 182):

Several states in each of which the combination can exist indefinitely, without any outside support, while appropriate outside stimuli (electric pulses) will transfer it from one equilibrium into another. These are the so-called trigger circuits, the basic one having two equilibria. ... The trigger circuits with more than two equilibria are disproportionately more involved. Thus, whether the tubes are used as gates or as triggers, the all-or-none, two equilibrium arrangements are the simplest ones. Since these tube arrangements are to handle numbers by means of their digits, it is natural to use a system of arithmetic in which the digits are also two valued. This suggests the use of the binary system." See "First Draft of a Report on the EDVAC" (von Neumann 1981).

The idea of the binary system was in connection with Leibniz's ontology. From only two absolutely simple concepts, God and Nothingness, all other concepts may be constructed, the world, and everything within it. And then Leibniz came to see that what is crucial in what he had written is the alternation between God and Nothingness. And for this, the numbers 0 and 1 suffice. According to Chaitin (2004):

[...] all of information theory derives from Leibniz, for he was the first to emphasize the creative combinatorial potential of the 0 and 1 bit, and how everything can be built up from this one elemental choice, from these two elemental possibilities. So, perhaps not entirely seriously, I should propose changing the name of the unit of information from the bit to the leibniz!

formal language in his famous work *Begriffsschrift, eine der arithmetischen nachgebildete Formelsprache des reinen Denkens* (1879). The first complete calculus for this language was presented by Hilbert (1928) in 1927. Hilbert's calculus was proved to be complete by Gödel in 1930 (cf. page 88 of this work).

The problem of decidability was already present in Hilbert's Paris lecture in the question of Diophantine equations.⁵⁷ In the 10th problem Hilbert asked about a finite procedure of finding an answer whether a given Diophantine equation had or did not have a solution.⁵⁸ The opinion that in the 10th problem the idea of Entscheidungsproblem⁵⁹ was present (e.g. Penrose) could be justified by the fact, that

it is a corollary of the methods used to give a negative solution to Hilbert's tenth problem that the question of whether any given Turing machine will eventually halt, and hence the Entscheidungsproblem can be encoded as a Diophantine problem. (Davis, Matijasevic & Robinson 1976)

There is one-to-one correspondence of the Turing machines and the Diophantine equations.⁶⁰

Decidability pertains to a non-finite countable class of questions that are characterized by a finite amount of information. Such a class is decidable if and only if there exists a finitely described (mechanical) procedure that in a finite number of steps enables to answer any question of the class YES or NO.

The substance of the decidability relays on the existence of the only one method that is applicable to any question of a considered class. In the case of Diophantine equations since the times of Diophant the answer

⁵⁷ A Diophantine equation is an equation $P(x_1, \dots, x_k) = 0$, where P is a polynomial with integers coefficients and the variables x_i ranger over integer. As an example we could use a famous Fermat's equations: $x^n + y^n = z^n$, where $x, y, z, n \in \mathfrak{N}$ – Fermat claimed that for any $n > 2$ the equation is unsolvable.

⁵⁸ Hilbert's 10th problem (is there a finite process which determines if a polynomial equation is solvable in integers?) has been solved in negative by a young Russian mathematician Yuri Matiyasevich (1970). He used some earlier results of Martin Davis, Hilary Putnam and Julia Robinson. Matiyasevich and Robinson have – despite the Cold War – collaborated on the problem. Julia Robinson was close to the end result. See (Davis 1973, Davis & Hersh 1973, Davis 1982, Matiyasevich 1993).

⁵⁹ German for “decision problem”. The first documented case of using the term “Entscheidungsproblem” was by Behmann (1921, p. 47) in his announcement at the meeting of mathematical society in Göttingen in May 1921.

⁶⁰ Matiyasevich's solution that there is a Diophantine equation that has not got a solution and that this fact cannot be proved is equivalent to the Gödel theorem. The interrelations between the Universal Turing Machine and Diophantine equations lead Chaitin towards discovering random structure of mathematics.

YES or NO has been given for many subclasses and for many particular equations. Such an algorithm does exist for the solution of the first-order Diophantine equations. Moreover, there is an algorithm of finding solutions of any of the Diophantine equations. The problem of decidability of the class of the Diophantine equations concerns finding a method that enables to give an answer YES or NO to the question: has a given equation got a solution at all (not: the solution). This question has been raised as the 10th problem.

A formal system is decidable if and only if there is a mechanical procedure that in the case of any sentence in the language of the system after a finite number of steps enables to give an answer YES or NO to the question if the sentence is a theorem of the system. The problem of decidability of a formal system was stated in works of Schröder (1895), Löwenheim (1915) and Hilbert (1918).

The first though vague formulation of the problem of decidability appeared in the works of Hilbert (1900, 1918). Directly the problem was formulated by Behmann (1922, p. 166):

Es soll eine ganz bestimmte allgemeine Vorschrift angegeben werden, die über die Richtigkeit oder Falschheit einer beliebig vorgelegten mit rein logischen Mitteln darstellbaren Behauptung nach einer endlichen Anzahl von Schritten zu entscheiden gestattet, oder zum mindesten dieses Ziel innerhalb derjenigen – genau festzulegenden – Grenzen verwirklicht werden, innerhalb deren seine Verwirklichung tatsächlich möglich ist.

A completely definite general procedure should be given when after a finite number of steps it is possible to decide about truth of falsehood of any given statement that could be formulated by using logical means, or at least that goal may be realized in – precisely stated – frames where its realization is indeed possible.

The question of decidability of the first-order logical calculus, i.e. the classical problem of decidability, what presently is denoted as “*Entscheidungsproblem*”, was formulated in 1928 by Hilbert in *Grundzüge der theoretischen Logik* (1928), a book written together with Ackerman.⁶¹ In the part entitled “Das Entscheidungsproblem im Funktionenkalkül und seine Bedeutung” (The decidability problem in functional calculus and its significance) we read (1928, p. 72):

⁶¹ In the year of book’s publication there took place a congress of mathematicians in Bologna. In his lecture at the congress Hilbert presented a question of completeness. Nothing was said about *Entscheidungsproblem*.

Das Entscheidungsproblem ist gelöst, wenn man ein Verfahren kennt, das bei einem vorgelegten logischen Ausdruck durch endlich viele Operationen die Entscheidung über die Allgemeingültigkeit⁶² bzw. Erüllbarkeit erlaubt.⁶³

The problem of decidability is solved if a procedure is known when for any given logical expression after a finite number of operations it is possible to decide about validity⁶⁴ or satisfiability.⁶⁵

The importance of decidability of the first-order logical calculus is based on Hilbert's thesis (not explicitly formulated by him) that all mathematics could be expressed in the first-order language. The Pascal theorem serves as an example. Hilbert showed how that theorem and questions concerning its logical connections in geometry would be reduced to a problem of the first-order logical calculus. Geometrical notions are replaced by implicit definitions and in result symbols are devoid of their geometrical meaning, see (Hilbert & Ackerman 1928, p. 74–76). After the discussion of the example he wrote (Hilbert & Ackerman 1928, p. 77):

Änliche Überlegungen gelten natürlich für jedes beliebige Axiomensystem. Similar considerations are valid for any axiomatic system.

In the conclusion it is clearly stated that:

... das Entscheidungsproblem muß als das Hauptproblem der mathematischen Logik bezeichnet werden.

the Entscheidungsproblem has to be appointed as the central problem of mathematical logic.

Hilbert's point was that if we came to possessing of such an effective procedure applicable to any first-order formula, then ignorance would be banished

⁶² Bei dem Problem der Allgemeingültigkeit handelt es sich um die folgende Frage: *Wie kann man beim einem beliebigen vorgelegten logischen Ausdruck, der keine individuellen Zeichen enthält, feststellen, ob der Ausdruck bei beliebigen Einsetzungen für die vorkommendem Variablen eine richtige Behauptung darstellt oder nicht?*

⁶³ Bei dem Problem der Erüllbarkeit handelt es sich um die Frage (Hilbert & Ackerman 1928, p. 73): *ob es überhaupt eine Einsetzung für die Variablen gibt, so daß durch den betreffenden Ausdruck eine richtige Behauptung dargestellt wird.*

⁶⁴ In the problem of validity there arises the following question: *How can man for any given logical expression, that do not contain individual symbols, establish if this expression for any substitution for occurring variables is a true sentence or not?*

⁶⁵ In the problem of satisfiability there appears the following question: *is there at all any substitution for the variables such that the corresponding sentence is true.* – Let us remark that the notions of validity and satisfiability are dual: a sentence is valid iff its negation is not satisfiable. The notion of validity is preferred in proof theory but the notion of satisfiability is preferred in model theory.

from mathematics forever. The view was shared by Bernays and Schönfin-
kel (1928):

Das zentrale Problem der mathematischen Logik, welches auch mit den Fragen
der Axiomatik im engsten Zusammenhang steht, ist das Entscheidungspro-
blem.

