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ON STRUCTURES AND THEIR ADEQUACY*

1. Introduction

While asking about the adequacy of our knowledge we most often ask about the adequacy of these sentences which represent this knowledge. We normally understand by this that the manner in which sentences describe objects to which they refer must be in accordance with what the objects really are. A precise reply to the question of adequacy requires thus the acceptance of a possibly broad notion of ontology. By ontology, following on from J. Perzanowski ([2004], p. 93), we shall understand the general theory of all essential possibilities. Within this framework we shall be able to speak about an ontology of the world (metaphysics), an ontology of language, an ontology of meanings, formal ontology, etc.

How to formulate questions about adequacy? Let us begin with quoting certain comments offered by R. Wójcicki, who – pointing to the work by H. Putnam [1989], as an introduction to the problem of adequacy, writes as follows: "*... problems of adequate representation of knowledge constitute, at present, one of the very intensively discussed issues of the so-called cognitive science*" ([1991], p. 85), and "*A lot of confusion around the definition of truth results from either mixing up truth with adequacy, or the mistaken view that either of these notions is redundant*" ([1996], p. 69).

So far the theory of adequacy, which would refer to the above-indicated problem area, has not yet been built. The most significant philosophical categories and the relations between them to determine a formulation of the problem of adequacy in the sense proposed by Wójcicki [1991] are indicated in Fig. 1. According to the diagram presented in it, posing the problem of adequacy of knowledge in science is a result of the two-way

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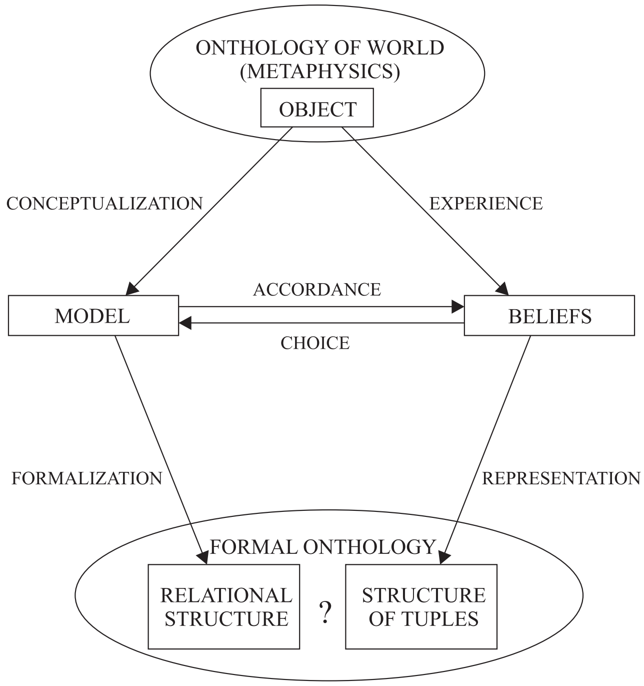
character of cognition: theoretical conceptualization and experimental verification.

Conceptualization in research into a concrete object leads to an appearance of its descriptive *models* understood as a certain kind of systems of representation of knowledge. These systems are imposed by linguistic communication or technical means (generally, through certain creations of culture) in such a way that the formalization of the processing of information on the object examined becomes possible on the grounds of a formal ontology, being most often the set theory or a mathematical theory built over the set theory (e.g. the theory of differential equations). Formal ontology is then treated here as a certain idealization of the ontology of the world. A result of the formalization of the knowledge of the object is the determination of the *relational structure* that reflects the features, properties and behaviour of the object (e.g. determination of the system of differential equations with appropriate boundary conditions).

Figure 1

A question about adequacy: Is the relational structure describing the object examined in accordance with the structure of tuples referring to the states of things which characterize the object by means of belief?



By means of experiments the knowledge about the states of things reflecting the features, properties and behavior of the object examined is collected. In this way *beliefs* are formed, which – in the language – are expressed by statements referring to the states of things which were discovered. Beliefs are represented on the grounds of formal ontology by means of expressions corresponding to *structures of tuples*, that is sets of ordered systems (*tuples*) of objects reflecting features, properties and behaviour of the object examined.

