Anna Gomolińska*
University of Białystok

ROUGH INFORMATION GRANULES IN SOCIAL AGENT SYSTEM MODELLING**

Abstract. The aim of the paper is to present rough granular methodology as a promising and serviceable tool in modelling of both social agents, interacting under uncertain and incomplete information, as well as systems of such agents. In this article, we focus upon information granules and fundamentals of the rough set theory. An agent's architecture in terms of rough information granules, discussed in part in [23], and other special issues concerning social agent system modelling will be considered elsewhere.

1. Introduction

Modelling and analysis of social agent systems have been attracted many researchers not only in social sciences but also in scientific disciplines and research fields to which investigation of social phenomena was primarily considered as irrelevant, to mention computer science. The fundamental notion focusing the attention of a vast number of researchers is *agent*. Apart from traditional interests in studying individual or collective human agents, institutions, companies, and the nature, there is a rapidly growing interest in investigation of populations of animals (e.g., ants, bees, or birds), artificial agents like robots or computer programs, and various kinds of collective agents (e.g., teams consisted of human agents and intelligent computer programs). In general, an agent is an *autonomous* object able to act on behalf of others or for itself.

A number of types of agents can be observed, and it would be too hard – or even hardly possible – to give a complete typology here. As regarding agents' attitude to activity, *pro-active* (or *goal-oriented*) and *reac-*

* This work is dedicated to Ewa Roszkowska with whom the author collaborated on the Burns generalized game theory. Many thanks go to Andrzej Skowron for valuable discussions on modelling of multiagent systems from the rough-set standpoint.

** The research reported in this paper is a part of the project entitled *Undecidability and Algorithmic Intractability in the Social Sciences*, supported by the Polish Ministry of Science, grant no. 2H01A03025.

ISBN 978–83–7431–108–3 ISSN 0860-150X

tive agents can be distinguished. Pro-active agents are goal-driven, whereas the causes for activity of reactive agents are events. We can also say about goal-oriented agents that they act according to *consequentialist* (or *instrumental-rationality*) modality as they mainly or solely pay attention to the realization of their goals and to achieving the best outcomes. To the contrary, *normatively-oriented* agents pay attention to or judge on the basis of norms and qualities of actions. There are also other action modalities like routine (habitual) modality, symbolic communication and rituals, play, and emotionally-driven acting [7, 9, 12]. *Social* agents, i.e., agents which interact with others can be *cooperative* or not. However, by *socially-oriented* agents we mean agents taking into account interests of others, i.e., agents which are not pure egoists. By a *cognitive* agent, we understand an agent with cognitive faculties like understanding, reason, judgment, perception, learning, adaptation, and so on. Such agents can have *beliefs* and *knowledge*. Where reason or adaptation are, for example, of particular interest, we can speak of *reasoning* agents or *adaptive* agents, respectively. Agents relying mainly on reason in their acting are called *rational*. By an *intelligent* agent we mean an agent granted or having developed some form of intelligence, so intelligent agents are cognitive as well. Modelling of *emotionally-embedded* agents and *intention-driven* agents are also of interest in the present agent technology [13, 14, 15]. From the standpoint of social sciences, studying of social systems of human agents is, of course, of the primary concern. On the other hand, computer scientists are mainly interested in systems of artificial agents or in mixed systems constituted both of human and artificial agents.

Real-life social systems and advanced systems of artificial or half-artificial agents are complex systems being subject to frequent changes and modifications. Usually, agents have to act and interact under incomplete, uncertain, and vague information about the situation of interaction, other participants, and even themselves. Also, agents have limited abilities (i) to identify the situation of interaction, the involvement of particular agents in this situation, the roles to be played, and the goals to be achieved, (ii) to predict the behaviour of others and the results of interaction, (iii) to make proper decisions, (iv) to perform the actions determined, etc., so the issue of modelling of such systems seems to be highly complicated or even intractable. As a matter of fact, traditional tools of mathematics, logic, and computer science turned out to be insufficient and often inadequate for the purpose of modelling of complex systems of agents working under uncertain and incomplete information. Soft computing methods like, e.g., rough sets, fuzzy sets, and neural networks are designed to deal just with imprecise, vague, and incomplete information. Usage of these or similar methods seems to be a reasonable and proper thing to do. In this paper, we propose a rough granular approach, where methods and tools of granular computing, based on recent achievements of rough set theory, are used.

The notion of a *rough set* was introduced by Pawlak in the early 80's of the 20th century as a result of research on approximate classification of objects in *information systems* [39, 40, 41, 42, 44]. The theory and the methodology of rough sets, worked out by Pawlak and other researchers, are suitable for dealing with vague and incomplete information, and are applicable to a vast number of problems like data and concept analysis, classification and, in particular, decision making, generation of classification and association rules, synthesis and analysis of complex objects, ontology formation, analysis of conflicts, reasoning under uncertainty, and many others. Rough set methods are based on Boolean reasoning, and the typical example of a rough set problem to which Boolean reasoning can be employed is computation of minimal reducts of the set of attributes of an information system. This problem is known to be NP-hard [58]. Since this one and many other problems faced in knowledge engineering are computationally hard, people search for heuristics which are computationally much more attractive although they provide us with suboptimal and approximate solutions only.

In general, we aim at modelling of social, cognitive, and communicating agents, either natural or artificial. From our perspective, agents and their systems are complex objects which can be represented by rough information granules. In this article, we just present information granules as a promising tool for agent system modelling. Some attempts to use information granules in social agent system modelling have been already made, see, e.g., research articles on representation of human agents – either individual or collective ones – and on explanation of social interactions of such agents in terms of rule complexes [7, 9, 10, 11, 12, 20, 22, 23]. As regarding the agent architecture, an agent may be viewed as a complex structure built of rough information granules, and similarly for a multiagent system. Rules of some knowledge representation language play an important role here as, on one hand, they can be used as labels for information granules and, on the other hand, they are themselves objects to form information granules. An agent's knowledge and belief bases, value and norm systems, judgment system, classification procedures, and (inter)action modules can be modelled as information granules consisting of rules. Schemes of interaction and systems of agents can also be viewed as such granules. However, representation of agents and their systems in terms of granules is only one issue in our agenda. The list of important questions is much longer, for instance, how agents reason, judge, and understand in an approximate way, how they follow rules, determine (inter)actions, and interact, how agents communicate with and understand one another, as well as how they derive procedures for collective classification of objects (and in particular, decision making). Formation of rough ontologies of concepts is another interesting research problem [37, 64]. It is already known that using sensory data as the only source of information is insufficient for learning complex concepts, so agents are assumed to have some domain knowledge as well. Examples of sources of domain knowledge are teachers, socio-cultural traditions, codes, handbooks, various know-how books, and web pages.

Throughout the paper, the power set of a set X, the cardinality of X, and the Cartesian product $X \times X$ will be denoted by $\wp X$, $\#X$, and X^2, respectively. As usual, $\circ$ will denote composition of relations. The propositional connectives of conjunction and disjunction will take the precedence of implication and double implication. A family $\mathcal{X}$ of non-empty subsets of X is a covering of X if $\bigcup \mathcal{X} = X$. A covering $\mathcal{X}$ is called a partition of X in case for any $Y, Z \in \mathcal{X}$, $Y \cap Z \neq \emptyset$ implies $Y = Z$.

Given a set X and a binary relation $\varrho \subseteq X^2$, the *image* and the *co-image* of a set $Y \subseteq X$, $\varrho^{\rightarrow} Y$ and $\varrho^{\leftarrow} Y$, respectively, are defined along the standard lines. In particular, for any $x \in X$,

$$\varrho^{\rightarrow}\{x\} = \{y \in X \mid (x, y) \in \varrho\} \ \ \& \ \ \varrho^{\leftarrow}\{x\} = \{y \in X \mid (y, x) \in \varrho\}. \qquad (1)$$

Every *reflexive* relation ϱ on X (i.e., every $\varrho \subseteq X^2$ such that $\forall x \in X. (x, x) \in \varrho$) determines a covering of X, viz., $\{\varrho^{\leftarrow}\{x\} \mid x \in X\}$. In the context of approximation spaces, reflexive relations are called *similarity* relations as well. Furthermore, ϱ is *symmetric* if $\forall x, y \in X.((x, y) \in \varrho \Rightarrow (y, x) \in \varrho)$. It is worth mentioning that ϱ is symmetric if and only if ϱ is equal to its converse relation ϱ^{-1}. Relations being both reflexive and symmetric are referred to as *tolerance* relations. Obviously, every tolerance relation is also a similarity relation. In the sequel, ϱ is *transitive* if $\forall x, y, z \in X.((x, y) \in \varrho \ \& \ (y, z) \in \varrho \Rightarrow (x, z) \in \varrho)$. Transitive tolerance relations are known as *equivalence* relations. In this case, $\varrho^{\leftarrow}\{x\}$ is called the *equivalence class* of x and denoted by $[x]_\varrho$ (or, simply, $[x]$ if ϱ is understood)[1]. The covering of X determined by ϱ, $\{[x]_\varrho \mid x \in X\}$, is the partition of X. Moreover, there is a one-to-one correspondence between the set of all equivalence relations on X and the set of all partitions of X. Within the framework of rough sets and Pawlak's information systems, equivalence relations on the universe of objects are understood as *indiscernibility* relations.

---

[1] Note that $\varrho^{\leftarrow}\{x\} = \varrho^{\rightarrow}\{x\}$ by symmetry of ϱ.

The rest of the paper is organized as follows. In Sect. 2, a general notion of an information granule is introduced. Granules in Pawlak's information systems and neighbourhoods of objects are discussed in Sect. 3. Formulas and rules of a description language as labels for information granules are presented in Sect. 4. Next, rule complexes and, in particular, rough classifiers are recalled in Sect. 5. Section 6 contains remarks on collective agents and, in particular, economic clusters. In the sequel, the classical Pawlak rough set model is described in Sect. 7. Section 8 is devoted to two of several extensions of this model. The last section contains a brief summary.

2. Information Granules

The notion of an *information granule* (or simply, *granule*) was introduced by Zadeh in the context of fuzzy set analysis of complex systems [77], and elaborated in a series of research articles [52, 53, 61, 62, 63, 65, 66, 72, 78, 80]. According to Zadeh's proposal, an *information granule* is a clump of objects of some class, drawn together on the basis of indiscernibility, similarity, or functionality. In fact, the term 'indiscernibility' may be dropped since the property of being indiscernible is a particular, "perfect" case of being similar.