The central problem of mathematical logic, which is also most closely related
to the questions of axiomatics, is the Entscheidungsproblem.

Bearing in mind Leibniz's idea of *Ars Magna*, the Entscheidungsproblem
may be characterized as follows (Börger, Grädel & Gurevich 1997, p. 4):

In the framework of first-order logic, an *ars inveniendi* exists: the collection of
valid first-order formulae is recursively enumerable, hence there is an algorithm
that lists all valid formulae. The classical decision problem can be viewed as
the *ars iudicandi* problem in the first-order framework. It can be sharpened to
a yes/no question: Does there exist an algorithm that decides the validity of
any given first-order formula?

The important difference between the earlier stated problem of decida-
bility, e.g., by Löwenheim (1915) and Behmann (1922), and Hilbert's *Ent-
scheidungsproblem* consists in generality. Hilbert (1928, pp. 77–78) directly
pointed out the known particular solutions of the decidability. If Hilbert's
thesis was supposed to be solving *Entscheidungsproblem*, it would comprise
all the questions of decidability and

Auch Fragen der *Widerspruchsfreiheit* würden sich an Hand des Entsch-
eidungsverfahrens lösen lassen. (Hilbert & Ackerman 1928, p. 76)

By the decidability procedure the question of consistency could be solved, too.

Hilbert's belief that science is alive as long as there are unsolved pro-
blems seems to be in contradiction with the idea of decidability: if everything
can be solved by calculation, then in science the development is fictitious.
Already Leibniz remarked that:

It is unworthy of excellent men to lose hours like slaves in the labor of calcu-
lation which could safely be regulated to anyone else if machines were used.

To answer this question, we have to take into account the fact that besides
sentences there are notions. Hilbert (1992, p. 8) wrote:

... vielmehr zeigt sich, daß die Begriffsbildungen in der Mathematik beständig
durch Anschauung und Erfahrung geleitet werden, so daß im großen und
ganzen die Mathematik ein willkürfreies, geschlossenes Gebilde darstellt.

... moreover it turns out that in mathematics formation of concepts is directed by intuition and experience, hence in size and whole mathematics is a free of arbitrariness, closed composition.

The mathematical cognition is accomplished in the process of forming of concepts:

Es bilden also die verschiedenen vorliegenden mathematischen Disziplinen notwendige Glieder im Aufbau einer systematischen Gednakenentwicklung, welche von einfachen, naturgemäß sich bietenden Fragen anhebend, auf einem durch den Zwang innerer Gründe im wesentlichen vorgezeichneten Wege fortschreitet. Von Willkür ist hier keine Rede. Die Mathematik ist nicht wie ein Spiel, bei dem die Aufgaben durch willkürlich erdachte Regeln bestimmt werden, sondern ein begriffliches System von innerer Notwendigkeit, das nur so und nicht anders sein kann. (Hilbert 1992, p. 9)

4. Entscheidungsproblem

One can characterize the problem of decidability as follows: Given a collection of assumptions A stated in some logical system, and a statement s , is it possible to decide on the basis of some universal computation method (a decision method) whether s is or not inferrable from A ? If for a given system S such a universal computational method exists, then the system is decidable. For many logical systems, several such procedures existed before Church's and Turing's publications, but not for the system of the first-order logic.⁶⁶

The first-order logic is complete. If we suppose – what Hilbert did – that any human reasoning may be expressed in this logic, then the decidability of the first-order logic means that all human reasoning can be reduced to a calculation. For this reason the Entscheidungsproblem has been characterized by Hilbert as the fundamental problem of mathematical logic.

In order to solve the question of the Entscheidungsproblem, a concept of calculus or – in other words – an effective (a mechanical) procedure has to be defined. Solving the Entscheidungsproblem (and the 10th problem of Diophantine equations) required a precise formalization of what mechanical computation is. An attempt to define an intuitive notion rises the following questions:

⁶⁶ See (Kneale & Kneale 1962, pp. 724–737).

1. which already definite (in order to avoid *ignotum per ignotum*) notions should be used to be as close as possible to the intuition of the notion of effective procedure?
2. is there only one or more intuitively acceptable notions of effective procedure?

In the beginning, in order to answer these questions, let us consider a methodological problem of the procedure that shall be used. In formal sciences by definition one basically understands a nominal definition.⁶⁷ Nominal definitions tell us about the correct usage of names. There are two kinds of nominal definitions. The meaning of the defined term of a certain language \mathcal{L} is reported in this language. In this case the defined expression is definable in the language \mathcal{L} . In another case a nominal definition introduces to a language \mathcal{L} a new word to replace (usually as a short) an expression of \mathcal{L} . This consists in introducing a new symbol or notation by assigning a meaning to it. A real definition or conceptualization is a description of an object in such a way that it is a description of this and only this object. Neither the case of nominal definition nor the case of real definitions are taken into account here. The exclusion of real definition needs some explanation. The real definition is conceived in such a way that a real definition of an object presupposes the existence of the object. Perhaps an argumentation is possible that there is an object “effective procedure” that is independent of our cognition. Nevertheless, we have to take into account the fact that a real definition characterizes or not a defined object, *tertium non datur*. Such a definition is right or wrong. Thus, the correctness of a real definition does not depend in any way on our conventions or on our methods of cognition. It is not the case with the concept of effective procedure.

Besides nominal definition and conceptualization there is a third procedure that may be used to precise concepts, namely, explication. It seems that explication is the proper procedure to analyze calculus. Roughly speaking, an explication is a detailed analysis of a concept used to explicate means to unfold; to give a detailed explanation of; to develop the implications of; and to analyze logically.⁶⁸

Explication was described by Carnap (1950, p. 3) as follows:

By the procedure of explication we mean the transformation of an inexact, prescientific concept, the *explicandum*, into a new exact concept, the *explicatum*. Although the *explicandum* cannot be given in exact terms, it should be made as clear as possible by informal explanations and examples.

⁶⁷ For the first time rules of definition were formulated by Stanisław Leśniewski (1931).

⁶⁸ See e.g. (Trzęsicki 2000, pp. 366–369).

The task of explication consists in transforming a given more or less inexact concept into an exact one or, rather, in replacing the first by the second. We call the given concept (or the term used for it) the *explicandum*, and the exact concept proposed to take the place of the first (or the term proposed for it) the *explicatum*. The *explicandum* may belong to everyday language or to a previous stage in the development of scientific language. The *explicatum* must be given by explicit rules for its use, for example, by a definition which incorporates it into a well-constructed system of scientific either logico-mathematical or empirical concepts.

As an example of explication the definition of truth by Tarski⁶⁹ is pointed out (Carnap 1950, p. 5).

I am looking for an explication of the term ‘true’, not as used in phrases like ‘a true democracy’, ‘a true friend’, etc., but as used in everyday life, in legal proceedings, in logic, and in science, in about the sense of ‘correct’, ‘accurate’, ‘veridical’, ‘not false’, ‘neither error nor lie’, as applied to statements, assertions, reports, stories, etc. This explanation is not yet an explication; an explication may be given by a definition within the framework of semantical concepts, for example, by Tarski’s definition of ‘true’ [...].

Carnap (1950, p. 5) indicates the following requirements that a concept must fulfil to be an adequate explicatum for a given explicandum:

1. similarity to the explicandum,
2. exactness,
3. fruitfulness,
4. simplicity.

These conditions in more detailed form are as follows (Carnap 1950, pp. 6–7):

1. The explicatum is to be similar to the explicandum in such a way that, in most cases in which the explicandum has so far been used, the explicatum can be used; however, close similarity is not required, and considerable differences are permitted.
2. The characterization of the explicatum, that is, the rules of its use (for instance, in the form of a definition), is to be given in an exact form, so

⁶⁹ The idea that the notion of effective procedure is the same kind notion as truth, function, limit and so on is exploited by Mendelson in his argumentation for the Church-Turing thesis.

My viewpoint can be brought out clearly by arguing that CT is another in a long list of well-accepted mathematical and logical “theses” and that CT may be just as deserving of acceptance as those theses. Of course, these theses are not ordinarily called “theses”, and that is just my point. See (Mendelson 1990).

as to introduce the explicatum into a well-connected system of scientific concepts.