The model of the object, which is formed as a result of conceptualization, can be intended, i.e. chosen in such a way that the beliefs are in accordance with it (see Fig. 1). Then we can say that the model is adequate to the beliefs. All the models of a given object that are adequate to the same set of beliefs referring to this object are called *adequate models*. Accepting that E is any object, we can repeat after Wójcicki [1991], that it is an *adequate model of object* p iff the conditions E1–E7 as formulated below are satisfied:

E1. There has been determined *potential scope of applicability* E to p, i.e. a set of statements Ξ referring to E, which (see E2) can be translated into statements concerning p.

E2. There has been determined an effective procedure – *interpretation code* I – which allows translating any sentence of set Ξ into a sentence concerning p.

E3. There has been determined a set of procedures Δ, which allow effective decibility of any sentence $\alpha \in \Xi$.

E4. There has been determined the *real scope of applicability*, for short – *scope of applicability of* E *to* p, i.e. a set of statements $\Xi^* \subseteq \Xi$ such that for any sentence $\alpha \in \Xi^*$ the equivalence called condition of adequacy of model E is satisfied:

$$(*) \qquad \alpha \Leftrightarrow \mathrm{I}(\alpha).$$

E5. All the sentences belonging to the set of interpretation $\mathrm{I}(\Xi^*)$ are empirically decidable.

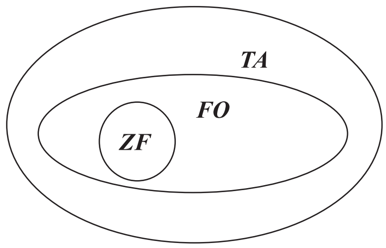
E6. There has been determined together with E, in practice, an infinitely numerous set of models Σ, satisfying the conditions E1–E3,

E7. For each model belonging to Σ there is determined the probability of this model being adequate.

Precise checking whether the condition of adequacy for the model of the object examined is satisfied, is possible only on the ground of formal ontology. Then, the formula which states being a *relational structure* – corresponding to being a descriptive model – is well defined. So is the formula which states being a *structure of tuples* – corresponding to being belief. Moreover, the formula settling the unambiguous relation between the object satisfying the first formula and the object satisfying the second one is defined in such a way when that knowing the first object, the second one could be determined.

Let us pay attention to intuitions which are a motivation to introduce formal ontology on the grounds of the set theory. These intuitions refer to the following cognitive schemata: the first stage of cognition, on the way of abstraction, of any object is the distinguishing of elements of this object and all the tuples which bind these elements. In this way, we get to know the *structure of tuples* of the object (the structure of the object) separately from the relations that allow determining tuples, which bind the elements of the object. We recognize these relations at the second stage of cognition of the object. The set of all the tuples belonging to these relations is a kind of set of generators (a *base structure*) forming all the tuples determining the structure of the object. According to intuition, the tuples binding elements of the object are allowed connections of elements of the object examined when they are determined by accessible cognitive means. Elements that are directly available to cognition are represented by a one-element tuple.

Figure 2



As regards the above-mentioned intuitions, the following question may be asked: How, having given relations, that is having imposed relational structure, can all the tuples be generated out of the structure of tuples characterizing the object (see Fig. 1.)? An answer to this question, on the ground of formal ontology, allows us to solve the problem of the adequacy (accordance) of the descriptive model with beliefs. The present work is an attempt at formulating basis of a formal theory of adequacy ***TA***. The theory of adequacy refers to the concept of adequacy in the sense proposed by Wójcicki, yet it takes into account the above intuitions. The formulated theory also makes references to certain notions of adequacy applied in logic and philosophy (ontology). Theory ***TA*** will be built over formal ontology ***FO*** developed over the set theory ***ZF*** (see Fig. 2).

2. A structure of tuples determined by the relational structure

Let U be any established, nonempty set of objects called the ***universe***.

Definition 1.

Each finished sequence $\langle \alpha_1, \alpha_2, \dots, \alpha_n \rangle \in U^n$ *of elements* $\alpha_1, \alpha_2, \dots, \alpha_n$ *of the universe* U*, where* $n \geq 1$ *and* U^n *is the Cartesian product of* n *sets* U*, is called a* ***tuple****. If the tuple has* n*-elements, then the number* n *is called its* ***length****.*

Definition 2.