The definition of a granule leaves a lot of freedom as regarding the interpretation. By way of example, a set of objects X is often viewed as an information granule if all members of this set are similar to a given distinguished object, say u. Calling X an information granule is justified since all elements of X function as 'objects similar to u'. From another standpoint, X is treated as a granule formed on the basis of similarity, yet similarity is understood as similarity with respect to u. Also, a granule may consist of objects of varied sorts like rules and sets of rules. *Rule complexes* used to model agents as well as their systems and interactions in *generalized game theory*, proposed and developed by Burns and his collaborators [7, 9, 10, 11, 12, 20, 22, 23], are examples of such heterogeneous information granules. The inner structure of granules does not matter, either. From the mathematical point of view, granules may be ordinary sets of objects, ordered sets of objects, sequences of objects, tables of objects, and so on. However, not every set of objects is treated as a granule. What really matters is the reason (viz., similarity or functionality) for drawing objects together.

The idea of computing with granules of objects instead of single objects, also attributed to Zadeh [77, 78, 79, 80], has found a number of followers (see, e.g., [22, 31, 35, 52, 53, 75, 81]). Recently, *granular computing* is a dynamically developing and expanding research area which makes use of various methods and tools, e.g. those offered by rough set theory, fuzzy set theory, or neural networks. Methods of granular computing not only serve the purpose of modelling of complex objects like agents and systems of agents but they are also suitable for specification and solving of a variety of research problems related to functioning of such objects. *Computing with words* (CW for short), appearing in the literature in connection with granular computing, is a step in performing computations on granules. Words are simply linguistic labels for information granules, for example, a formula may serve as a label for the set (i.e., granule) of all objects satisfying the formula. In computing with words, we manipulate with labels of granules instead of numbers, and – as pointed out by Zadeh [79] –

> "[t]here are two major imperatives for computing with words. First, computing with words is a necessity when the available information is too imprecise to justify the use of numbers. And second, when there is a tolerance for imprecision which can be exploited to achieve tractability, robustness, low solution costs and better rapport with reality. Exploitation of the tolerance for imprecision is an issue of central importance in CW."

Objects to be modelled by granules are creatures or productions which undergo various changes and modifications, and therefore transformation of granules is an important issue in granular computing. Such questions are addressed as how given granules are related to one another (inclusion, closeness, similarity), formation of new granules from elementary objects and/or granules already obtained (aggregation, decomposition), expansion, contraction, and revision of granules. As a result of aggregation, a new granule is obtained from given granules. To the contrary, decomposition of a granule leads to a collection of granules derived from the primary granule. The last three forms of transformation are named by us after well-known operations of belief change [3, 18]. Briefly speaking, expansion of a granule X consists in extending it with an object or a granule of objects as opposite to contraction of X, where some objects or granules are removed from X. Revision of X by an elementary or compound object is a composition of contraction with expansion. In particular, revision can take the form of replacement, where an object being a member or a part of X is replaced by another object. Operations on granules are constrained by conditions specifying numerical or symbolic values of variables. So, for instance, a constraint may say that the result of an aggregation of granules should be a granule close to the degree at least t to a given exemplary granule X, where t is a real number from the unit interval, or that the resulting granule should differ from X as little as possible.

The definition of the notion of an information granule covers a broad class of cases. Examples of information granules are sets of similar rows in Pawlak information systems or, in other terms, sets of similar objects of the system, sets of neighbourhoods of objects, concepts in approximation spaces and approximations of these concepts, pairs of sets of objects satisfying premises and conclusion(s) of a rule, respectively, sets of rules like algorithms and classifiers, rule complexes representing agents and games [20, 22, 23], computational grids (i.e., collections of distributed resources which can be used to execute large-scale computations), coalitions and teams of cooperating agents, and economic clusters [6, 56, 57]. In the following sections, we give more details about some cases of information granules just mentioned.

3. Granules in Pawlak's Information Systems

The notion of an *information system* (IS for short) was introduced by Pawlak [39, 42]. In a recent formulation, an information system is a pair $\mathcal{A} = (U, A)$ of finite non-empty sets of objects and attributes, respectively. The set of all objects considered, U, is the universe of $\mathcal{A}$. Henceforth, elements of U (resp., A) will be denoted by u (resp., a), possibly with sub/superscripts. Each attribute a is viewed as a mapping $a : U \mapsto V_a$ which assigns to every object u, a value $a(u) \in V_a$. Let $V = \bigcup\{V_a \mid a \in A\}$. Any pair (a, v), where $a \in A$ and $v \in V_a$ is called a *descriptor*. From a different standpoint, every object u may be viewed as a mapping $u : A \mapsto V$ such that $u(a) \in V_a$, i.e., $U \subseteq \prod\{V_a \mid a \in A\}$. From the mathematical point of view, every object u is, thus, a set of descriptors $u = \{(a, a(u)) \mid a \in A\}$. Therefore, granulation of objects on the basis of information contained in $\mathcal{A}$ and formation of granules from sets of descriptors of $\mathcal{A}$ are equivalent.

As well-known, every set of attributes $B \subseteq A$ induces an equivalence relation of B-*indiscernibility* of objects, ind_B, defined for any objects $u, u' \in U$ by

$$(u, u') \in \mathrm{ind}_B \Leftrightarrow \forall a \in B.a(u) = a(u'). \tag{2}$$

In other words, $(u, u') \in \mathrm{ind}_B$ if and only if for every $a \in B$, $(a, u(a)) = (a, u'(a))$, i.e., if and only if $u_{|B} = u'_{|B}$. The equivalence class of u, $[u]_B$, consisting of all objects which cannot be discerned from u by means of attributes of B, is a B-*elementary* information granule.

Mappings on U, assigning information granules to objects of U, may be called *granulation mappings.* Granulation mappings of the form $\Gamma : U \mapsto \wp U$, where for every $u \in U$, $u \in \Gamma u$, are referred to as *uncertainty mappings* [60]. $[\cdot]_B$ which assigns to every object u, its equivalence class $[u]_B$ is an example of an uncertainty mapping. Now, we recall a pretty universal recipe for construction of uncertainty mappings described, e.g., in [45]. To this end, suppose that for every attribute $a \in B \subseteq A$, a metric and a threshold function, $\delta_a, f_a : (V_a)^2 \mapsto [0, +\infty)$, respectively, are given. Define a family of uncertainty mappings $\Gamma_a : U \mapsto \wp U$, parameterized by attributes of B, as follows, for any objects u, u':

$$u' \in \Gamma_a u \; \Leftrightarrow \; \delta_a(a(u), a(u')) \leq f_a(a(u), a(u')) \tag{3}$$

In words, u' is a member of the granule assigned to u by the mapping Γ_a if and only if the distance between the values of a at u, u', given by δ_a, does not exceed the threshold value $f_a(a(u), a(u'))$. Next, a global uncertainty mapping, Γ, may be defined as the intersection of the mappings Γ_a, i.e.,

$$\Gamma u \stackrel{\text{def}}{=} \bigcap \{\Gamma_a u \mid a \in B\}. \tag{4}$$

Each Γ_a induces a tolerance relation ϱ_{Γ_a}, given by

$$(u', u) \in \varrho_{\Gamma_a} \stackrel{\text{def}}{\Leftrightarrow} u' \in \Gamma_a u, \tag{5}$$

which is not an equivalence relation in general. Therefore, the relation ϱ_Γ, defined by

$$(u', u) \in \varrho_\Gamma \stackrel{\text{def}}{\Leftrightarrow} u' \in \Gamma u, \tag{6}$$

is merely a tolerance relation as well.

A *decision system* is an IS of the form $\mathcal{A} = (U, C \cup D)$, where C, D are non-empty disjoint sets of *condition* and *decision attributes*, respectively. Clearly, $\mathcal{A}$ is the result of merging of ISs $\mathcal{C} = (U, C)$ and $\mathcal{D} = (U, D)$. The data table corresponding to $\mathcal{A}$ is referred to as a *decision table* [39, 42, 43]. In most cases, D consists of only one decision attribute, say d. Then, we often write $\mathcal{A} = (U, C, d)$ instead of $\mathcal{A} = (U, C \cup \{d\})$.

As the set of descriptors $\{(a, a(u)) \mid a \in A\}$ represents the information accessible to an agent about an object u, information systems contain the available information about a collection of objects. From the standpoint of functionality, these information systems as well as their representations in the form of data tables serve as granules comprising information about classes of objects.

Example 3.1. *By way of illustration, let us consider a decision system* $\mathcal{A} = (U, A)$*, where* U *consists of* 8 *objects (scholarship applications), denoted by* $0, \ldots, 7$*, and* A *consists of* 6 *condition attributes* $a_0, \ldots, a_5$ *(*4 *numerical and* 2 *nominal) and two decision attributes* d_0, d_1 *(*1 *numerical and* 1 *nominal). Every object is specified in terms of such attributes as: studies (*a_0*), year of studies (*a_1*), mean result (*a_2*), income of the family per month per person (*a_3*), age (*a_4*), and place of residence (*a_5*). The range of* a_0 *is* $V_{a_0} = \{\mathtt{m}, \mathtt{ch}, \mathtt{ph}\}$*, where* `m` *stands for 'mathematics',* `ch` *– 'chemistry',* `ph` *– 'philosophy';* $V_{a_1} = \{1, 2, 3\}$*;* $V_{a_2} \subseteq [3.0, 5.0]$*;* $V_{a_3} \subseteq [800, 1200]$*;* $V_{a_4} = \{19, \ldots, 22\}$*; and* $V_{a_5} = \{\mathtt{c}, \mathtt{t}, \mathtt{v}\}$*, where* `c` *stands for 'city',* `t` *– 'town', and* `v` *– 'village'. Furthermore, the decision attribute* d_0 *specifies whether or not the application is accepted and the form of a scholarship awarded. Its range is* $V_{d_0} = \{\mathtt{s0}, \mathtt{s1}, \mathtt{s2}, \mathtt{s3}\}$*, where* `s0` *represents the negative decision,* `s1` *is for 'social scholarship',* `s2` *– 'research scholarship', and* `s3` *– 'both social and reseach scholarship'. The second decision attribute,* d_1*, specifies the amount of money awarded per month, and* $V_{d_1} \subseteq [0, 500]$*. System* $\mathcal{A}$ *is presented in Table 1.*