3. The explicatum is to be a fruitful concept, that is, useful for the formulation of many universal statements (empirical laws in the case of a non-logical concept, logical theorems in the case of a logical concept).
4. The explicatum should be as simple as possible; this means as simple as the more important requirements 1, 2 and 3.

Ch. Morris divided semiotics, a general theory of sign, into:

1. syntactics,
2. semantics

and

3. pragmatics.

Exact terms do not need explication. Since the explicated term is inexact, its explication cannot be evaluated as right or wrong. A proposed solution may be less or more satisfactory than another one.

Syntactics concerns relations between signs, semantics has as its subject relations between signs and reality. Pragmatics analyzes relations between sign and its user. The concept of (formal) proof is fundamental for syntactics. Semantics is based on the notion of truth. In classical semantics Tarski's definition of truth is assumed. The notion of effective procedure characterizes our ability of operating signs, thus it is basic for pragmatics.

What is expected in the case of the Entscheidungsproblem is the explication of the effective procedure. The truth table test is such a method for the propositional calculus. Different models of computation were introduced. 1936 saw an independent development of the three influential models of computation, aimed at doing the following: the lambda calculus, Turing machines and recursive functions. An explication of effective procedure was done by Church (1935, 1936*b*, 1936*a*) whose solution presupposed the definition of the intuitive notion of "mathematical function". Church's explication has been based on the notion of "lambda-definable-function" introduced by Church (1932, 1936*b*, 1936*a*) and Stephen Kleene (1935). A few months later, independently of Church, it was done by Turing (1936–37).⁷⁰ Another

⁷⁰ A fascinating biography of Turing by Andrew Hodges (1983) may be recommended. Turing, a co-founder of computer science, is shown as a man and scientist. Turing was much more than a mathematician, he was a specialist in electronics and signal processing even while the subjects were still unrecognised. There is a website maintained by Hodges: <http://www.turing.org.uk/> It is worth remarking that as Hilbert's problem gave stimulus on theoretical as codebreaking of Enigma on practical level of computer science. (Enigma – used by the Nazis an improved, military version of the commercial machine created by a German electrical engineer Scherbius in 1918.) A contribution of Polish mathematicians to solving the mystery of ENIGMA has to be pointed out here.

definition of the notion of “effective procedure” was done by Kleene and Post (1936) and by others later, e.g. by Markov (1955). The concepts of “recursive functions”, studied by Jacques Herbrand (1932) and Kurt Gödel (1934), gave rise to the idea of general recursiveness that was the base of the explication proposed by Herbrand, Gödel and Kleene. The definition starting from the notion of binormality was proposed by Post. Each proposal was well-defined and each seemed to correspond with what we would intuitively regard as an effective process (Copeland 2005, Deutsch 1985, Deutsch 1997, Kleene 1967, Marciszewski 2005). Unlike others, Turing’s concept was as much an instruction manual on how such a device might be built as it was a formalism for studying computation. That link between abstract computability and physical computability made the Turing machines quickly become the standard model of computation.⁷¹

Alan Turing was first introduced to Hilbert’s problem during the spring term of 1935 while attending a lecture on the foundations of mathematics given by Max Newman at Cambridge. Turing was extremely intrigued by the problem and immediately set out to prove that there was no algorithm that could satisfy the Entscheidungsproblem in an interesting and ingenious manner. Turing’s draft paper arrived on Max Newman’s desk at the same time as a copy of the *American Journal of Mathematics* containing “An Unsolvability Problem in Elementary Number Theory” by Alonzo Church. In his article “On Computable Numbers, with an Application to the Entscheidungsproblem” (1936–37) Turing explicitly referred to Gödel (1931), Church (1936*b*), and the 1931 edition of (Hilbert & Ackerman 1928). There was no rivalry between Turing and Church. Turing came to Princeton and some of his best work was accomplished there under Church’s direction.⁷²

Turing’s analysis of computation aimed at determining the mental processes fundamentally required by an individual performing a computation. Turing focused on the actual thought-process that took place when an individual attempted to develop an algorithm. He utilized an example of a simple mathematical calculation to create a case study for the actual thought processes taking place when a person performed a calculation. In *On Computable Numbers with an Application to the Entscheidungsproblem* Turing noted (1936–37, p. 117):

⁷¹ The Turing machine inspired the first real computer in 1943, called *Colossus*, and later the modern computer.

⁷² Turing arrived to Princeton for a two-year stay. Shortly before, in 1936, Alonzo Church founded *The Journal of Symbolic Logic*. Kurt Gödel, S. C. Kleene, and J. B. Rosser were all to be found in Princeton, New Jersey. The United States had become a world center for cutting-edge research in mathematical logic. See (Davis 1995).

We may compare a man in the process of computing a real number to a machine...

That was the point that Turing was to emphasize, in various forms, again and again. These words resonated with Wittgenstein's words (*italics in original*):

These (Turing's) machines are *humans* who calculate. (Wittgenstein 1980, 1096)

According to Shanker (1987), that remark summarized the key feature of Wittgenstein's reaction to Turing.⁷³ Later Wittgenstein wrote (1994, p. 234):

If calculating looks to us like the action of a machine, it is the human being doing the calculation that is the machine.

Turing (1950, pp. 454–455) asserted that the essential constituents of computational procedure performed by a person during a calculation could be replicated by a machine. He believed that the true nature of human mind is mechanical. According to him:

The 'skin of an onion' analogy is also helpful. In considering the functions of the mind or the brain we find certain operations which we can express in purely mechanical terms. This we say does not correspond to the real mind: it is a sort of skin which we must strip off if we are to find the real mind. But then in what remains, we find a further skin to be stripped off, and so on. Proceeding in this way, do we ever come to the 'real' mind, or do we eventually come to the skin which has nothing in it? In the latter case, the whole mind is mechanical.

According to him it was easy to imagine that a person performing a computation could be replaced by a machine. He noted (1950):

The idea behind digital computers may be explained by saying that these machines are intended to carry out any operations which could be done by a human computer.

An effective procedure can be performed by an idealized, infinitely patient mathematician working with an unlimited supply of paper, pencils and time

⁷³ Turing attended a few lectures of Wittgenstein's course on the foundations of mathematics. The status of a contradiction and the nature of a proof were the subject of disputes.

– but without insight. Turing’s paper *Computing Machinery and Intelligence* (1950) can be taken as saying that even a mathematician working with insight cannot exceed the power of the Turing machine. Turing experienced with construction of a computer.⁷⁴ He was much more than a mathematician, he was a specialist in electronics and signal processing even while the subjects were still unrecognized.

The Turing machine⁷⁵ was one of the simplest results. It was considered to be a formal counterpart to an effective procedure or an algorithm. Church (1937) highlighted that Turing:

proposes as a criterion that an infinite sequence of the digits 0 and 1 be ‘computable’ that it shall be possible to devise a computing machine, occupying a finite space and with working parts of finite size, which will write down the sequence to any desired number of terms if allowed to run for a sufficiently long time. As a matter of convenience, certain further restrictions are imposed on the character of the machine, but these are of such a nature as obviously to cause no loss of generality – in particular, a human calculator, provided with a pencil and paper and explicit instructions, can be regarded as a kind of Turing machine.

Hodges, the biographer of Turing, in the updating to (1983)⁷⁶ holds that Turing (AMT) “might well have been disappointed by the lack of interest in his work on the part of the mathematical world in general, but it is worth adding that Church was wholehearted in recommending and adopting AMT’s definition of computability. Given that AMT was a young unknown outsider crashing into Church’s field this was not something he could have taken for granted. As regards the tricky question of priority, Church wrote:”

In an appendix, the author [i.e. AMT] sketches a proof of the equivalence of ‘computability’ in his sense and ‘effective calculability; in the sense of the present author [i.e. Church’s definition using the lambda-calculus.] The author’s result concerning the existence of uncomputable sequences was also anticipated, in terms of effective calculability, in the cited paper [i.e. Church’s paper]. His work was, however, done independently...

⁷⁴ The first computer was named *Colossus* (1943). *Colossus* was a secret enterprize of the World War II located at Bletchley Park (‘Station X’). Max Newman and Bill Tutte were primarily responsible for its construction, while Turing was a consultant for programming. 11 *Colossus* machines were working on German codes. For more, refer to <http://www.codesandciphers.org.uk/lorenz/colossus.htm>

⁷⁵ By Turing the machine was named a logical computing machine, or “*a*-machine” (automatic machine). It has subsequently become known as the “Turing machine”. Turing used the term “machine”, while “Turing machine” was first used by Church (1937).