Let t, t_1 *be tuples.*

$$t_1 \varepsilon t \Leftrightarrow t_1 \textit{ is a subsequence of } t.$$

The expression "$t_1 \varepsilon t$" is read: the tuple t_1 is a ***subtuple*** of the tuple t.

Let us accept a certain convention concerning one-element tuples and introduce certain notation:

1) For any $\alpha \in U$, a tuple $\langle \alpha \rangle$ will be identified with α.
2) For any set Z, the family of all subsets of Z is denoted by $\boldsymbol{P}(Z)$.
3) For any momentary set Z the set of all tuples of elements of Z is denoted by $S(Z)$, i.e.

   $S(Z) = \bigcup_{i \in N} Z^i$, where Z^i is the Cartesian product of i sets U.

Definition 3.

Any subset of the set $S(U)$ *is called a* ***structure of tuples of elements of the universe*** U*.*

In subsequent parts of this paper, instead of writing: *a structure of tuples of elements of the universe* U, we shall write short from: *a structure of tuples*.

The connections between the objects of the universe U are usually determined by means of relations defining the relational structure with the universe U. These relations are subsets of U^n for any established n; the one-argument relations are identified with certain distinguished subsets of the universe U. The family of these relations (certain sets of tuples of elements of the universe U) – called a *base* here – is characterized by the following definition:

Definition 4.

a) *The nonempty family of sets* $\boldsymbol{B} \subseteq \boldsymbol{P}(S(U))$ *is a* ***base*** *iff*

$$\forall X \in \boldsymbol{B}\ \exists n \in N\ (X \subseteq U^n).$$

b) *The nonempty family of sets* $\boldsymbol{B}_1$ *is a* ***subbase*** *of the base* $\boldsymbol{B}$ *iff*

$$\forall X \in \boldsymbol{B}_1 \ \exists Y \in \boldsymbol{B} \ (X \subseteq Y).$$

c) *The set* $BS^{\boldsymbol{B}} = \bigcup \boldsymbol{B}$ *is called a* ***base structure*** *for the base* $\boldsymbol{B}$*.*

Thus, the base structure is a certain structure of tuples to which belong all the tuples determined by relations from the base $\boldsymbol{B}$.

Fact 1.

If the base $\boldsymbol{B}_1$ *is a subbase of the base* $\boldsymbol{B}_2$*, then* $BS^{\boldsymbol{B}_1} \subseteq BS^{\boldsymbol{B}_2}$*.*

Further along in the work, we will understand by a relational structure an ordered system $\boldsymbol{Re} = \langle U^{\boldsymbol{Re}}, \{R_k\}_{k\in N}, I\rangle$, where $U^{\boldsymbol{Re}}$ is a nonempty subset of the universe U, $\{R_k\}_{k\in N}$ is a nonempty family of distinguished relations determined in $U^{\boldsymbol{Re}}$, and I is a distinguished subset of $U^{\boldsymbol{Re}}$.

Fact 2.

For any relational structure $\boldsymbol{Re} = \langle U^{\boldsymbol{Re}}, \{R_k\}_{k\in N}, I\rangle$*, the family of sets* $\boldsymbol{B}(\boldsymbol{Re}) = \{R_k\}_{k\in N} \cup \{I\}$ *is a base.*

Definition 5.

Let $\boldsymbol{Re} = \langle U^{\boldsymbol{Re}}, \{R_k\}_{k\in N}, I\rangle$ *be a relational structure. Then the base* $\boldsymbol{B}(\boldsymbol{Re}) = \{R_k\}_{k\in N} \cup \{I\}$ *is called a* ***base of relational structure Re****.*

We can determine a certain relational structure for any structure of tuples. This relational structure will be defined in the following way:

Definition 6.

Let $S \subseteq S(U)$*. We denote*

a) $U(S) = \{\alpha \in U : \exists t \in S \ \exists i \in N \ (t = \langle \alpha_1, \ldots, \alpha_i, \ldots, \alpha_n\rangle \wedge \alpha_i = \alpha)\}$.
$U(S)$ is a set of certain elements of the universe U, being – at the same time – elements of tuples of the structure S.

b) $\boldsymbol{R}(S) = \{R : \exists t \in S \ (R = \{t\})\}$.
$\boldsymbol{R}(S)$ is a set of one-element relations composed from particular tuples of the structure S.

c) *The relational structure* $\boldsymbol{Re}(S) = \langle U(S), \boldsymbol{R}(S), \emptyset\rangle$ *is called the* ***relational structure for structure of tuples*** *S.*

Fact 3.