Table 1

The decision system $\mathcal{A}$

u	a_0	a_1	a_2	a_3	a_4	a_5	d_0	d_1
0	m	2	3.45	950	20	c	s1	200
1	m	1	4.10	820	19	v	s3	400
2	ph	1	4.30	850	19	v	s3	420
3	ch	2	3.70	1130	20	c	s0	0
4	ph	3	4.55	1050	22	t	s2	300
5	m	2	3.75	980	20	v	s1	180
6	ch	3	4.40	1200	21	t	s2	250
7	ch	2	3.95	850	21	c	s3	380

With each a_i*, we can associate an uncertainty mapping* Γ_{a_i}*, and similarly for the decision attributes. For example, let*

$$u' \in \Gamma_{a_i} u \overset{\text{def}}{\Leftrightarrow} \begin{cases} a_i(u') = a_i(u) & \text{if } i = 0, 1, 4, 5, \\ |a_i(u') - a_i(u)| \leq 0.25 & \text{if } i = 2, \\ |a_i(u') - a_i(u)| \leq 75 & \text{if } i = 3. \end{cases} \tag{7}$$

Γ_{a_i} *generate partitions of* U *for* $i = 0, 1, 4, 5$*, viz.,* $\Gamma_{a_0}^{\rightarrow} U = \{\{0, 1, 5\}, \{2, 4\}, \{3, 6, 7\}\}$*,* $\Gamma_{a_1}^{\rightarrow} U = \{\{1, 2\}, \{0, 3, 5, 7\}, \{4, 6\}\}$*,* $\Gamma_{a_4}^{\rightarrow} U = \{\{1, 2\}, \{0, 3, 5\},$

$\{6,7\},\{4\}\}$, and $\Gamma_{a_5}^{\rightarrow}U = \{\{0,3,7\},\{4,6\},\{1,2,5\}\}$. Thus, e.g. in the case of Γ_{a_0}, the elementary information granules are the equivalence classes $\{0,1,5\}$, $\{2,4\}$, and $\{3,6,7\}$. Values of the uncertainty mappings Γ_{a_2} and Γ_{a_3} are given in Table 2. These mappings generate coverings of U which, nevertheless, are not partitions. Mappings Γ_{a_i} can be aggregated in many ways. For instance, we can derive the following uncertainty mappings Γ, Γ' (see Table 2):

$$\Gamma u \stackrel{\text{def}}{=} \Gamma_{a_0} u \cap \Gamma_{a_2} u \ \ \& \ \ \Gamma' u \stackrel{\text{def}}{=} \Gamma_{a_2} u \cap \Gamma_{a_5} u \tag{8}$$

Observe that $\Gamma^{\rightarrow}U$ is a partition of U, whereas $\Gamma'^{\rightarrow}U$ is merely a covering.

Table 2

Values of Γ_{a_2}, Γ_{a_3}, Γ, and Γ'

u	$\Gamma_{a_2}u$	$\Gamma_{a_3}u$	Γu	$\Gamma' u$
0	$\{0,3\}$	$\{0,5\}$	$\{0\}$	$\{0,3\}$
1	$\{1,2,7\}$	$\{1,2,7\}$	$\{1\}$	$\{1,2\}$
2	$\{1,2,4,6\}$	$\{1,2,7\}$	$\{2,4\}$	$\{1,2\}$
3	$\{0,3,5,7\}$	$\{3,6\}$	$\{3,7\}$	$\{0,3,7\}$
4	$\{2,4,6\}$	$\{4,5\}$	$\{2,4\}$	$\{4,6\}$
5	$\{3,5,7\}$	$\{0,4,5\}$	$\{5\}$	$\{5\}$
6	$\{2,4,6\}$	$\{3,6\}$	$\{6\}$	$\{4,6\}$
7	$\{1,3,5,7\}$	$\{1,2,7\}$	$\{3,7\}$	$\{3,7\}$

3.1. Neighbourhoods of Objects

As mentioned in the preceding section, an uncertainty mapping Γ on U assigns to every object u, a granule of information Γu. One can easily see that the corresponding binary relation ϱ_Γ, defined like in (6), is a similarity relation. Objects similar to a given object may be viewed as close to that object. As a consequence, Γu may be treated as a neighbourhood of u. Obviously, a number of various uncertainty mappings on U may be taken into account at the same time, say $\Gamma_0, \ldots, \Gamma_m$. These mappings may be aggregated like we did before (cf. (4)) or they may be considered separately. In the latter case, with every object u, there is associated a global information granule $\Gamma u = (\Gamma_0 u, \ldots, \Gamma_m u)$ being, in fact, a sequence of neighbourhoods of u.

Example 3.2. *In connection with the previous example, note that the following global granules* $\Gamma''0 = (\{0,1,5\}, \{0,3,5,7\}, \{0,3\}, \{0,5\}, \{0,3,5\}, \{0,3,7\})$ *and* $\Gamma''1 = (\{0,1,5\}, \{1,2\}, \{1,2,7\}, \{1,2,7\}, \{1,2\}, \{1,2,5\})$*, being sequences of neighbourhoods generated by* Γ_{a_i} *(*$i = 0, \ldots, 5$*, respectively), can be associated with objects* $0, 1$.

There arises a question how this happens that so many uncertainty mappings can come into play. First, different subsets of attributes B may lead to different uncertainty mappings. Secondly, starting with an uncertainty mapping Γ_0 and taking into account (6), we can derive a sequence of uncertainty mappings Γ_i $(i = 0, 1, \ldots)$ such that for any objects u, u':

$$u' \in \Gamma_i u \overset{\text{def}}{\Leftrightarrow} (u', u) \in \underbrace{\varrho_{\Gamma_0} \circ \ldots \circ \varrho_{\Gamma_0}}_{i+1} \tag{9}$$

In particular, $u' \in \Gamma_1 u$ if and only if $(u', u) \in \varrho_{\Gamma_0} \circ \varrho_{\Gamma_0}$, i.e., if and only if there is u'' such that $(u', u'') \in \varrho_{\Gamma_0}$ and $(u'', u) \in \varrho_{\Gamma_0}$, i.e., if and only if there is u'' such that $u' \in \Gamma_0 u''$ and $u'' \in \Gamma_0 u$. ϱ_{Γ_0} is reflexive, so is an arbitrary composition of ϱ_{Γ_0} with itself. Therefore, $\Gamma_0 u, \Gamma_1 u, \Gamma_2 u, \ldots$ may be understood as the granules of the nearest neighbours, the 1-step neighbours, the 2-step neighbours, and so on. It can happen that for some number i, the relation induced by Γ_i is an equivalence relation[2]. Since any further iteration does not lead to new uncertainty mappings, the procedure may be stopped.

Example 3.3. *Continuing Example 3.1, it is worth noting that* $\varrho_{\Gamma_{a_3}} \circ \varrho_{\Gamma_{a_3}}$ *is an equivalence relation, so the process of granule production may be stopped at* $i = 1$*. The corresponding partition of* U *is equal to* $\{\{0,4,5\}, \{1,2,7\}, \{3,6\}\}$*. For example, the granule of the 1-step neighbours of the object* 0 *equals to* $\{0,4,5\}$.

4. Formulas and Rules as Labels of Granules

Adapting slightly Zadeh's idea of computing with words, formulas and rules may be viewed as linguistic labels representing information granules. Computation on granules is performed with help of the labels attached to these granules.

In information systems, the *descriptor language* is a formal language commonly used to express and reason about properties of objects and con-

[2] This always holds for finite U and ϱ_{Γ_0} being a tolerance relation.

cepts [39, 42, 45]. We briefly present this language starting with an information system $\mathcal{A} = (U, A)$ as earlier. For the sake of simplicity, elements of $A \cup V$ are identified with their names playing the role of constant symbols. Moreover, these symbols are the only terms. Commas and the round parentheses are auxiliary symbols. As primitive propositional connectives we may take $\wedge$ (conjunction) and $\neg$ (negation), whereas the remaining connectives of disjunction, material implication, and double implication, $\vee, \rightarrow$, and $\leftrightarrow$, respectively, can be defined by means of $\wedge, \neg$ along the classical lines. Descriptors, viz., pairs of the form (a, v), where $a \in A$ and $v \in V_a$, are atomic formulas. Compound formulas are formed from the atomic ones as usual. Formulas are denoted by α, β, possibly with sub/superscripts. The relation of (crisp) satisfiability of formulas for objects, $\models$, is understood in line with the classical Tarskian approach, viz., for any formulas $(a, v), \alpha, \beta$ and any object u,

$$\begin{aligned} u \models (a, v) &\overset{\text{def}}{\Leftrightarrow} a(u) = v, \\ u \models \alpha \wedge \beta &\overset{\text{def}}{\Leftrightarrow} u \models \alpha \;\;\&\;\; u \models \beta, \\ u \models \neg\alpha &\overset{\text{def}}{\Leftrightarrow} u \not\models \alpha. \end{aligned} \tag{10}$$

With every formula α, there is associated an information granule $\|\alpha\|$ of objects satisfying this formula, called the (crisp) *meaning* of α. In other words, α is a label for the granule consisting of objects satisfying α. Thus,

$$\begin{aligned} &\|(a, v)\| = \{u \mid a(u) = v\}, \\ &\|\alpha \wedge \beta\| = \|\alpha\| \cap \|\beta\|, \\ &\|\neg\alpha\| = U - \|\alpha\|. \end{aligned} \tag{11}$$

In the classical case, the satisfiability of formulas is extended to the satisfiability of sets of formulas[3] in such a way that for any set of formulas X and any object u,

$$u \models X \overset{\text{def}}{\Leftrightarrow} \forall \alpha \in X.u \models \alpha. \tag{12}$$

Hence, the information granule associated with X, called the (crisp) *meaning* of X, is the set

$$\|X\| = \bigcap\{\|\alpha\| \mid \alpha \in X\}. \tag{13}$$

In [25], we propose and discuss various rough forms of satisfiability of formulas and sets of formulas for objects. For instance, given an uncertainty mapping $\Gamma : U \mapsto \wp U$ and a threshold value $t \in [0, 1]$, a formula α is said to

[3] We use the same symbol $\models$ for simplicity.

be *satisfied for* u *to the degree at least* t, written $u \models_t \alpha$, if and only if the degree of inclusion of the granule Γu in the crisp meaning of α, measured by a rough inclusion function (see Sect. 7), is not less than t.