⁷⁶ See <http://www.turing.org.uk/book/update/part3.html>

In a letter (22 April 1937) to Church Bernays wrote:

He [Turing] seems to be very talented. His concept of computability is very suggestive and his proof of equivalence of this notion with your λ -definability gives a stronger conviction of the adequacy of these concepts for expressing the popular meaning of ‘effective calculability’.

In *Grundlagen der Mathematik* there are two references to Turing. In the first one we read (1970, p. 356):

Bei den Kriterien der Widerlegbarkeit, die wir aus dem Herbrandschen Satz entnommen haben, wurde der Allgemeinbegriff der berechenbaren Funktion benutzt.

For Gödel, who was unconvinced by Church’s paper, Turing’s proposal was very well-justified. He considered Turing’s work as a successful analysis of “mechanical procedure”. Gödel (1965, p. 2) stated unequivocally that Turing had analyzed “mechanical procedure” in a satisfactory way:

The most satisfactory way, in my opinion, is that of reducing the concept of finite procedure to that of a machine with a finite number of parts, as has been done by the British mathematician Turing.

For someone with as exacting standards of correctness as Gödel had, this was a high praise.

Gurevich (1995, p. 130) esteems Turing’s argumentation as very convincing. For him it is “a beautiful piece of speculative philosophy”. The idea of computability executed by the Turing machine is very fertile and has originated many ideas such as von Neumann’s classical computer. In his paper there appeared very fruitful notions of “input-output”, “memory”, “compiler/interpreter”, “finite-state machine”, “coded program”, and “algorithm”. Turing’s definition of computability remains a classic paper in the elucidation of an abstract concept into a new paradigm. It means that the question 1 posed on page 101 is answered.

Alonzo Church’s lambda calculus and Steven Kleene’s recursive functions were arguably more elegant, but it was the mechanical action of Turing’s machines that most agreed intuitively about how people calculate. It also makes the Turing machines a natural object for studying even more powerful models of computation. Church (1937, pp. 42–43) reviewed Turing’s paper comparing the Turing machine to other concepts:⁷⁷

⁷⁷ Church omitted Post’s concept of binormality. For Post (1965, pp. 408, 419) himself

As a matter of fact, there is involved here the equivalence of three different notions: computability by a Turing machine, general recursiveness in the sense of Herbrand-Gödel-Kleene, and the λ -definability in the sense of Kleene and the present reviewer. Of these, the first has the advantage of making the identification with effectiveness in the ordinary (not explicitly defined) sense evident immediately – i.e., without the necessity of proving preliminary theorems. The second and third have the advantage of suitability for embodiment in a system of symbolic logic.

For Post (1944, p. 462), the new notion was worth studying:

But apart from the question of importance, these formalisms bring to mathematics a new and precise mathematical concept, that of the general recursive function of Herbrand-Gödel-Kleene, or its proved equivalents in the developments of Church and Turing. It is the purpose of this lecture to demonstrate by example that this concept admits of development into a mathematical theory much as the group concept has been developed into a theory of groups.

It quickly turned out that not only the proposals of Turing and Church but other formalisms for describing effective computability (register machines, Emil Post's systems, combinatory definability, Markov algorithms, formal grammars, μ -recursive functions) were functionally equivalent to each other. They differed only in how functions were to be computed. It was proved that all different models of computation defined the same class of computable functions. It means that the question 2 on page 101 is answered, too. Church and Turing have proved – roughly speaking – that there are such formulas of the first-order that are not recursive (Church) or computable by the Turing machine (Turing). Roughly speaking, it has been proved that for the predicate calculus there is no procedure as the truth table test for the propositional calculus. Neither of these theorems alone answer Hilbert's Entscheidungsproblem.

Let us recall here that the Entscheidungsproblem is the problem of an effective mechanical procedure that applies to a formula of the first-order language after a finite number of steps gives YES or NO as the answer to the question: is the given formula valid? To solve this problem, two facts have to be established:

it was only a working hypotheses:

[...] for full generality a complete analysis would have to be made of all the possible ways in which the human mind could set up finite processes for generating sequences. [...] we have to do with a certain activity of the human mind as situated in the universe. As activity, this logico-mathematical process has certain temporal properties; as situated in the universe it has certain spatial properties.

1. a precise formal notion(s) of effective procedure,
2. a thesis that links the notion(s) with the Entscheidungsproblem.

The notion of computability by the Turing machine fulfills all the conditions of correct explication (page 102). Thus it is necessary to discuss the fact 2. It should be stated that the notion of computability by the Turing machine (or its equivalent) is the notion of the effective procedure of the Entscheidungsproblem. In other words, the adequacy of computability by the Turing machine with the intuitive notion of effective method in Hilbert's decidability problem it should be stated. The fact is out of reach for any formal proof. It only may and should be justified.⁷⁸

The Church-Turing thesis⁷⁹ in its most common form states that every effective computation can be carried out by the Turing machine or – in other words – any computation solvable by a precisely stated set of instruction (i.e., an algorithm) can be run on the Turing machine (or a digital process computer). The thesis states that the Turing computability precisely captures the intuitive notion of computability. It may be paraphrased as saying that the notion of effective or mechanical method is captured by the Turing machine.⁸⁰ In *On Computable Numbers...* a different formulation of the thesis may be found, e.g.:

[T]he “computable numbers” include all numbers which would naturally be regarded as computable. (1936–37, p. 249)

It is my contention that these operations [the primitive operations of a Turing machine] include all those which are used in the computation of a number. (1936–37, p. 232)

⁷⁸ For a survey of the discussion and solutions of the question see (Murawski 2005).

⁷⁹ For the first time the question of adequacy, which is the subject of the thesis emerged in a talk between Church and Gödel in Princeton in 1934. It was formulated for the first time on a session of the American Mathematical Society by Church on 19 April, 1935 (Church 1936*b*, §7). Turing's mathematically equivalent formulation was published, together with the notion of the Turing machine, in (Turing, 1936–37). It was dubbed “thesis” by Kleene (1936, p. 232):

So Turing's and Church's theses are equivalent. We shall usually refer to them both as *Church's thesis*, or in connection with that one of its ... versions which deals with “Turing machines” as *the Church-Turing thesis*.

This terminology is used also in (Kleene, 1943) and (Kleene 1952, p. 317). Cf. (Murawski 2005).

⁸⁰ Strictly speaking, it is Turing's thesis. Church's thesis states the same about Church's concept of effective method. For more about the Church-Turing thesis and its evolution from its modest origins to its current elevated status see (Copeland 2005).

For J. Martin (2003, p. 353):

To say that the Turing machine is a general model of computation is simply to say that any algorithmic procedure that can be carried out at all (by a human, a team of humans, or a computer) can be carried out by a TM [Turing machine].

Turing proved that there were functions that could not be computed by the machine invented by him. It is clear when we consider that the set of functions from \mathbb{N} to \mathbb{N} is uncountable while the set of Turing machines is countable. Both Turing and Church showed the examples of functions which could not be computed. The most known and discussed of these is Turing's halting function which takes a pair: a natural number representing a Turing machine and its input, returning 1 if the machine halts on its input and 0 if it does not. From this, we also get the existence of a specific uncomputable set (the halting set): $\{n \mid n \text{ represents a Turing machine/input pair that halts}\}$.⁸¹

There is a strong connection between Gödel's and Turing's results. Gödel was looking for the type of model to represent formal systems. In 1936 he praised the Turing machine as allowing a "precise and unquestionably adequate definition of the general concept of [a] formal system" (1965). A formal system can be specified as Turing machine that semi-computes a set of formulas provable in this system. A formal system may be conceived as a recursively enumerable set of axioms with recursively enumerable rules of inference. Gödel's Incompleteness Theorem can therefore be completely specified, stating that no consistent formal system of this type can prove all truths of arithmetic, or, that the set of true formulas of arithmetic is not recursively enumerable. Gödel, in his Incompleteness Theorem, proved that each consistent formal system had its own unprovable statement. It was not excluded that the statement could be proved in another system. Turing (and Church) pointed out the 'absolutely' undecidable function whose values could be proven by no consistent formal system. Thus Turing's proof of the uncomputability of the halting function also extended Gödel's Incompleteness Theorem.

⁸¹ The halting function is partially computable. In general, a function, $f: X \rightarrow \{0, 1\}$ is semi-computable (recursively enumerable) by the Turing machines if and only if there is a Turing machine which transforms input, x , to $f(x)$ whenever $f(x) = 1$ and either returns 0 or diverges when $f(x) = 0$. A function, $f: X \rightarrow \{0, 1\}$ is co-semi-computable (co-recursively enumerable) by Turing machines if and only if there is a Turing machine which transforms input, x , to $f(x)$ whenever $f(x) = 0$ and either returns 1 or diverges when $f(x) = 1$.