Let $S \in S(U)$*. Then*

a) $\boldsymbol{B}(\boldsymbol{Re}(S)) = \boldsymbol{R}(S)$,

b) $BS^{\boldsymbol{B}(\boldsymbol{Re}(S))} = S$.

Now, we shall give some intuitions connected with the next definition referring to the **structure $S^{\boldsymbol{Re}}$ of tuples determined by any relational**

structure *Re*. The set (structure) of all connections (tuples) of the object examined is a set of connections available to cognition and determined by the relational structure ***Re***. The available connections are obtained in the following way: (C1) from connections of a set of generators, (C2) by joining, to produce chains, the connections obtained earlier, (C3) if a certain tuple is an allowed connection of distinguished elements with a given element by means of other elements, forming an allowed connection with this element, then also the distinguished elements form an allowed connection with it, and (C4) if a certain tuple represents an available to cognition (allowed) connection of elements, which are directly available to cognition, with a certain element, then this element is also directly available to cognition.

Let us first accept the definition of the *composition of two tuples*.

Definition 7.

Let $r = \langle \alpha_1, \alpha_2, \ldots, \alpha_j, \beta_1, \beta_2, \ldots, \beta_k \rangle \in S(U)$, $s = \langle \beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_l \rangle \in S(U)$ *and* $\langle \beta_1, \beta_2, \ldots, \beta_k \rangle \in S(U)$ *for* $j, k, l \geq 1$, *where* k *is the greatest number of common elements of* r *and* s. *Then the* ***composition of tuples*** r *and* s (*symbolically:* $r \bullet s$) *is the tuple* $\langle \alpha_1, \alpha_2, \ldots, \alpha_j, \beta_1, \beta_2, \ldots, \beta_k, \gamma_1, \gamma_2, \ldots, \gamma_l \rangle$.

Definition 8.

Let $BS^{\boldsymbol{B}(\boldsymbol{Re})}$ *be the base structure for the base* $\boldsymbol{B}(\boldsymbol{Re})$. *The set* $S^{\boldsymbol{Re}}$ *is a* ***structure of tuples determined by the relational structure Re*** *iff*

$$S^{\boldsymbol{Re}} = \bigcap \{S \subseteq S(U) : S \textit{ satisfies the conditions } \text{C1-C4}\},$$

i.e. $S^{\boldsymbol{Re}}$ *is the smallest set among the sets* $S \subseteq S(U)$ *satisfying the following conditions:*

(C1) ***(generators)***:

$$BS^{\boldsymbol{B}(\boldsymbol{Re})} \subseteq S,$$

(C2) ***(composition of tuples)***:

$$\forall t, s \in S \ (\exists r \in S(U) \ (t \bullet s = r) \Rightarrow t \bullet s \in S),$$

(C3) ***(reduction of tuples)***:

for any $\beta_1, \ldots, \beta_k, \alpha, \alpha_1, \ldots, \alpha_j, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_l \in U$, *where* $j, k, l \geq 1$,
if $\langle \beta_1, \ldots, \beta_k, \alpha \rangle \in S$ *and* $\langle \alpha_1, \ldots, \alpha_j, \beta_1, \ldots, \beta_k, \alpha, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_l \rangle \in S$
then $\langle \alpha_1, \ldots, \alpha_j, \alpha, \alpha_{j+1}, \alpha_{j+2}, \ldots, \alpha_l \rangle \in S$,

(C4) ***(detachment)***:

for any $\beta_1, \ldots, \beta_k, \alpha \in U$, *where* $k \geq 1$,
if $\langle \beta_1, \ldots, \beta_k, \alpha \rangle \in S$ *and* $\forall i = 1, \ldots, k \ (\beta_i \in S)$, *then* $\alpha \in S$.

A structure of tuples $S^{\boldsymbol{Re}}$ is a nonempty set, because the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re})} \neq \emptyset$.

From the above definition and Fact 1 there follows:

Fact 4.