Example 4.1. *Consider a formula* $\alpha = ((a_1, 2) \wedge (a_3, 850)) \vee (a_5, \mathrm{v})$ *of the descriptor language for the decision system* $\mathcal{A}$ *from Example 3.1 which says that the year of studies of a student is* 2 *and the income is* 850*, or a student comes from a village. Then,* $\|\alpha\| = (\|(a_1, 2)\| \cap \|(a_3, 850)\|) \cup \|(a_5, \mathrm{v})\| = (\{0, 3, 5, 7\} \cap \{2, 7\}) \cup \{1, 2, 5\} = \{1, 2, 5, 7\}$*. Let us assume that for any object* u_i *(*$i = 0, \ldots, 7$*) and* $t \in [0, 1]$*,* $u_i \models_t \alpha$ *if and only if* $\#(\Gamma u \cap \|\alpha\|)/\#\Gamma u \geq t$*. Hence, if* $t > 1/2$*, then* $u_i \models_t \alpha$ *for* $i = 1, 5$*; if* $0 < t \leq 1/2$*, then* $u_i \models_t \alpha$ *for* $i = 1, 2, 3, 4, 5, 7$*; and, finally,* $u_i \models_0 \alpha$ *for every* i*.*

A *rule* is a pair $r = (P_r, C_r)$ of finite sets of formulas P_r and C_r of *premises* and *conclusions* of r, respectively. By assumption, the set of conclusions is non-empty. Rules will be denoted by r with sub/superscripts if needed. Sets P_r and C_r are information granules built of formulas on the basis of functionality. On the other hand, 'P_r' and 'C_r' are labels for the granules consisting of all objects of U which satisfy the sets P_r, C_r, respectively. In data mining and knowledge discovery (KDD), induction of *classification* (in particular, *decision*) rules as well as *association* rules is of particular interest [1, 2, 4, 28, 29, 30, 32, 34, 36, 69, 71]. For the lack of space, we give a few details about decision rules only. Such rules are typically extracted from decision systems with one decision attribute. The set of premises of a decision rule r consists of descriptors with different attribute symbols[4] and the set of conclusions of r contains a single descriptor of the form (d, v) only, where d is the decision attribute and $v \in V_d$.

In general, premises and conclusions of a rule may be understood differently, viz., application of rules may be treated as a game between two agents. One agent checks whether or not a rule is applicable to an object and communicates the result to the second agent. Then, the latter agent applies the rule to the object. Therefore, let us consider two forms of satisfiability of sets of formulas for objects, $\models_1, \models_2$, and the corresponding forms of meaning $\|\cdot\|_1, \|\cdot\|_2$. The *meaning* of a rule r is defined as the pair $\|r\|_{1,2} = (\|P_r\|_1, \|C_r\|_2)$ of information granules $\|P_r\|_1, \|C_r\|_2$, so r is a label for an information granule being an ordered pair of granules. Furthermore, we can say that r is *applicable to* an object u if and only if $u \models_1 P_r$, i.e., $u \in \|P_r\|_1$. More generally, r is *applicable* if and only if there is u such

[4] A conjunction of such descriptors is called a *template*.

that $u \models_1 P_r$, i.e., $\|P_r\|_1 \neq \emptyset$. More about rules and their rough applicability can be found in [24, 26].

Example 4.2. *From the decision system considered in Example 3.1, one can derive the following exemplary rules:* $r_1 = (\{(a_1, 1), (a_5, \mathrm{v})\}, \{(d_0, s_3)\})$, $r_2 = ({(a_4, 19), (a_5, \mathrm{v})}, \{(d_0, s_3)\})$, *and* $r_3 = (\{(a_1, 1) \vee (a_4, 19), (a_5, \mathrm{v})\}, \{(d_0, s_3)\})$. *Clearly, the latter rule is obtained from the first two rules, being decision rules, by combination of premises* $(a_1, 1), (a_4, 19)$. *By way of example,* r_2 *informally says that if a student is 19 and comes from a village, then he/she will be awarded a social and scientific scholarship. Note that* $\|(a_1, 1)\| = \|(a_4, 19)\| = \|(a_1, 1) \vee (a_4, 19)\| = \{1, 2\}$, $\|(a_5, \mathrm{v})\| = \{1, 2, 5\}$, *and* $\|(d_0, s_3)\| = \{1, 2, 7\}$. *Moreover,* $\|P_{r_1}\| = \|P_{r_2}\| = \|P_{r_3}\| = \{1, 2\}$ *and* $\|C_{r_1}\| = \|C_{r_2}\| = \|C_{r_3}\| = \{1, 2, 7\}$. *Hence, each and every rule* r_i $(i = 1, 2, 3)$ *is applicable in the crisp sense to objects* $1, 2$, *and it is valid in* $\mathcal{A}$ *since* $\|P_{r_i}\| \subseteq \|C_{r_i}\|$.

5. Rule Complexes

Informally speaking, *rule complexes* are sets having a nested structure and built of rules over some language [20, 22]. They are examples of complexes of objects, where rules are taken as objects. It should be emphasized that neither complexes of objects nor even rule complexes are information granules by definition. However, they may be viewed as such granules if the objects forming them are clustered on the basis of some kind of similarity or functionality. For instance, rules building an agent's value complex are drawn together for they all represent or at least are related to the agent's values and norms. Another example of a rule complex, being an information granule, is a *rough classifier* briefly described in Section 5.1.

Let us overview the notion of a *complex* of objects[5]. A program with embedded procedures is both a prototypical rule complex as well as an example of an information granule formed on the basis of functionality. In the definition below, empty complexes of objects are excluded for the sake of convenience only. Given a non-empty set of objects U, the class of all complexes of objects upon U (or, simply, complexes of objects if U is understood), $\mathcal{C}(U)$, is the least class of sets which contains U and is closed under the following formation rules: (CPL1) every non-empty subset of a complex of objects is a complex of objects; (CPL2) every non-empty set of

[5] In [22], the term 'complex of points' was used.

complexes of objects is a complex of objects; and (CPL3) the set-theoretical union of any non-empty set of complexes of objects is a complex of objects. A characteristic feature of complexes of objects is the possibility of multiple occurrences of objects in the same complex of objects although every such complex is actually a set.

In the domain of complexes of objects, the usual set-theoretical notions of an element and a subset of a set can be generalized to the notions of a *generalized element* and a *subcomplex* of a complex of objects. First, an object or a complex of objects x is a generalized element (or *g-element*) of a complex of objects X, $x \in_g X$, if x occurs in X, i.e.,

$$x \in_g X \overset{\text{def}}{\Leftrightarrow} x \in X \vee \exists n \in \mathbb{N}.\exists X_0, \ldots, X_n.x \in X_0 \in \ldots \in X_n \in X. \quad (14)$$

Given $t \in [0,1]$ and a family of *rough membership functions*[6] $\mu_X : U \mapsto [0,1]$, where X is any complex of objects upon U, a rough version of g-membership can be obtained. We can say that an object u is a *generalized element* (or *g-element*) *to the degree* t of a complex of objects X, $\mu^g_X(u) = t$, if $\mu_X(u) = t$ or there exists a complex of objects Y being a g-element of X that $\mu_Y(u) = t$, i.e.,

$$\mu^g_X(u) = \max\{\mu_Y(u) \mid Y = X \ \vee \ Y \in_g X\}. \quad (15)$$

In the sequel, a complex of objects X is referred to as a subcomplex of a complex of objects Y, $X \sqsubseteq Y$, if $X = Y$ or X can be obtained from Y by removing some g-elements of Y, copies of the empty set which may appear after removing g-elements, and useless parentheses. For instance, let $Y = \{\{\{\{u,u'\},u\},u''\},u'\}$ be the original complex of objects. By removing u,u' from $\{u,u'\}$ as well as u'', we obtain $Y' = \{\{\{\emptyset,u\}\},u'\}$ which, nevertheless, is not a complex of objects. At the next step, we remove $\emptyset$ to arrive at $Y'' = \{\{\{u\}\},u'\}$, already being a complex of objects and a subcomplex of Y. For simplicity, we may decide to remove the external parentheses in the expression '$\{\{u\}\}$' to obtain $X = \{\{u\},u'\}$ being another subcomplex of Y.

Every element is also a g-element and every non-empty subset is a subcomplex of a complex of objects, yet the converse may not hold in general. Both being a g-element and being a subcomplex are transitive. Note also that every complex of objects being a g-element of a complex of objects X is a subcomplex of X.

[6] The notion of a rough membership function goes back to Pawlak and Skowron [46]. More details can be found in Sect. 8.1.

Example 5.1. *Consider the universe U from Example 3.1, granulated by Γ. Sets $X_1 = \{1,2\}$, $X_2 = \{1,3,4\}$, $X_3 = \{2, X_2\}$, and $X_4 = \{1,2,X_1,X_3\}$ are exemplary complexes of objects upon U. In the case of X_4, its elements are* 1, 2, *X_1, and X_3. Apart from the elements just mentioned, the g-elements of X_4 are* 3, 4, *and X_2. Examples of subcomplexes of X_4 are its non-empty subsets, $\{1,\{2\},\{3,4\}\}$, and $\{1,2,\{2,\{4\}\}\}$. Assuming that for any object u and a set of objects X, $\mu_X(u) = \#(\Gamma u \cap X)/\#\Gamma u$, we obtain*

$$\mu^g_{X_4}(u_i) = \begin{cases} 0 & \text{if } i = 0,5,6, \\ 0.5 & \text{if } i = 2,3,4,7, \\ 1 & \text{if } i = 1. \end{cases} \tag{16}$$

5.1. Rough Classifiers

A decision system contains information about classification of a given set of objects U into some, pairwise disjoint classes $C_0, \ldots, C_m$ ($m \in \mathbb{N}$) of objects. The classification task consists in mapping any object examined, regardless it belongs to U or not, to exactly one of these classes. Where $m = 1$, the classification resolves into decision making whether or not an object belongs to C_0 (or, equivalently, C_1). Rough set techniques and other soft computing technology provide a number of algorithmic tools to extract decision rules from decision systems for the purpose of classification of unseen objects [4, 5, 28, 29, 30, 36, 69]. Mining classification rules of a satisfactory quality from a decision system $\mathcal{A} = (U, A, d)$ resolves itself into computation of suitable *reducts* of A. Informally speaking, a reduct of A is a set of attributes $B \subset A$ such that the classification of objects under the set of attributes B is almost the same as in the case of the primary set of attributes A. Most of the methods offered by the rough set theory to determine exact reducts[7] are based on computation of prime implicants of a Boolean function what is NP-hard [58]. Therefore, various heuristics are proposed to compute approximate reducts which are suboptimal, yet obtained in a shorter time.