The thesis and the theorems by Church and Turing should not be confused, though. Instead they should be taken together for they are relevant to the Entscheidungsproblem. The Church-Turing thesis is essential to prove that certain mathematical functions/sets/reals are uncomputable.

Let us remark that the Church-Turing thesis has been a subject for many speculations and in many cases it is misunderstood. Jack Copeland (1997) refers to many papers and books where the Church-Turing thesis is misstated.

The Church-Turing thesis tells about finitistic formalism, thus it does not pertain to formalism in general. The infinitistic methods are beyond the thesis. Hilbert aimed to restrict the methods of proof theory to finitistic ones. He allowed non-finitistic methods in mathematics in general. For example, by using transfinite induction Gentzen has proved the consistency of arithmetic, cf. page 92.

Because all the formal definitions of effective computability have been shown to describe essentially the same set of functions, it is now generally assumed that the Turing-Church thesis is correct.⁸²

Both the theorem, claiming that there are problems uncomputable by the Turing machine, and the thesis, stating that the notion of computability by the Turing machine is the correct accurate rendering of the effective procedure,⁸³ solve the Entscheidungsproblem in negative. Though Turing and Church discovered the result independently of one another, Turing's solution is more satisfying than that of Church's. It was acknowledged by Church himself.

The theorem of Turing (or Church) does not say that there are any unsolvable questions. The theorem only says that there is not an effective procedure (if we suppose that the notion of the effective procedure is exhausted by the concept of the Turing machine) that makes possible to solve any question. Thus we may hope that any given question may be solved (earlier or later). But we do not suppose that one method will be found that will solve any question. Gödel, who proved the incompleteness theorem, believed that in case of any formula we would be able to prove or disprove it. There is no effective procedure that works for any formula of the first-order calculus but it is not excluded that there are decidable classes of formulas,

⁸² Any formal description of the effective method is accompanied by a proper formulation of the thesis. See (Kleene 1943, Markov 1955, Boolos & Jeffrey 1980, Kalmár 1959, Kreisel 1970, Hofstadter 1979, Lewis & Papadimitriou 1981, Savage 1998, Wood 1987). About the arguments that are used to support the thesis and against it, see (Murawski 2005, 2004a, Cooper 2005).

⁸³ Cf. (Turing 1969, p. 7).

e.g. the one-place predicate calculus is decidable. It was also proved that there exist many classes of problems, few of which are very easily defined, which do not take an effective procedure to solve them.⁸⁴

The negative solution of the Entscheidungsproblem in conjunction with Gödel's theorems was seen as a sufficient reason for the abolishment of Hilbert's program in mathematics. Hilbert's program towards a decision procedure for all fields of mathematics was proved to be impossible. The Hilbertian dream remains in total ruins. The original hopes for Hilbert's programme lie in tatters. However, even if it is true, a vast amount was learned about the fundamental nature of computation. One of the greatest accomplishments of Turing's work is that he has answered Hilbert's question developing innovative ideas related to the development of a universal computing machine. Alan Turing's creation of a theoretical computing machine served as a theoretical framework for the modern computer. Before Turing three components – machine, program and data – were distinguished.

Turing's universal machine showed that the distinctness of these three categories is an illusion. (Davis 2000, p. 165)

Thus, it is the development of a system that allows of the integration and “fluidity” among these three components. It has become fundamental to the contemporary computer science, cf. (Davis 2000, p. 165).

Gödel, Church and Turing set limits to Hilbert's program, but there remains much of value in continuing Hilbert-like programs (e.g. nonstandard analysis). The development of computer science has led to a rebirth of Hilbert's proof theory, where its methods play a significant role. A modified Hilbert's program is a base for the development of proof theory, metamathematics, and decision theory (computability theory). The theory of algorithms (recursive functions) enables us to give the exact definition of the “formal system”: a system is formal if and only if there is an algorithm for checking correctness of inferences in this system. In formal systems the standards of inferences must be described precisely enough to enable checking of proofs by a computer.

The consequences of Hilbert's program, unexpected by himself, are summarized by Chaitin (2004):

As I said, formal systems did not succeed for reasoning, but they succeeded wonderfully for computation. So Hilbert is the most incredible success in the world, but as technology, not as epistemology.

⁸⁴ For the most comprehensive treatment available in a book form of the classical decision problem of mathematical logic and of the role of the classical decision problem in modern computer science see (Börger et. al. 1997).

[...]

To end, let me quote from a posthumous collection of essays by Isaiah Berlin, *The Power of Ideas*, that was just published: “Over a hundred years ago, the German poet Heine warned the French not to underestimate the power of ideas: philosophical concepts nurtured in the stillness of a professor’s study could destroy a civilization.” So beware of ideas, I think it’s really true. Hilbert’s idea of going to the limit, of complete formalization, which was for epistemological reasons, this was a philosophical controversy about the foundations of mathematics – are there foundations? And in a way this project failed, as I’ve explained, because of the work of Gödel and Turing. But here we are with these complete formalizations which are computer programming languages, they’re everywhere!

The period between the two World Wars was a remarkable time in philosophy and particularly in logic and the foundations of mathematics. Hilbert was the one to raise the most important and fruitful problems for the future of mathematics. Let us refer to the beginning of the text where Hilbert was called a man of problems (Hilbert 1932–1935, vol. 3, p. 405). Though simple, Hilbert’s starting points were always important. David Hilbert died in 1943. He lived to see the end of the great mathematical dynasty at the Georg-August University of Göttingen. His funeral was attended by fewer than a dozen people, only two of whom were his fellow academics. He had never seen a computer. What is more, he never imagined how big the consequences of his program of finitistic foundation of mathematics were to be.

5. Beyond the Church-Turing Thesis

Church and Turing have formally defined a boundary to what is possible to calculate or compute algorithmically, but they did not necessarily define an ultimate boundary. Turing himself was the first person who had recognized the limitations of the Turing machine (Copeland & Proudfoot 1999). His choice machine (*c*-machine) (Turing 1936–37) and unorganized machine (*u*-machine) (Turing 1969) did not survive later scrutiny. In Turing’s Ph.D. thesis (1939) the *a*-machine (the Turing machine) is augmented with an oracle, a function which determines the value of a function that cannot be computed by the Turing machine (*o*-machine).⁸⁵

⁸⁵ For Davis (2004):

It is perfectly plain in the context of Turing’s dissertation, that *O*-machines were introduced simply to solve a specific technical problem about definability of sets of natural numbers. There is not the faintest hint that Turing was making a proposal about a machine to be built. ... It makes no sense to imagine that he was thinking about actual machines to compute the uncomputable. Cf. (Cooper 2005).

The Turing machine still leaves room for the speculation about a possibility of a quantum-mechanical, non-mechanical, or super-mechanical computation. Everybody who tries to cross the boundary set by the Church-Turing thesis may feel encouraged by Hilbert's words:

It is by the solution of problems that the investigator tests the temper of his steel; he finds new methods and new outlooks, and gains a wider and freer horizon. (Hilbert 1900)

For many years, attempts to find more powerful models than the Turing machines gave no positive results. The first efforts to weaken the Church-Turing thesis were unsuccessful. The discussion was focused on logical and mathematical problems. Laszlo Kalmár's argument is an example of one of the most known and widely discussed such attempts, cf. (Kalmár 1959, pp. 72, 79). The thesis was also questioned by Peter Rózsa (1959), Jean Porte (1960), G. Lee Bowie (1973), Elliot Mendelson (1963) and others. Several models were proposed to replace the Turing Machine as a universal model of computation. Under discussion there remain models that communicate with the outside world during their computations, models that appeal to the laws of physics, models that can manipulate real numbers, and parallel computers. Artificial neural networks, cellular automata, evolutionary computing, L-systems (multicellular organisms), swarm computing, molecular computing, and many others are still under considerations.

If we were to find out that the Church-Turing thesis was true (or false), it would not say anything about what is computable by mathematicians through insight or the types of computation that are achievable through natural processes.