Let $\boldsymbol{Re1}$ *and* $\boldsymbol{Re2}$ *be two relational structures of the same universe. If the base* $\boldsymbol{B}(\boldsymbol{Re1})$ *is a subbase of the base* $\boldsymbol{B}(\boldsymbol{Re2})$, *then* $S^{\boldsymbol{Re1}} \subseteq S^{\boldsymbol{Re2}}$.

3. Adequacy of structures

One of the most important conditions of the adequacy of structures is their homomorphism. The adequacy (agreement) of known conceptual constructions in logic and mathematics is often determined by means of this notion. Let us pay more attention to this notion.

Let two relational structures $\boldsymbol{Re1} = \langle U_1, \boldsymbol{R}_1, I_1\rangle$ and $\boldsymbol{Re2} = \langle U_2, \boldsymbol{R}_2, I_2\rangle$ be given, where $U_1, U_2 \subseteq U$. Let $\boldsymbol{B}(\boldsymbol{Re1})$ be the base of $\boldsymbol{Re1}$ and $\boldsymbol{B}(\boldsymbol{Re2})$ be the base of $\boldsymbol{Re2}$. Let $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ are structures of tuples determined by the relational structures $\boldsymbol{Re1}$ and $\boldsymbol{Re2}$, respectively.

We say that the relational structures $\boldsymbol{Re1}$ and $\boldsymbol{Re2}$ are *similar* (have the same signature), if there exists a function of *interpretation*

$$int : \boldsymbol{B}(\boldsymbol{Re1}) \longrightarrow \boldsymbol{B}(\boldsymbol{Re2})$$

such that:

1. $\forall A \in \boldsymbol{R}_1$ [($int(A) \in \boldsymbol{R}_2$) and tuples belonging to the relation A and $int(A)$ have the same length]
2. $int(I_1) = I_2$.

The relational structures $\boldsymbol{Re1}$ and $\boldsymbol{Re2}$ are *homomorphic* (cf. Marciszewski [1987], p. 164), if they are similar and there exists such a function

$$h : U_1 \longrightarrow U_2,$$

mapping the set U_1 in the set U_2, such that:

1. If $A \in \boldsymbol{R}_1$, then for any tuple $\langle \alpha_1, \ldots, \alpha_n\rangle \in A$, $\langle h(\alpha_1), \ldots, h(\alpha_n)\rangle \in int(A)$,
2. If $i \in I_1$, to $h(i) \in I_2$.

The criterion of homomorphism of the relational structures $\boldsymbol{Re}(S_1)$ and $\boldsymbol{Re}(S_2)$, for the structures of the tuples $S_1 \subseteq S(U)$ and $S_2 \subseteq S(U)$ (see Definition 6), is as follows:

Theorem 1.

Let two structures of tuples $S_1 \subseteq S(U)$ and $S_2 \subseteq S(U)$ be given and also let the function $f : S(U) \longrightarrow S(U)$ be given. If the function f satisfies the conditions:

1) $f(S_1) \subseteq S_2$ *(The image of the structure of tuples S_1 with respect to the function f is included in the structure of tuples S_2),*

and for any tuple $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in S_1$

2) $f(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle) = \langle f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n) \rangle \in S_2$[1],

then the relational structures $\boldsymbol{Re}(S_1)$ and $\boldsymbol{Re}(S_2)$ are homomorphic.

Proof.

Let the relational structures $\boldsymbol{Re}(S_1) = \langle U(S_1), \boldsymbol{R}(S_1), \emptyset \rangle$ and $\boldsymbol{Re}(S_2) = \langle U(S_2), \boldsymbol{R}(S_2), \emptyset \rangle$ be given.

Let us determine the function of interpretation $int : \boldsymbol{R}(S_1) \longrightarrow \boldsymbol{R}(S_2)$ for the relational structures $\boldsymbol{Re}(S_1)$ and $\boldsymbol{Re}(S_1)$ in the following way:

(1) $\forall t \in S_1 \ (\{t\} \in \boldsymbol{R}(S_1) \Rightarrow int(\{t\}) = \{f(t)\})$.

It follows from assumption 1) that $\forall t \in S_1 \ (f(t) \in S_2)$. Hence, making use of the way in which we determine the family of the relation $\boldsymbol{R}(S_2)$, we obtain $\forall t \in S_1 \ (\{f(t)\} \in \boldsymbol{R}(S_2))$. Hence, on the basis of Formula (1) we have:

(2) $\forall \{t\} \in \boldsymbol{R}(S_1) \ (f(t) \in int(\{t\}) \in \boldsymbol{R}(S_2))$.