Having obtained a set of decision rules is not sufficient for the purpose of classification. We need to have at our disposal methods for judging applicability of rules, for carrying these rules into effect, and for solving conflicts and inconsistencies which possibly can occur. In this way, we arrive at the notion of a *rough classifier* for a concept or a set of concepts. Any such classifier is a set of decision rules together with meta-rules (methods) for

[7] That is, reducts which precisely preserve the primary classification.

rough judgment of applicability, rough rule following, and resolving conflicts among rules. Clearly, a rough classifier is a granule of information based on functionality.

6. Collective Agents

Examples of collective agents are coalitions, teamworks, nations, states, institutions, companies, associations, school classes, economic clusters, and swarms. A collective agent is a collection of individual agents and/or other collective agents together with interconnections among the members as well as interaction rules and procedures, plans, values and norms, judgmental procedures, objectives and goals, intentions, commitments, beliefs and knowledge, consciousness, action modalities, etc. which are shared by the group members[8]. A collective agent is more than a system of agents just because of the assumption of sharing. A distinction is made about being shared and being common. In the former case, members of a collective agent need not to reflect on what they share, and they may not be conscious that others share or not share something. On the other hand, being a common value, rule, belief, or anything is a very robust feature. In this case, members of a collective agent not only share the thing but also they are conscious of their sharing, they are conscious that others are conscious of their sharing, and so on [8, 13, 14, 15, 17].

In our terms, collective agents are compound information granules consisting of two sorts of objects: (i) individual or other collective agents and (ii) systems of shared rules, values, beliefs, interaction procedures, and so on. Rule complex is just a mathematical tool which seems to be suitable for modelling and analyzing collective agents.

6.1. Economic Clusters

Among collective agents one can distinguish *economic clusters*. The notion of an *economic cluster*, going back to Porter [56, 57], has gained much interest since its introduction in the 90's of the 20th century (see, e.g., [6], other web pages, and articles on economic clusters). It is a vague concept related to a social and economic phenomenon that in spite of globalization of market and excellent opportunities for making businesses via internet,

[8] Clearly, what is shared among the members of a collective agent – and to which extent – depends on the agent and the situation of interaction, e.g., members of the group may share a goal or some interaction rules only.

geographical location is still fundamental to competition of companies. According to Porter who considers a variety of economic clusters,

> "[c]lusters are geographic concentrations of interconnected companies and institutions in a particular field."

Obviously, economic clusters are not merely associations of similar objects like companies in some industry sector. They comprize heterogenous objects like governmental or private institutions (including universities and research centres), various associations, and a number of companies[9]. The world well-known clusters are the California wine cluster, Silicon Valley, Hollywood, the finance cluster on Wall Street, the Massachusetts biotechnology cluster, the Italian leather fashion cluster, the Dutch transportation and flower clusters, and the cluster of built-in kitchens and appliances in Germany. Other typical examples of clusters are tourism clusters functioning successfully in a large number of countries. There is a conviction that cluster formation is an essential ingredient of economic development in the 21st century. From our perspective, economic clusters are information granules based on functionality. In an economic cluster, objects (i.e., institutions and companies) are interdependent and interconnected by (in)formal links and are located in a relatively short distance from one another. Both competition and cooperation occur. Companies in a cluster aim both at their own economic outcomes as well as the development of the region. Last but not least, objects draw a substantial productive advantage from their nearness and interconnections.

7. Pawlak's Rough Approximation of Sets of Objects

The foundations of the theory of rough sets were laid by Pawlak in the early 80's of the 20th century [40, 41, 42, 44]. The starting point was the notion of an information system recalled in Sect. 3. The classical *Pawlak rough set model* is based on an equivalence relation of indiscernibility of objects.

In this section, we concisely present Pawlak's rough set model based on an arbitrary equivalence relation ϱ on a non-empty set of objects U considered as the universe of discourse. Objects are considered as equivalent if they cannot be discerned from one another. As earlier, the equivalence

[9] Clusters can comprize hundreds of companies.

class $[u]$ of an object u is an elementary information granule. Any subset of U is viewed as a *concept*. Concepts being set-theoretical unions of elementary granules are referred to as *definable*. Even if a concept is not definable in this sense, it can be approximated from the inside and the outside by a pair of concepts which are already definable. In this way, we arrive at the famous notions of the *lower* and *upper rough approximations* of a concept. For any concept $X \subseteq U$, the lower and upper rough approximations of X, $\mathrm{low}^{\cup}X$ and $\mathrm{upp}^{\cup}X$, respectively, are defined by

$$\mathrm{low}^{\cup}X = \bigcup\{[u] \mid [u] \subseteq X\} \;\;\&\;\; \mathrm{upp}^{\cup}X = \bigcup\{[u] \mid [u] \cap X \neq \emptyset\}. \quad (17)$$

$\mathrm{low}^{\cup}X$, viewed as the *positive region* of X, is the largest definable concept included in X. In other words, $\mathrm{low}^{\cup}X$ consists of all objects u which certainly belong to X since all objects indiscernible from u are members of X. On the other hand, $\mathrm{upp}^{\cup}X$ is the least definable concept containing X or, from another standpoint, $\mathrm{upp}^{\cup}X$ is the set of all objects u which possibly belong to X as at least one object indiscernible from u is a member of X. The complement of $\mathrm{upp}^{\cup}X$ may be viewed as the *negative region* of X. The difference

$$\mathrm{bnd}^{\cup}X \stackrel{\mathrm{def}}{=} \mathrm{upp}^{\cup}X - \mathrm{low}^{\cup}X \quad (18)$$

is called the *boundary region* of X. A concept X is *exact* if its boundary region is empty; otherwise X is referred to as *rough*. It turns out that a concept is definable if and only if it is exact. Moreover, the pair (U, ϱ) is called a *rough approximation space*. It is worth noting that the lower and upper rough approximations of a concept may also be given by

$$\mathrm{low}X = \{u \mid [u] \subseteq X\} \;\;\&\;\; \mathrm{upp}X = \{u \mid [u] \cap X \neq \emptyset\}, \quad (19)$$

respectively, whereas the boundary region of X may be defined as $\mathrm{bnd}X = \mathrm{upp}X - \mathrm{low}X$. Indeed, one can easily see that

$$\begin{aligned} \bigcup\{[u] \mid [u] \subseteq X\} &= \{u \mid [u] \subseteq X\}, \\ \bigcup\{[u] \mid [u] \cap X \neq \emptyset\} &= \{u \mid [u] \cap X \neq \emptyset\}. \end{aligned} \quad (20)$$

Let us recall fundamental properties of the lower and upper rough approximations.

Proposition 7.1. *For any concepts X, Y, it holds:*

(a) $\mathrm{low}X \subseteq X \subseteq \mathrm{upp}X$

(b) $\mathrm{low}\emptyset = \mathrm{upp}\emptyset = \emptyset \;\;\&\;\; \mathrm{low}U = \mathrm{upp}U = U$

(c) $X \subseteq Y \Rightarrow \mathrm{low}X \subseteq \mathrm{low}Y$ & $\mathrm{upp}X \subseteq \mathrm{upp}Y$

(d) $\mathrm{low}(X \cup Y) \supseteq \mathrm{low}X \cup \mathrm{low}Y$ & $\mathrm{low}(X \cap Y) = \mathrm{low}X \cap \mathrm{low}Y$

(e) $\mathrm{upp}(X \cup Y) = \mathrm{upp}X \cup \mathrm{upp}Y$ & $\mathrm{upp}(X \cap Y) \subseteq \mathrm{upp}X \cap \mathrm{upp}Y$

(f) $\mathrm{upp}X = U - \mathrm{low}(U - X)$

(g) $\mathrm{low}(\mathrm{low}X) = \mathrm{upp}(\mathrm{low}X) = \mathrm{low}X$ & $\mathrm{upp}(\mathrm{upp}X) = \mathrm{low}(\mathrm{upp}X) = \mathrm{upp}X$

Some comments can be handy. According to (a), the lower rough approximation of a concept is included in that concept, and every concept is included in its upper rough approximation. In virtue of (b), both the empty set and the whole universe are exact (and definable) concepts. Due to (c), the operations of lower and upper rough approximations are monotone. (d) says that the lower rough approximation of the union of two concepts contains the lower rough approximations of both concepts, whereas the lower rough approximation of the intersection of two concepts equals to the intersection of the lower rough approximations of those concepts. (e) is an analogous property for the upper rough approximation, viz., the upper rough approximation of the union of two concepts coincides with the union of the upper rough approximations of these concepts, and the upper rough approximation of the intersection of two concepts is included in the intersection of the upper rough approximations of the concepts. By (f), the operations of lower and upper rough approximations are dual. Finally, (g) says that the lower and upper rough approximations of a concept are already exact (and also definable) concepts.

Example 7.2. *(Continuation of Example 3.1.) The uncertainty mapping* Γ *induces the partition* $P = \{\{0\}, \{1\}, \{2, 4\}, \{3, 7\}, \{5\}, \{6\}\}$ *of* U*. Starting with* P*, one can compute the lower and upper rough approximations of the concept* $X = \{1, 2, 7\}$ *as the sets* $\{1\}$ *and* $\{1, 2, 3, 4, 7\}$*, respectively. Since the boundary region of* X *is equal to* $\{2, 3, 4, 7\}$*,* X *is rough and undefinable in terms of information granules constituting* P*. Examples of exact and definable sets are* $\{1\}$ *and* $\{1, 2, 3, 4, 7\}$*.*

8. Two Generalizations of Pawlak's Approach

In this section, we recall two important generalizations of the Pawlak rough set model. First, Ziarko [82, 83] introduced a model known as the *variable-precision rough set model* (VPRS-model for short) and based on an indiscernibility relation, where concepts are approximated in terms of variable-precision positive and negative regions instead of lower and upper rough approximations. Next, Skowron and Stepaniuk [59, 60] presented a rough set model based on a tolerance relation[10], where concepts are approximated by means of rough lower and upper approximations, yet defined differently than in the classical case. In Skowron–Stepaniuk's framework, *rough inclusion functions* (RIFs for short) play an important role and the fundamental notion is that of a *parameterized approximation space*. The classical Pawlak's model was extended and refined in several other directions as well. For instance, approximation spaces based on arbitrary non-empty binary relations are investigated in [19]. The reader interested in other developments is referred, e.g., to [21, 27, 47, 48, 51, 55, 67, 68, 69, 70, 71, 73, 74, 76].