Turing aimed to describe the process of calculation by an idealized mathematician. Here the "idealized" means that the mathematician does not take into account the meanings of things that are the subject of simple operations and that his action does not depend on his will, desire and emotions. It is supposed that any operation needs the same amount of time and that these operations form a finite sequence. In other words, the Turing machine is equivalent to the finitistic formalization in the Hilbertian sense. The processes of calculation that are performable by contemporary digital computers are of that kind.⁸⁶ The Church-Turing thesis does not say

⁸⁶ We do not take into account the fact that there is a difference between what is theoretical and that what is physical (practical). Even if the nature "calculates" mathematics as it is performed by the Turing machine, there could be a difference when a chance of some faults is not excluded. The Turing machine is a theoretical device, our computers are not, they are physical devices. Though operating as the Turing machine, a compu-

anything about how a human being thinks, though Turing maintained that his machine is as powerful as the human mind. Moreover, it says nothing about the formalization in general. Thus, the question whether Leibniz's and others' dream of the substitution of human thinking by a calculation is performable by the Turing machine is open as long as it is not proved (logically, psychologically, empirically?) that all (logically) sound reasonings are simulable by the Turing machine. It is a question of creation of artificial intelligence. There are two types of approaches to the task (one does not exclude another):

1. the development of software that would be able to perform those activities that are normally thought to require intelligence,
2. the construction of a machine having this ability.

Any of the approaches depends on the Turing-Church thesis. It is not excluded that to achieve our goal either of the ways the thesis has to be rejected.

Turing believed in the possibility of the construction of a true thinking machine. He generally shied away from metaphysical questions but not in case of that issue. Arguing that the creation of such a machine would be similar to the creation of a soul, he wrote (1950):

In attempting to construct such machines we should not be irreverently usurping his [God] power of creating souls, any more than we are in the procreation of children, [...] Rather we are, in either case, instruments of his will providing mansions for the souls that he creates.

The Church-Turing thesis is strongly influenced by the philosophical and scientific environment of the time it was formulated in.

The contemporary electronic computer is a great achievement of engineers. There are good reasons to believe that the progress in the construction of computers will not be restricted to the electronic scheme. Calculation that is performable by the contemporary computers harnesses the phenomenon of electricity. Are there any reasons to reject to harness other natural phenomena to perform calculation? The other natural processes should not be excluded as phenomena that are able to calculate. Nevertheless, one may repeat after Davis, see (Cooper 2005):

Of course, even assuming that all this really does correspond to the actual universe in which we live, there is still the question of whether an actual device to take advantage of this phenomenon is possible.

ter as the output may have data that differs from what is calculated by a (theoretical) Turing machine. Some natural processes could be such that the cause-result relation is non-recursively calculated. Such a process is not simulable by the Turing machine. Such a "computer" is able to do more than the Turing machine can.

Till now the problem of an interface⁸⁷ is not solved but it does not mean that the problem will not be solved in the future. Of course, a lot of ingenuity needs to be put into making natural phenomena compute more than now. We may as well agree with Davis: “But the theoretical question is certainly of interest.” To support this idea, Cooper (2005) gives an analogy:

The domestication of horses around five or six thousand years ago brought a revolution in transportation, only achieved through a creative interaction between humans and the natural world. At that time, trying to understand the principles underlying the equine organism in order to synthesise an artificial horse was unthinkable. But a few thousand years later there was enough understanding of scientific basics to underpin the invention of the “iron horse”, leading, amongst other things, to the opening up of many previously isolated parts of the world to people with no riding skills whatsoever.

Even if it is true that Turing machine does what an idealized mathematician does, it does not mean that the Turing machine calculates everything what is calculable. There are several examples of computations that cannot be performed by the Turing machine (Copeland 1998, Siegelmann 1995, *Hypercomputation* 2006, Burgin 1999, 2004*b*, Copeland 2002*b*, Copeland & Proudfoot 2005).

However, our way of thinking may be changed. We may say that to calculate means the same as to be (deterministically) processed by the Nature. Generally speaking, this kind of argumentation is based on the scheme: natural processes are effective; the notion of effectiveness of the Church-Turing thesis is not adequate, since the Turing machine is not able to simulate all of them. It is argued that there are effective processes of the physical world that are not restricted to those that meet a notion of “effectiveness” as being embodied in Turing-computability, these processes are not fully describable by algorithms that fulfil the Church-Turing thesis. It is unsound to argue the other way round: that one first formulates some definition of effectiveness and then argues backwards that the physical world ought to conform to it, e.g., Rosen (1999, p. 160)⁸⁸ maintains that:

⁸⁷ David Pearson (1996) comments on the manner in which we are accustomed to electronic computers:

[...] if we had a crystal-lattice computer today, we simply wouldn't know how to use it. Programming is so well-developed under the von Neumann model that is virtually impossible to remove it from our thinking. Obviously we can simply program a CA [cellular automaton] to emulate a standard von Neumann machine [...] that will certainly be done [...]. But that does not take advantage of the added power the CA has. To use it efficiently, we must be prepared to change our programming model.

⁸⁸ Robert Rosen was Professor Emeritus of Biophysics at Dalhousie University and the author of books including *Life Itself* (compiling twenty articles on the nature of life and on

Indeed, we shall take the view that material object-systems, as things in the external world, and whose behaviors are governed by webs of causal entailments, constitute the true province of effective processes. That is, the notion of effectiveness has to get imported into language via modeling relations arising in material nature, through encodings/decodings from these. Accordingly, any attempt to characterize effectiveness independent of these, from within language (e.g., mathematics) alone, and most particularly, in terms of syntax alone, is perilous. But that is exactly the substance of propositions such as Church's Thesis – namely, that causal entailments in the external world must conform to some notion of effectiveness drawn from inside the language itself. However, it is the other way around.

Are there any (deterministic) processes of the universe that are more powerful than the Turing machine? The answer to the question is not the subject of explication, it is rather a subject of empirical exploration of processes that are performable by the nature. Models of computation that compute more than the Turing machine are assertions about the nature of physics and their truth or falsity rests in the underlying structure of our universe rather than being claims in philosophy or mathematics.

The Church-Turing Thesis tells about the procedure of calculation carried out by a human being. It does not say anything about the “calculation” realized in the nature by physical or biological processes. The idea that the universe is a big Turing machine is interesting for many reasons. But the idea that the universe is more powerful than the Turing machine is even more interesting. There are some good reasons to believe in the nature hypercomputes, i.e. that nature computer exceeds the Turing machine not only in size.

The idea of supercalculation or supermachine is based on questioning some of the attributes of the Turing machine.⁸⁹ Different concepts of supercomputation are a result of combination of omitting various attributes of the Turing machine. Two types of supercomputation may be distinguished:

1. based on the difference between real or ideal human thinking and the Turing machine,
2. based on the difference between real computer and Turing machine or the idea that some (quantum- or biological)-processes “calculate” not only more efficiently but also differently than the Turing machine.

the objective of the natural sciences, this remarkable book complements Robert Rosen's groundbreaking *Life Itself* – a work that influenced a wide range of philosophers, biologists, linguists, and social scientists.), *Principles of Mathematical Biology, and Principles of Measurement*.

⁸⁹ For more on super- and hypercomputation see (Davis 2004, Copeland 2002a, Copeland 2002b, *Hypercomputation* 2006).

Under discussion there are machines with various expanded abilities, possibly with the ability to compute directly on real numbers, the ability to carry out uncountably many computations simultaneously, or the ability to carry out computations with exponentially higher complexity. Theoretical models such as probabilistic, oracle, and quantum computers are being studied.

The Turing machine performs one deterministic step at a time. The steps form a sequence and the time needed for execution is a linear function of their number. This model resembles, on a basic level, the von Neumann computers in use today. John von Neumann is one of the first scientists that considered cellular automata, too. Cellular automata are abstract mathematical models for computation that, unlike the Turing machines, operate in full parallelism. Neither the sequentiality of steps nor the constant time of operations seems to be an indispensable attribute of calculus. The calculation may be performed in parallelism and/or the time needed for a step may (exponentially) decrease with the number of steps. If each step is performed in half the time taken by the previous step, the computation will last twice as long as the first step (Copeland 1998, Copeland 2002a). A cellular automaton might operate by manufacturing a clone of itself and similarly it could increase its speed exponentially. Though parallel computers have been promised for many years, there are significant problems to overcome in their development. “Conventional machines have always overtaken the parallel computers in speed before these problems could be overcome.”⁹⁰ Is a cellular automaton only a theoretical device or – as it is the case of the Turing machine – there are possible empirical devices (constructed or natural) that simulable it?⁹¹

At this stage, none of such devices seem physically plausible, and so hypercomputers are likely to remain a mathematical fiction. The question whether the hypercomputation is a proleptic computer science, or it is of mainly philosophical interest still seems to be open. In the next section we will speculate on the idea of hypercomputation based on quantum physics and molecular biology.⁹² Maybe a quantum- or DNA-hypercomputer

⁹⁰ See (Pearson 1996), cf. (Petrov 2005).