Let us now define the function $h : U(S_1) \longrightarrow U(S_2)$ in the following way:

(3) $\forall \alpha \in U(S_1) \ (h(\alpha) = f(\alpha))$.

The function h is well determined since for any $\alpha \in U(S_1)$, $f(\alpha) \in U(S_2)$, because from assumption 2) it follows that $f(\alpha)$ is a one-element tuple $\langle f(\alpha) \rangle$, composed from this element and, being identified with this tuple, belongs to $U(S_2)$. Using again assumption 2) and formulas (3) and (2), as well as the definition of homomorphism of structures, we can conclude that the function h establishes the homomorphism of the structure $\boldsymbol{Re}(S_1)$ into the structure $\boldsymbol{Re}(S_2)$. ∎

Theorem 2.

Let two structures of tuples $S_1 \subseteq S(U)$ and $S_2 \subseteq S(U)$ be given. If the relational structures $\boldsymbol{Re}(S_1)$ and $\boldsymbol{Re}(S_2)$ are homomorphic, then there exist a function $f : S(U) \longrightarrow S(U)$ satisfying the conditions:

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[1] Let us remind that the one-element tuples are identified with their elements, thus $f(\langle \alpha_i \rangle) = f(\alpha_i)$, for any $i \in N$.

1) $f(S_1) \subseteq S_2$,

and for any tuple $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in S_1$

2) $f(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle) = \langle f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n) \rangle \in S_2$.

Proof.

Let the function $h : U(S_1) \longrightarrow U(S_2)$ establish the homomorphism of the above-given relational structures $\boldsymbol{Re}(S_1)$ and $\boldsymbol{Re}(S_2)$.

Let the following function $H : S(U) \longrightarrow S(U)$, be an expansion of the function of homomorphism h:

a) $H(\alpha) = h(\alpha)$, for any $\alpha \in U(S_1)$,

b) $H(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle) = \langle h(\alpha_1), h(\alpha_2), \ldots, h(\alpha_n) \rangle$, for any $n \geq 1$ and $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in S_1 = BS^{\boldsymbol{B}(\boldsymbol{Re}(S_1))}$

c) If $t \notin (U(S_1) \cup S_1)$, than $H(t) = t$.

Let us note that for any $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in S_1$, $\langle h(\alpha_1), h(\alpha_2), \ldots, h(\alpha_n) \rangle$ belongs to a certain relation of the base $\boldsymbol{B}(\boldsymbol{Re}(S_2))$, and hence also to the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re}(S_2))} = S_2$ (Fact 3b). From the assumptions a) and b) we have:

$$H(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle) = \langle H(\alpha_1), H(\alpha_2), \ldots, H(\alpha_n) \rangle \in S_2.$$

Thus: $H(S_1) \subseteq S_2$, and for the function H, the conditions 1) and 2) of the thesis of the theorem being proved are satisfied. ■

In the light of the above-presented theorems, the following definition is well justified:

Definition 9.

Let two structures of tuples $S_1 \subseteq S(U)$ *and* $S_2 \subseteq S(U)$ *be given and also let the function* $f : S(U) \longrightarrow S(U)$ *be given. The function* f *establishes a* ***homomorphism*** *of the structure of tuples* S_1 *into the structure of tuples* S_2 *(the structures of tuples* S_1 *and* S_2 *are* ***homomorphic****) iff*

1) $f(S_1) \subseteq S_2$

and for any tuple $\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle \in S_1$

2) $f(\langle \alpha_1, \alpha_2, \ldots, \alpha_n \rangle) = \langle f(\alpha_1), f(\alpha_2), \ldots, f(\alpha_n) \rangle \in S_2$.

From Theorem 1, Theorem 2 and Definition 9 there follows the following theorem:

Theorem 3.

Let two structures of tuples $S_1 \subseteq S(U)$ *and* $S_2 \subseteq S(U)$ *be given. The structures of tuples* S_1 *and* S_2 *are homomorphic iff the relational structures* $\boldsymbol{Re}(S_1)$ *and* $\boldsymbol{Re}(S_2)$ *are homomorphic.*

Theorem 4.