8.1. Rough Inclusion Functions and Rough Membership Functions

Broadly speaking, a rough inclusion function (RIF) upon U is a mapping $\kappa : (\wp U)^2 \mapsto [0,1]$ which measures the degree of inclusion of a concept in a concept of U. Polkowski and Skowron have axiomatically characterized RIFs within *rough mereology*, a formal theory of the notion of being-part-to-degree [49, 50, 52, 53, 54]. The most known example of a RIF is the *standard* one, $\kappa^{£}$, defined for finite universes. The idea behind this notion goes back to Łukasiewicz [33]. For any finite concepts X, Y, $\kappa^{£}(X,Y) = \#(X \cap Y)/\#X$ if X is non-empty, and $\kappa^{£}(\emptyset, Y) = 1$. Clearly, the definition may be extended to the case of infinite second arguments.

In our approach, every RIF κ is assumed to fulfil rif_1 and rif_2 given below:

$$\mathrm{rif}_1(\kappa) \overset{\mathrm{def}}{\Leftrightarrow} \forall X, Y.(\kappa(X,Y) = 1 \Leftrightarrow X \subseteq Y)$$
$$\mathrm{rif}_2(\kappa) \overset{\mathrm{def}}{\Leftrightarrow} \forall X, Y, Z.(Y \subseteq Z \Rightarrow \kappa(X,Y) \leq \kappa(X,Z))$$

The above conditions are in accordance with the axioms of rough mereology. One can easily see that $\kappa^{£}$ satisfies both of them. Other exemplary properties which might also be postulated are

$$\mathrm{rif}_3(\kappa) \overset{\mathrm{def}}{\Leftrightarrow} \forall X \neq \emptyset.\kappa(X, \emptyset) = 0,$$
$$\mathrm{rif}_4(\kappa) \overset{\mathrm{def}}{\Leftrightarrow} \forall X \neq \emptyset.\forall Y.(\kappa(X,Y) = 0 \Leftrightarrow X \cap Y = \emptyset),$$
$$\mathrm{rif}_5(\kappa) \overset{\mathrm{def}}{\Leftrightarrow} \forall X \neq \emptyset.\forall Y.\kappa(X,Y) + \kappa(X, U - Y) = 1.$$

Like in the fuzzy set theory, one can measure the degree of membership of an object in a concept of U. Rough membership functions just serve the pur-

[10] This assumption was relaxed to the case of a similarity relation later on.

pose. In this way, the crisp notion of membership in a set is extended to the case of reasoning under vague information within the rough set theory [46]. Given a concept X in a rough approximation space (U, ϱ), where U is finite and ϱ is an indiscernibility relation, the *rough X-membership function* is defined in [46] as a mapping $\mu_X : U \mapsto [0,1]$ such that for any $u \in U$,

$$\mu_X(u) = \#([u] \cap X)/\#[u], \tag{21}$$

i.e., $\mu_X(u) = \kappa^{\pounds}([u], X)$. This definition can be extended to the case of an arbitrary universe of objects U, any RIF κ upon U, and any uncertainty mapping $\Gamma : U \mapsto \wp U$. Namely, for any concept X, the rough X-membership function, $\mu_X : U \mapsto [0,1]$, may be defined as follows, for any object u:

$$\mu_X(u) = \kappa(\Gamma u, X) \tag{22}$$

82. The VPRS Model

Ziarko refined Pawlak's rough set model by introducing degrees of precision of approximation [82, 83]. In Ziarko's framework, concepts are approximated in terms of the s-negative regions and the t-positive regions of concepts, where the significance threshold parameters s, t satisfy $0 \leq s < t \leq 1$. The original VPRS model is based on an equivalence relation, say ϱ, of indiscernibility of objects. For every target concept $X \subseteq U$, the existence of a prior probability value $\Pr(X)$ is assumed. The value can be estimated from a finite data sample $U' \subseteq U$ by $\Pr(X) = \kappa^{\pounds}(U', X)$. $\Pr(X)$ is understood as the probability of the event that an object belongs to X. With every equivalence class $[u]$, viewed as an elementary information granule as earlier, there is associated a conditional probability value $\Pr(X|[u])$, understood as the probability that an object u' belongs to X provided that u' is indiscernible from u. For finite equivalence classes, this probability is estimated by $\Pr(X|[u]) = \kappa^{\pounds}([u], X)$. Instead of lower and upper rough approximations, variable-precision negative and positive regions of concepts are considered. In detail, the *s-negative region*, the *t-positive region*, and the *(s,t)-boundary region* of a concept X, written $\mathrm{neg}^{\cup}_s X$, $\mathrm{pos}^{\cup}_t X$, and $\mathrm{bnd}^{\cup}_{s,t} X$, respectively, are defined by

$$\begin{aligned}
\mathrm{neg}^{\cup}_s X &= \bigcup\{[u] \mid \Pr(X|[u]) \leq s\}, \\
\mathrm{pos}^{\cup}_t X &= \bigcup\{[u] \mid \Pr(X|[u]) \geq t\}, \\
\mathrm{bnd}^{\cup}_{s,t} X &= \bigcup\{[u] \mid s < \Pr(X|[u]) < t\}.
\end{aligned} \tag{23}$$

In words, the s-negative (resp., t-positive) region of X is the union of all granules $[u]$ that the conditional probability $\Pr(X|[u])$ does not exceed (resp., is not less than) a given threshold value s (resp., t). The (s,t)-boundary region of X is the union of all remaining equivalence classes. For finite U, the above equalities take the following forms:

$$\begin{aligned} \mathrm{neg}_s^{\cup} X &= \bigcup\{[u] \mid \kappa^{\pounds}([u], X) \leq s\} \\ \mathrm{pos}_t^{\cup} X &= \bigcup\{[u] \mid \kappa^{\pounds}([u], X) \geq t\} \\ \mathrm{bnd}_{s,t}^{\cup} X &= \bigcup\{[u] \mid s < \kappa^{\pounds}([u], X) < t\} \end{aligned} \tag{24}$$

As in the case of Pawlak's lower and upper rough approximations, the variable-precision negative, positive, and boundary regions can be defined by

$$\begin{aligned} \mathrm{neg}_s X &= \{u \mid \kappa^{\pounds}([u], X) \leq s\}, \\ \mathrm{pos}_t X &= \{u \mid \kappa^{\pounds}([u], X) \geq t\}, \\ \mathrm{bnd}_{s,t} X &= \{u \mid s < \kappa^{\pounds}([u], X) < t\}, \end{aligned} \tag{25}$$

respectively, since $\mathrm{neg}_s^{\cup} X = \mathrm{neg}_s X$, $\mathrm{pos}_t^{\cup} X = \mathrm{pos}_t X$, and $\mathrm{bnd}_{s,t}^{\cup} X = \mathrm{bnd}_{s,t} X$. Moreover, if $s = 0$ and $t = 1$, then $\mathrm{neg}_s X = U - \mathrm{upp} X$ and $\mathrm{pos}_t X = \mathrm{low} X$, i.e., the s-negative region and the t-positive region become the negative region and the positive region, respectively.

The notions of variable-precision negative and positive regions of a concept can be generalized to the case of an arbitrary RIF κ and an arbitrary similarity relation ϱ. After necessary modifications of (24) and (25), we can obtain, for example,

$$\begin{aligned} \mathrm{neg}_s^{\cup} X &= \bigcup\{\varrho^{\leftarrow}\{u\} \mid \kappa(\varrho^{\leftarrow}\{u\}, X) \leq s\}, \\ \mathrm{pos}_t^{\cup} X &= \bigcup\{\varrho^{\leftarrow}\{u\} \mid \kappa(\varrho^{\leftarrow}\{u\}, X) \geq t\}, \\ \mathrm{bnd}_{s,t}^{\cup} X &= \bigcup\{\varrho^{\leftarrow}\{u\} \mid s < \kappa(\varrho^{\leftarrow}\{u\}, X) < t\}, \\ \mathrm{neg}_s X &= \{u \mid \kappa(\varrho^{\leftarrow}\{u\}, X) \leq s\}, \\ \mathrm{pos}_t X &= \{u \mid \kappa(\varrho^{\leftarrow}\{u\}, X) \geq t\}, \\ \mathrm{bnd}_{s,t} X &= \{u \mid s < \kappa(\varrho^{\leftarrow}\{u\}, X) < t\}. \end{aligned} \tag{26}$$

Unlike in the indiscernibility-based case, $\mathrm{neg}_s^{\cup} X$ and $\mathrm{neg}_s X$ need not to coincide, and analogously for the remaining pairs of concepts.

Example 8.1. *(Continuation of Example 3.1.) Suppose that the granulation of U is induced by the uncertainty mapping Γ_{a_2}, and the RIF considered is standard. In Table 3, we give the t-positive and t-negative regions of the concept $X = \{1, 2, 7\}$ for various values of t. As regarding the variable-precision* *boundary regions, we obtain, e.g.,* $\mathrm{bnd}_{s,t}X = \{4,5,6\}$ *for* $0 < s \leq 1/4$ *and* $1/3 < t \leq 1/2$.

Table 3

The t-positive and t-negative regions of X

t	$\mathrm{pos}_t X$	$\mathrm{neg}_t X$
$\{0\}$	U	$\{0\}$
$(0, 1/4]$	$\{1, \ldots, 7\}$	$\{0, 3\}$
$(1/4, 1/3]$	$\{1, 2, 4, 5, 6, 7\}$	$\{0, 3, 4, 5, 6\}$
$(1/3, 1/2]$	$\{1, 2, 7\}$	$\{0, 2, \ldots, 7\}$
$(1/2, 1]$	$\{1\}$	U

8.3. Parameterized Approximation Spaces

Skowron and Stepaniuk generalized Pawlak's rough set model to the case of similarity-based approximation of concepts. The notion of a parameterized approximation space, introduced in [59, 60] and elaborated in a series of articles, influenced the further research on the theory and the applications of approximation spaces. Let U be a non-empty set of objects, \$ be a list of tuning parameters to obtain a satisfactory quality of approximation, $\Gamma_\$: U \mapsto \wp U$ be a mapping called an *uncertainty* mapping and assigning to every object $u \in U$, a set of objects in some sense similar to u, and $\kappa_\$$ be a RIF upon U. It is assumed that $\forall u.u \in \Gamma_\$ u$, and indiscernibility is treated as a special case of similarity. Elementary granules of information are of the form $\Gamma_\$ u$, where u is an object of U.