⁹¹ To be precise, let us point out that the Turing machine is also a cellular automaton; depending on the structure, a cellular automaton is n -dimensional. According to the Kuratowski (1930) theorem for planar graphs, the minimum number of spatial dimensions that is sufficient to represent any possible graph is equal to 3) it follows that any n -dimensional cellular automaton may be equivalently reduced to at most 3-dimensional one (the universe is a 3-dimensional cellular automaton). The Turing machine is an 1-dimensional cellular automaton. See (Petrov 2005).

⁹² Under discussion there are also solutions based on optical realms (Li, Pan & Zheng 1998) and chemical reactions (Siehs & Mayer 1999).

would be able not only to compute the Entscheidungsproblem.⁹³ The Church-Turing thesis is questioned from the angle of quantum mechanics and biology.

In 1982 Richard Feynman (1982)⁹⁴ remarked that the computer technology was unable to simulate quantum systems effectively. He showed that simulation of quantum mechanical system on the Turing machine caused exponential slowdown of operations.

In 1985 David Deutsch (1985) deliberated over a theoretical model of computer based on quantum mechanics. He suggested that such a computer would be able to calculate problems that were not calculable by a traditional computer. That idea found a larger interest in 1994 when Peter Shor (1994)⁹⁵ discovered a new quantum algorithm of factorization of big numbers.⁹⁶ The initial promises in the nineties towards quantum computation are now far from their realization. Since the time a little progress has been made.

At present the quantum computing is still the subject of interest of many scientists and institutes. Paul Davies maintains that:⁹⁷

The nineteenth century was known as the machine age, the twentieth century will go down in history as the information age. I believe the twenty-first century will be the quantum age.

Let us only note that almost all important papers in the field of quantum information and computation can be found at the on-line archive arxiv.org under the Quantum Physics (quant-ph) subject group. Among many institutes which encourage the growth and development of the emerging field of quantum information science there are:

⁹³ The fact that for the Turing machine the Entscheidungsproblem is solved in negative does not contradict to the possibility that for any formula of the first-order predicate logic there is a procedure of deciding whether this formula is valid (satisfiable) or not.

⁹⁴ A physicist, Nobel Prize laureate, one of the founders of modern quantum mechanics.

⁹⁵ Peter Shor, a research and computer scientist at AT&T's Bell Laboratories in New Jersey.

⁹⁶ In a quantum computer, the fundamental unit of information, a quantum bit or qubit, is not binary but rather more quaternary in nature. A qubit can exist not only in a state corresponding to the logical state 0 or 1 as in a classical bit, but also in states corresponding to a blend or simultaneously as both 0 and 1, with a numerical coefficient representing the probability for each state. Shor's algorithm harnesses the power of quantum superposition to rapidly factor very large numbers of ~ 10200 digits and greater in a matter of seconds.

⁹⁷ See foreword to (Milburn 1997).

1. Institute for Quantum Information, Caltech;⁹⁸
<http://www.iqi.caltech.edu/>
2. Institute for Quantum Computing, University of Waterloo;
<http://www.iqc.ca/>
3. Quanta, MIT;
<http://www.media.mit.edu/quanta/>
4. Centre for Quantum Computation, UK;
<http://www.qubit.org/>
5. Group of Applied Physics, Geneva;
<http://www.gap-optique.unige.ch/>

The site of Institute for Quantum Information *Caltech* present the following:

Quantum information science (QIS) is a new field of science and technology which draws upon the disciplines of physical science, mathematics, computer science, and engineering. Its aim is to understand how fundamental physical laws can be harnessed to dramatically improve the acquisition, transmission, and processing of information.

The inspiration for QIS is the discovery that quantum mechanics can be exploited to perform important and otherwise intractable information-processing tasks. Quantum effects have already been used to create fundamentally unbreakable cryptographic codes, to teleport the full quantum state of a photon, and to compute certain functions in fewer steps than any classical computer can.

Even aside from its technological implications, QIS is an intellectually stimulating basic research field. Fundamental questions such as “What is the computational power of Nature?”, “Can measurement be reversed?” and “How much information can we learn?” continue to drive the field and inspire new research directions. We expect that QIS will have an extensive impact on how science is taught at the college and secondary level. We also expect that QIS paradigms will enable quantum physics to be understood better by a broad segment of the lay public. See: <http://www.iqi.caltech.edu/qis.html>

The idea of a biological computer was already considered by Turing. The advances both in molecular biology and in information science have stimulated the merging of two great discoveries of the 20th Century. In 1958 at the University of Illinois the Biological Computer Laboratory (BCL)

⁹⁸ This institute is sponsored by the National Science Foundation (NSF) in order to “recognizes the importance of QIS [Quantum Information Science] to the scientific, economic, and technological strength of our nation.”

was founded by Heinz von Foerster.⁹⁹ “BCL left behind a rich legacy. In its day, it was one of the few education institutions teaching cybernetics. Between 1958 and 1975, operating under 25 grants, the laboratory produced 256 articles and books, 14 master’s theses, and 28 doctoral dissertations. The topics covered epistemology, logic, neurophysiology, theory of computation, electronic music, and automated instruction. Most likely, the first parallel computers were built and exhibited there.” (See <http://www.ece.uiuc.edu/pubs/centhist/six/bcl.htm>)

In 1994 Adelman (1994)¹⁰⁰ suggested that DNA (deoxyribonucleic acids) could be used to solve mathematical problems. According to him the Hamiltonian path problem (the traveling salesman problem) may be encoded in DNA sequences. Each city was encoded as its own DNA sequence. The DNA sequences were set to replicate and create trillions of new sequences based on the initial input sequences in a matter of seconds (called DNA hybridization).

There are many advantages of DNA computers. Input, output and “software” are all composed of DNA, the material of genes, while DNA-manipulating enzymes are used as “hardware”. They have the potential to take computing to new levels, picking up where Moore’s Law leaves off. DNA computers perform calculations parallel to other calculations. Hence they are able to solve complex problems in hours, whereas electrical computers might take hundreds of years to complete them. More than 10 trillion DNA molecules can fit into an area no larger than 1 cubic centimeter. With this, a DNA computer could hold 10 terabytes of data and perform 10 trillion calculations at a time. DNA computer is incredible energy-efficient resource – the DNA sequences are created by simply “just add water” to initiate the “computation” – and cheap – as long as there are cellular organisms, there will be a supply of DNA, biochips can be made cleanly. Ordinary computers need absolutely correct information, a biological computer will come to the correct answer based on partial information, by filling in the gaps itself. Biological computers may “think for itself” because neurons are able to form their own connections from one to another (silicon computers only make the connections they are told to by the programmer).

⁹⁹ Born 1911 in Vienna. From 1962 to 1975 he was Professor of Biophysics and 1958–75 director of the Biological Computer Laboratory. Together with Warren McCulloch, Norbert Wiener, John von Neumann, and others, Heinz von Foerster was the architect of cybernetics.

¹⁰⁰ Leonard Adelman is a mathematician and computer scientist, biologist and information scientist. He was one of the inventors of the RSA (Rivest, Shamir, Adelman) public-key encryption system.

The original version of the biomolecular computer capable of performing simple mathematical calculations, was introduced by Shapiro and colleagues (Weizmann Institute of Science, Israel: <http://www.weizmann.ac.il/>) in 2001. An improved system, which uses its input DNA molecule as its sole source of energy, was reported in 2003 and was listed as the smallest biological computing device. In Shapiro's opinion:

It is clear that the road to realizing our vision is a long one; it may take decades before such a system operating inside the human body becomes reality. Nevertheless, only two years ago we predicted that it would take another 10 years to reach the point we have reached today.¹⁰¹

In the foreseeable future DNA computers will not be replacing the common old PC. Studying DNA computers may also lead us to a better understanding of a more complex computer – the human brain. Teuscher (2004) writes about its knowledge:

People have started thinking about the possibility that simulating the mind in silicon might be impossible – or at least impossible using today's methods. Should we first forget about computers and look closer at the gray stuff in the brain, since the actual knowledge of the brain is severely fragmented and many details are still not at all understood?

Despite these new ideas of computers it is generally believed today that the halting function is uncomputable on any model of computation. Next section will be devoted to a short presentation of an argument based on Chaitin's concept of randomness and the number Ω . In his opinion (Chaitin 2004):

... not only Hilbert was wrong, as Gödel and Turing showed. ... I want to summarize all of this. With Gödel it looks surprising that you have incompleteness, that no finite set of axioms can contain all of mathematical truth. With Turing incompleteness seems much more natural. But with my approach, when you look at program size, I would say that it looks inevitable. Wherever you turn, you smash up against a stone wall and incompleteness hits you in the face!