A parameterized approximation space is a triple $\mathcal{M}_\$ = (U, \Gamma_\$, \kappa_\$)$, where any concept $X \subseteq U$ can be approximated by its *lower* and *upper approximations*, $\mathrm{low}^{\mathrm{S}} X$ and $\mathrm{upp}^{\mathrm{S}} X$, respectively, such that

$$\mathrm{low}^{\mathrm{S}} X \stackrel{\mathrm{def}}{=} \{u \mid \kappa_\$(\Gamma_\$ u, X) = 1\} \ \& \ \mathrm{upp}^{\mathrm{S}} X \stackrel{\mathrm{def}}{=} \{u \mid \kappa_\$(\Gamma_\$ u, X) > 0\}. \quad (27)$$

According to this definition, $\mathrm{low}^{\mathrm{S}} X$ consists of all objects u that their elementary granules are included in X to the degree 1. On the other hand, $\mathrm{upp}^{\mathrm{S}} X$ is the set of all objects that their elementary granules are included in X to some positive degree. The *boundary region* of X, $\mathrm{bnd}^{\mathrm{S}} X$, can be defined as the set

$$\mathrm{bnd}^{\mathrm{S}} X = \{u \mid 0 < \kappa_\$(\Gamma_\$ u, X) < 1\}. \quad (28)$$

For simplicity, we shall omit \$ unless necessary.

In parameterized approximation spaces, concepts can also be approximated in a number of other ways, e.g. in line with Pawlak's rough approximation. By way of example, the lower and upper approximations of X may be given by

$$\mathrm{low}X \stackrel{\mathrm{def}}{=} \{u \mid \Gamma u \subseteq X\} \;\;\&\;\; \mathrm{upp}X \stackrel{\mathrm{def}}{=} \{u \mid \Gamma u \cap X \neq \emptyset\}, \tag{29}$$

respectively, whereas the boundary region of X may be defined as $\mathrm{bnd}X = \mathrm{upp}X - \mathrm{low}X$. By $\mathrm{rif}_1(\kappa)$, $\mathrm{low}^{\mathrm{S}}X = \mathrm{low}X$, yet $\mathrm{upp}^{\mathrm{S}}X = \mathrm{upp}X$ needs not to hold[11]. When κ is standard and $\Gamma^{\rightarrow}U$ is a partition of U, the lower and upper approximations defined by (27) coincide with Pawlak's lower and upper rough approximations, respectively.

Every uncertainty mapping Γ induces a reflexive relation ϱ_Γ on U such that $(u', u) \in \varrho_\Gamma \stackrel{\mathrm{def}}{\Leftrightarrow} u' \in \Gamma u$ (cf. (6)), and vice versa, starting with a reflexive relation ϱ on U understood as a relation of similarity of objects, an uncertainty mapping $\Gamma_\varrho : U \mapsto \wp U$, defined by $u' \in \Gamma_\varrho u \stackrel{\mathrm{def}}{\Leftrightarrow} (u', u) \in \varrho$, can be derived[12]. Therefore, we can think of a *similarity-based rough approximation space* as a structure $\mathcal{M} = (U, \Gamma, \kappa)$ above or $\mathcal{N} = (U, \varrho, \kappa)$, where ϱ is a reflexive relation on U.

Example 8.2. *Consider an approximation space* $M = (U, \Gamma', \kappa)$*, where* U *and* Γ' *are as in Example 3.1, and* κ *is a RIF such that for any non-empty concept* X *and any concept* Y*,* $\kappa(X, Y) = 0$ *if and only if* $\mathrm{upp}X \cap \mathrm{upp}Y = \emptyset$*. Hence,* $X \cap Y \neq \emptyset$ *implies* $\kappa(X, Y) > 0$*, but the converse may not hold. Indeed,* $\{3\}, \{1, 2, 7\}$ *are disjoint, whereas* $\mathrm{upp}\{3\} \cap \mathrm{upp}\{1, 2, 7\} = \{0, 3, 7\} \cap \{1, 2, 3, 7\} = \{3, 7\}$*, i.e.,* $\kappa(\{3\}, \{1, 2, 7\}) > 0$*. Furthermore, for any concept* X*,* $\mathrm{upp}X \subseteq \mathrm{upp}^{\mathrm{S}}X$*. However, the both forms of approximation are different. Namely,* $\mathrm{upp}\{1, 2, 7\} = \{u \mid \Gamma' u \cap \{1, 2, 7\} \neq \emptyset\} = \{1, 2, 3, 7\}$*, whereas* $\mathrm{upp}^{\mathrm{S}}\{1,2,7\} = \{u \mid \kappa(\Gamma' u, \{1,2,7\}) > 0\} = \{u \mid \mathrm{upp}(\Gamma' u) \cap \{1,2,3,7\} \neq \emptyset\} = \{0, 1, 2, 3, 7\}$*.*

9. Summary

In this paper, we discussed the possibility of using the methodologies of rough sets and granular computing to build a rule-based model of a multiagent system. The main assumption of our approach is that the universe

[11] Among others, the equality holds true if κ is standard.

[12] As a matter of fact, ϱ induces two different uncertainty mappings unless ϱ is symmetric. Apart from Γ_ϱ, we obtain Γ^*_ϱ such that $u' \in \Gamma^*_\varrho u \stackrel{\mathrm{def}}{\Leftrightarrow} (u, u') \in \varrho$.

of all objects considered is granulated into information granules, i.e. clusters of objects formed on the basis of similarity or functionality. According to our idea, both agents and their systems, being complex objects of some kind, may be viewed as information granules. Apart from a brief overview of various aspects regarding the modelling of agent systems, we presented several examples of information granules which can be useful in such modelling. Furthermore, the classical Pawlak model of rough sets and its two extensions were recalled. Various detailed questions concerning social agent system modelling, mentioned in Introduction, will be elaborated in the future work.

References

[1] R. Agrawal, T. Imieliński and A. Swami. Mining association rules between sets of items in large databases. In *Proc. ACM SIGMOD Int. Conf. on Management of Data*, pages 207–216, Washington, D.C., 1993.

[2] R. Agrawal and R. Srikant. Fast algorithms for mining association rules. In *Proc. 20th Conf. on Very Large Databases*, pages 487–499, Santiago, Chile, 1994.

[3] C. E. Alchourrón, P. Gärdenfors and D. Makinson. On the logic of theory change: Partial meet contraction and revision functions. *J. Symbolic Logic*, 50:510–530, 1985.

[4] J. Bazan. A comparison of dynamic and non-dynamic rough set methods for extracting laws from decision tables. In *[51]*, volume 1, pages 321–365. 1998.

[5] J. Bazan and A. Skowron. Classifiers based on approximate reasoning schemes. In *[16]*, pages 191–202. 2005.

[6] R. Breault. *The evolution of structured clusters*. 2000. http://www.ptb-magazine.com/July00/ptb700.clusters2.html.

[7] T. R. Burns, J. C. Caldas and E. Roszkowska. Generalized game theory's contribution to multi-agent modelling. In *[16]*, pages 363–384. 2005.

[8] T. R. Burns and E. Engdahl. The social construction of consciousness: Collective consciousness and its socio-cultural foundations; Individual selves, self-awareness, and reflectivity, parts I and II. *J. Consciousness Studies*, 5(1–2):67–85, 1998.

[9] T. R. Burns and A. Gomolińska. The theory of socially embedded games: The mathematics of social relationships, rule complexes, and action modalities. *Quality and Quantity Int. J. Methodology*, 34(4):379–406, 2000.

[10] T. R. Burns and A. Gomolińska. Socio-cognitive mechanisms of belief change: Applications of generalized game theory to belief revision, social fabrication, and self-fulfilling prophesy. *Cognitive Systems Research*, 2(1):39–54, 2001.

[11] T. R. Burns, A. Gomolińska and L. D. Meeker. The theory of socially embedded games: Applications and extensions to open and closed games. *Quality and Quantity Int. J. Methodology*, 35(1):1–32, 2001.

[12] T. R. Burns and E. Roszkowska. Generalized game theory: Assumptions, principles, and elaborations in social theory. *Studies in Logic, Grammar and Rhetoric*, 8(21):7–40, 2005.

[13] B. Dunin-Kęplicz and R. Verbrugge. Collective intentions. *Fundamenta Informaticae*, 51(3):271–295, 2002.

[14] B. Dunin-Kęplicz and R. Verbrugge. Evolution of collective commitments during teamwork. *Fundamenta Informaticae*, 56(3):329–371, 2003.

[15] B. Dunin-Kęplicz and R. Verbrugge. A tuning machine for cooperative problem solving. *Fundamenta Informaticae*, 63(2–3):283–307, 2004.

[16] B. Dunin-Kęplicz et al., editor. *Monitoring, Security, and Rescue Techniques in Multiagent Systems*. Springer, Berlin, 2005.

[17] R. Fagin. *Reasoning About Knowledge*. The MIT Press, Cambridge, MA, 1995.

[18] P. Gärdenfors, editor. *Belief Revision*. Cambridge University Press, 1992.

[19] A. Gomolińska. A comparison of Pawlak's and Skowron-Stepaniuk's approximation of concepts. To appear in LNCS *Transactions on Rough Sets*.

[20] A. Gomolińska. Rule complexes for representing social actors and interactions. *Studies in Logic, Grammar and Rhetoric*, 3(16):95–108, 1999.

[21] A. Gomolińska. Variable-precision compatibility spaces. *Electronical Notices in Theoretical Computer Science*, 82(4):1–12, 2003. http://www.elsevier.nl/locate/entcs/volume82.html.

[22] A. Gomolińska. Fundamental mathematical notions of the theory of socially embedded games: A granular computing perspective. In *[38]*, pages 411–434. 2004.

[23] A. Gomolińska. Modelowanie systemów wieloagentowych w terminach granul informacyjnych. In K. Trzęsicki, editor, *Ratione et Studio*, pages 127–145. University of Białystok, Białystok, 2005.

[24] A. Gomolińska. Rough validity, confidence, and coverage of rules in approximation spaces. *Transactions on Rough Sets III: Journal Subline, Lecture Notes in Computer Science*, 3400:57–81, 2005.

[25] A. Gomolińska. Satisfiability and meaning of formulas and sets of formulas in approximation spaces. *Fundamenta Informaticae*, 67(1–3):77–92, 2005.

[26] A. Gomolińska. Towards rough applicability of rules. In *[16]*, pages 203–214. 2005.