¹⁰¹ Further information and photos can be obtained online at: www.weizmann.ac.il/udi-PressRoom

6. The number Ω

The question concerning the notion of effectiveness and boundaries of effective calculability has to be formulated as follows: are there any non-calculable facts, even if the calculability is conceived as large as possible in any reasonable sense? The answer to this question is given by Chaitin.¹⁰²

In order to choose among possible theories, which have the same predictions and the data available cannot distinguish between them, Ockham's razor is a very useful tool. Ockham's razor is the principle proposed by William of Ockham in the fourteenth century:

Pluralitas non est ponenda sine neccesitate

which translates as

entities should not be multiplied unnecessarily.

Ockham's razor directs us to study the simplest of the theories in depth. The principle prefers the less complex and more universal theory. If we have some competing theories that essentially explain one and the same phenomenon, we must choose the simplest and the most universal theory possible.

Our intuitions concerning the complexity of a number (a sequence of 0 and 1) are well explained by the algorithmic information theory.¹⁰³ The complexity of a number is associated with the size of the smallest program that generates it. The algorithmic information content (algorithmic entropy) of a number n , $H(n)$ is the size of the shortest program which produces the number n .¹⁰⁴

A theory can be conceived as a program that has a data describing experiment as an input and the results of the experiment are seen as the output. In 1964 Solomonoff proposed to take the length of the shortest of such programs as the measure of the complexity of a scientific theory. The task of a theory is the compression of data. For Chaitin, this theory is

¹⁰² Gregory J. Chaitin is an Argentine-American mathematician and computer scientist. Beginning in the late 1960s, Chaitin made important contributions to algorithmic information theory. His work paralleled the earlier work of Kolmogorov in many respects. For more about Chaitin see his home page: <http://www.cs.auckland.ac.nz/CDMTCS/chaitin/>

¹⁰³ This theory was founded by Andrei Kolmogorov (1965) and Gregory Chaitin (1966, 1987, 1997).

¹⁰⁴ To remember a number of e.g., a telephone, we try to find the simplest way of doing it. Such a practice is satirized by Jaroslav Hasek: On his attempt to remember a secret number of a locomotive, the brave soldier Švejk put efforts to elaborate a highly complicated rule how to do it rather than try to remember it itself.

better because it is more compressed. Any information may be presented as a number and the number is a sequence of 0 and 1. Sequences of 0 and 1 can be generated by a program. The complexity of a sequence of 0 and 1 may be measured by the length of the shortest of such programs. A random number is such a number for which the length of the shortest of the programs that it generates is almost the same as the number, i.e. the length of its minimal program approaches the length of the number itself. If information (a number) is not random and we have to send it to someone using as few bits as possible, we could send the program to generate it. In case of random information it does not matter whether we send the information or the program: in any case the number of the sent bits is comparable. While this definition of randomness of number as length of minimal programs or compressibility seems promising, it suffers practically since determining the minimal program for an arbitrary number is equivalent to solving the halting problem and is thus uncomputable by Turing machines. It is clear that almost all numbers are random. But it is not possible to give an example of a random number. No one is able to prove that a given number is a random number. If a program P has generated the number there is a program P_1 that is no longer than P and such that P_1 generates a number n , than the program P would generate the number n . The random numbers are definable but they are not calculable.¹⁰⁵ In any case, the fact that we do not have a program generating such a number when it is no longer the same number is not a sufficient reason to say that this number is a random number. It is not excluded that in the future somebody will find such a program. It is only possible to prove that a given number is not a random number.

For recursive infinite strings, such as the binary expansion of π , the algorithmic information content is simply the size of the smallest program generating π . For a non-recursive string such as the binary expansion of τ (the Turing's constant), there is no finite program, so the algorithmic information content is infinite. The number π is globally compressible: there are finite programs to generate it. By definition there is not such a finite program that would be able to generate τ . τ is not globally compressible. Nevertheless, the number is compressible locally: there is a program which on the basis of information consisting of n digits allows to calculate initially 2^n digits of the number.

¹⁰⁵ Chaitin's considerations are based on Berry's paradox. Bertrand Russell (1908, p. 223) put it as follows:

The least integer not nameable in fewer than nineteen syllables' is itself a name consisting of eighteen syllables; hence the least integer not nameable in fewer than nineteen syllables can be named in eighteen syllables, which is a contradiction.

In 1987 Gregory Chaitin discovered the number Ω (the ‘halting probability’):

$$\Omega = \sum_{P \text{ halts}} 2^{-|P|}$$

Ω represented by a sequence of 0 and 1 so that for any place the probability of occurrence of 0 is $\frac{1}{2}$ and the same is true about 1. It is not possible to point out a rule that governs the succession of the digits. Any program, no matter which, if its complexity is n , can generate maximally only n elements of the sequence. It means Ω is incompressible globally as well as locally. Chaitin defines an infinite sequences s as random if and only if the information content of the initial segment, s_n , of length n eventually becomes and remains greater than n . By this definition, Ω is random while τ is not.

The number Ω can be described in terms of exponential diophantine equations with a parameter n . Thus, the question of Ω is moved into the domain of arithmetic. Chaitin has shown that in no consistent formal system of k bits in size, the proof whether or not the diophantine equation with a parameter n has infinitely many solutions for more than k different values of n . In predicting the information about the patterns of the solutions to the equation, no formal system can do better than chance. The incompleteness (Gödel) and undecidability (Turing) of formal system have been filled up by randomness by Chaitin. To conceive the issue, let us take as an example the question of defining of the set of prime numbers. It seems that there is no formula which in a compression form (the program that generates the formula should be significantly shorter than the program that produces the numbers) could “contain” only prime numbers. The set of prime numbers is random. If so, by the nature of randomness, the fact that the information about the set is not compressible could not be proved. The knowledge about the set of prime numbers is empirical/statistical by the nature of the prime numbers (if their set is random).

From the fact of randomness one can conclude that no matter which is the notion of effective procedure there are facts that could not be compressed. Moreover, such facts dominate over the facts that could be compressed. There are such numbers that no computer – biological, quantum or any other – is able to produce by a program that is essentially shorter than that number. There will always be facts that could not be calculated by whatever computer. For any program there is a finite number t that is the most complicated number that can be generated by that program. For any phenomenon there is a number that characterizes its complexity. For any theory/program there is a phenomenon when the number that characterizes

its complexity is greater than the number that could be generated by that program.

Ω is maximally unrecognizable. Ω marks the current boundary of what mathematics can achieve. It means that there are infinite number of facts that could not be inferred from the axioms of arithmetics. Gödel believed that the human mind is not able to mechanize all its intuitions. In 1989 a physicist Roger Penrose in *The Emperor's New Mind*¹⁰⁶ (1989) based on quantum mechanics argued that the human mind is able to go beyond the mechanical reasoning but no machine is able to do it. Chaitin reveals limits to what we can know.

Einstein said:

I want to know God's thoughts [...] I am not interested in this phenomenon or that phenomenon [...] I want to know God's thoughts – the rest are mere details.

To realize that dream he thought about theory of everything. It was his attempt to extend general relativity and unite the known forces in the universe. It was a project that hopefully would unlock the mind of God. As Hilbert's program was canceled by Gödel, Church and Turing, Einstein's idea was rejected by Heisenberg, Bohr and Schrödinger. They created quantum mechanics. A core element to their new interpretation of the world was that at a fundamental level, everything was unpredictable. For Einstein's

God does not play dice,

they replied:

Einstein, stop telling God what to do with his dice.

Hilbert believed in complete, consistent and decidable mathematics and Einstein believed in a theory of everything that in an elegant mathematical form describes the universe with absolute accuracy and predictability. Both Hilbert and Einstein were wrong. Mathematics as well as the world has a random structure. God plays dice both in mathematics and in the physical world.

Leibniz's words:

Sans les mathématiques on ne pénètre point au fond de la philosophie. Sans la philosophie on ne pénètre point au fond des mathématiques. Sans les deux on ne pénètre au fond de rien.

¹⁰⁶ The title is understandable to anybody who has read Andersen's tales.

may be translated/paraphrased as follows:

Without information science [mathematics] we cannot penetrate deeply into philosophy. Without philosophy we cannot penetrate deeply into information science [mathematics]. Without both we cannot penetrate deeply into anything.

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