[27] S. Greco, B. Matarazzo and R. Słowiński. Fuzzy similarity relation as a basis for rough approximations. *Lecture Notes in Artificial Intelligence*, 1424:283–289, 1998.

[28] J. W. Grzymała-Busse. Classification of unseen examples under uncertainty. *Fundamenta Informaticae*, 30(3–4):255–267, 1997.

[29] J. W. Grzymała-Busse. LERS – A data mining system. In O. Maimon and L. Rokach, editors, *The Data Mining and Knowledge Discovery Handbook*, pages 1347–1351. Springer, Heidelberg, 2005.

[30] J. W. Grzymała-Busse. Rule induction. In O. Maimon and L. Rokach, editors, *The Data Mining and Knowledge Discovery Handbook*, pages 255–267. Springer, Heidelberg, 2005.

[31] M. Inuiguchi, S. Hirano and S. Tsumoto, editors. *Rough Set Theory and Granular Computing*, volume 125 of *Studies in Fuzziness and Soft Computing*. Springer, Berlin Heidelberg, 2003.

[32] M. Kryszkiewicz. Fast discovery of representative association rules. *Lecture Notes in Artificial Intelligence*, 1424:214–221, 1998.

[33] J. Łukasiewicz. Die logischen Grundlagen der Wahrscheinlichkeitsrechnung. In L. Borkowski, editor, *Jan Łukasiewicz – Selected Works*, pages 16–63. North Holland, Polish Scientific Publ., Amsterdam London, Warsaw, 1970. First published in Cracow in 1913.

[34] H. S. Nguyen and S. H. Nguyen. Rough sets and association rule generation. *Fundamenta Informaticae*, 40(4):383–405, 1999.

[35] H. S. Nguyen, A. Skowron and J. Stepaniuk. Granular computing: A rough set approach. *Int. J. Comput. Intelligence*, 17(3):514–544, 2001.

[36] H. S. Nguyen and D. Ślęzak. Approximate reducts and association rules – correspondence and complexity results. *Lecture Notes in Artificial Intelligence*, 1711:137–145, 1999.

[37] S. H. Nguyen et al. Layered learning for concept synthesis. *Transactions on Rough Sets I: Journal Subline, Lecture Notes in Computer Science*, 3100:187–208, 2004.

[38] S. K. Pal, L. Polkowski and A. Skowron, editors. *Rough-Neural Computing: Techniques for Computing with Words.* Springer, Berlin Heidelberg, 2004.

[39] Z. Pawlak. Information systems – Theoretical foundations. *Information Systems*, 6(3):205–218, 1981.

[40] Z. Pawlak. *Rough Sets, Basic Notions.* Number 431 in ICS PAS Report. Warsaw, 1981.

[41] Z. Pawlak. Rough Sets. *Int. J. Computer and Information Sciences*, 11:341–356, 1982.

[42] Z. Pawlak. *Systemy informacyjne. Podstawy teoretyczne.* Wydawnictwo Naukowo-Techniczne, Warsaw, 1983.

[43] Z. Pawlak. Rough sets and decision tables. *Lecture Notes in Computer Science*, 208:186–196, 1985.

[44] Z. Pawlak. *Rough Sets. Theoretical Aspects of Reasoning About Data.* Kluwer, Dordrecht, 1991.

[45] Z. Pawlak, L. Polkowski and A. Skowron. Rough sets and rough logic: A KDD perspective. In *[55]*, pages 583–646. 2001.

[46] Z. Pawlak and A. Skowron. Rough membership functions. In M. Fedrizzi, J. Kacprzyk and R. R. Yager, editors, *Advances in the Dempster-Shafer Theory of Evidence*, pages 251–271. John Wiley & Sons, New York, 1994.

[47] J. F. Peters. Approximation space for intelligent system design patterns. *Engineering Applications of Artificial Intelligence*, 17(4):1–8, 2004.

[48] L. Polkowski. *Rough Sets: Mathematical Foundations.* Physica-Verlag, Heidelberg, 2002.

[49] L. Polkowski and A. Skowron. Rough mereology. *Lecture Notes in Artificial Intelligence*, 869:85–94, 1994.

[50] L. Polkowski and A. Skowron. Rough mereology: A new paradigm for approximate reasoning. *Int. J. Approximated Reasoning*, 15(4):333–365, 1996.

[51] L. Polkowski and A. Skowron, editors. *Rough Sets in Knowledge Discovery*, volume 1–2. Physica-Verlag, Heidelberg, 1998.

[52] L. Polkowski and A. Skowron. Towards adaptive calculus of granules. In *[81]*, volume 1, pages 201–228. 1999.

[53] L. Polkowski and A. Skowron. Rough mereological calculi of granules: A rough set approach to computation. *Int. J. Comput. Intelligence*, 17(3):472–492, 2001.

[54] L. Polkowski and A. Skowron. Rough mereology in information systems. A case study: Qualitative spatial reasoning. In *[55]*, pages 89–135. 2001.

[55] L. Polkowski, S. Tsumoto and T. Y. Lin, editors. *Rough Set Methods and Applications: New Developments in Knowledge Discovery in Information Systems*. Physica-Verlag, Heidelberg, 2001.

[56] M. E. Porter. Clusters and the new economics of competition. *Harvard Business Review*, 76(6):77–90, 1998.

[57] M. E. Porter. *On Competition*. Harvard Business School Press, Boston, MA, 1998.

[58] A. Skowron and C. M. Rauszer. The discernibility matrices and functions in information systems. In R. Słowiński, editor, *Intelligent Decision Support – Handbook of Applications and Advances of the Rough Sets Theory*, pages 331–362. Kluwer, Dordrecht, 1992.

[59] A. Skowron and J. Stepaniuk. Generalized approximation spaces. In *Proc. 3rd Int. Workshop on Rough Sets and Soft Computing, San Jose, USA, November 1994*, pages 156–163, 1994.

[60] A. Skowron and J. Stepaniuk. Tolerance approximation spaces. *Fundamenta Informaticae*, 27(2–3):245–253, 1996.

[61] A. Skowron and J. Stepaniuk. Information granules in distributed environment. *Lecture Notes in Artificial Intelligence*, 1711:357–365, 1999.

[62] A. Skowron and J. Stepaniuk. Towards discovery of information granules. *Lecture Notes in Artificial Intelligence*, 1704:542–547, 1999.

[63] A. Skowron and J. Stepaniuk. Information granules and rough-neural computing. In *[38]*, pages 43–84. 2004.

[64] A. Skowron and J. Stepaniuk. Ontological framework for approximation. *Lecture Notes in Artificial Intelligence*, 3641:718–727, 2005.

[65] A. Skowron, J. Stepaniuk and S. Tsumoto. Information granules for spatial reasoning. *Bull. Int. Rough Set Society*, 3(4):147–154, 1999.

[66] A. Skowron, R. Swiniarski and P. Synak. Approximation spaces and information granulation. *Transactions on Rough Sets III: Journal Subline, Lecture Notes in Computer Science*, 3400:175–189, 2005.

[67] R. Słowiński and D. Vanderpooten. Similarity relation as a basis for rough approximations. In P. P. Wang, editor, *Advances in Machine Intelligence and Soft Computing*, volume 4, pages 17–33. Duke University Press, 1997. Also in *ICS Research Report* 53/95, Warsaw University of Technology, Warsaw, 1995.

[68] R. Słowiński and D. Vanderpooten. A generalized definition of rough approximations based on similarity. *IEEE Transactions on Data and Knowledge Engineering*, 12:331–336, 2000.

[69] J. Stefanowski, *Algorytmy indukcji reguł decyzyjnych w odkrywaniu wiedzy*, volume 361 of *Rozprawy*. Wydawnictwo Politechniki Poznańskiej, Poznań, 2001.

[70] J. Stefanowski and A. Tsoukiàs. Incomplete Information Tables and Rough Classification. *Int. J. Comput. Intelligence*, 17(3):545–566, 2001.

[71] J. Stepaniuk. Knowledge discovery by application of rough set models. In *[55]*, pages 137–233. 2001.

[72] J. Stepaniuk. Tolerance information granules. In *[16]*, pages 305–316. 2005.

[73] S. Tsumoto et al., editor. *Proc. 4th Int. Conf. Rough Sets and Current Trends in Computing (RSCTC'2004), Uppsala, Sweden, June 2004*, volume 3066 of *Lecture Notes in Artificial Intelligence*. Springer, Berlin Heidelberg, 2004.

[74] S. K. M. Wong, L. S. Wang and Y. Y. Yao. On modelling uncertainty with interval structures. *Int. J. Computational Intelligence*, 11:406–426, 1995.

[75] Y. Y. Yao. Granular computing. In *Proc. of CRSSC'2004*, pages 1–5, 2004, ISSN 1002-137X.

[76] Y. Y. Yao, S. K. M. Wong and T. Y. Lin. A review of rough set models. In T. Y. Lin and N. Cercone, editors, *Rough Sets and Data Mining: Analysis of Imprecise Data*, pages 47–75. Kluwer, Boston London Dordrecht, 1997.

[77] L. A. Zadeh. Outline of a new approach to the analysis of complex system and decision processes. *IEEE Trans. on Systems, Man, and Cybernetics*, 3:28–44, 1973.

[78] L. A. Zadeh. Fuzzy sets and information granularity. In M. Gupta, R. Ragade and R. Yager, editors, *Advances in Fuzzy Set Theory and Applications*, pages 3–18. North-Holland, Amsterdam, 1979.

[79] L. A. Zadeh. Fuzzy logic = computing with words. *IEEE Trans. on Fuzzy Systems*, 4(2):103–111, 1996.

[80] L. A. Zadeh. Toward a theory of fuzzy information granulation and its certainty in human reasoning and fuzzy logic. *Fuzzy Sets and Systems*, 90:111–127, 1997.

[81] L. A. Zadeh and J. Kacprzyk, editors. *Computing with Words in Information/Intelligent Systems*. Physica-Verlag, Heidelberg, 1999.

[82] W. Ziarko. Variable precision rough set model. *J. Computer and System Sciences*, 46(1):39–59, 1993.

[83] W. Ziarko. Probabilistic decision tables in the variable precision rough set model. *Int. J. Comput. Intelligence*, 17(3):593–603, 2001.

Anna Gomolińska
University of Białystok
Department of Mathematics
Akademicka 2, 15-267 Białystok
anna.gom@math.uwb.edu.pl