Let two relational structures $\boldsymbol{Re1} = \langle U_1, \boldsymbol{R}_1, I_1\rangle$ *and* $\boldsymbol{Re2} = \langle U_2, \boldsymbol{R}_2, I_2\rangle$ *be given, where* $U_1, U_2 \subseteq U$*. If the relational structures* $\boldsymbol{Re1}$ *and* $\boldsymbol{Re2}$ *are homomorphic, then also the base structures* $BS^{\boldsymbol{B}(\boldsymbol{Re1})}$ *and* $BS^{\boldsymbol{B}(\boldsymbol{Re2})}$ *are homomorphic.*

Proof.

Let the function $h : U_1 \longrightarrow U_2$ establish the homomorphism of the relational structures $\boldsymbol{Re1}$ and $\boldsymbol{Re2}$.

Let the following function $H : S(U) \longrightarrow S(U)$ be an expansion of the function of homomorphism h:

a) $H(\alpha) = h(\alpha)$, for any $\alpha \in U_1$,

b) $H(\langle\alpha_1, \alpha_2, \dots, \alpha_n\rangle) = \langle h(\alpha_1), h(\alpha_2), \dots, h(\alpha_n)\rangle$, for any $n \geq 1$ and $\langle\alpha_1, \alpha_2, \dots, \alpha_n\rangle \in BS^{\boldsymbol{B}(\boldsymbol{Re1})}$

c) If $t \notin (U_1 \cup BS^{\boldsymbol{B}(\boldsymbol{Re1})})$, than $H(t) = t$.

Let us note that for any $\langle\alpha_1, \alpha_2, \dots, \alpha_n\rangle \in S_1$, $\langle h(\alpha_1), h(\alpha_2), \dots, h(\alpha_n)\rangle$ belongs to a certain relation of the base $\boldsymbol{B}(\boldsymbol{Re2})$, and hence also to the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re2})}$. From the assumptions a) and b) we have:

$$H(\langle\alpha_1, \alpha_2, \dots, \alpha_n\rangle) = \langle H(\alpha_1), H(\alpha_2), \dots, H(\alpha_n)\rangle \in BS^{\boldsymbol{B}(\boldsymbol{Re2})}.$$

Thus: $H(BS^{\boldsymbol{B}(\boldsymbol{Re1})}) \subseteq BS^{\boldsymbol{B}(\boldsymbol{Re2})}$. Moreover, from the manner of determining the function H it follows that $H(\langle\alpha_1, \alpha_2, \dots, \alpha_n\rangle) = \langle H(\alpha_1), H(\alpha_2), \dots, H(\alpha_n)\rangle$, for any $\langle\alpha_1, \alpha_2, \dots, \alpha_n\rangle \in BS^{\boldsymbol{B}(\boldsymbol{Re1})}$. Hence, on the basis of Definition 9, the function H establishes the homomorphism of the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re1})}$ into the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re2})}$. ■

Definition 10.

Two structures of tuples $S_1 \subseteq S(U)$ *and* $S_2 \subseteq S(U)$ *are called* ***isomorphic*** *(symbolically:* $S_1 \approx_{isom} S_2$*) iff*

1) *there exists a function establishing a homomorphism of the structure of tuples* S_1 *into the structure of tuples* S_2 *and*

2) *the function* f *is a one to one function and*

3) *the function* f^{-1} *– the inverse of* f*, establishes a homomorphism of the structure of tuples* S_2 *into the structure of tuples* S_1*.*

The condition of homomorphism is one of the conditions of the adequacy of structures. In order to put the problem of adequacy in a more general way, it is necessary to expand the theory of adequacy being formulated by a new axiom and a definition. Let us first introduce the axiom of the adequacy.

Axiom 1. (*of adequacy*)

For any one-argument formulas α, β, and the two-argument formula φ, for any objects X, Y

$$\alpha(X) \longmapsto_{\varphi} \beta(Y) \Leftrightarrow$$
$$\varphi(X,Y) \wedge \forall V\ \forall Z\ (\varphi(V,Z) \Rightarrow \alpha(V) \wedge \beta(Z)) \wedge \forall Z\ (\varphi(X,Z) \Rightarrow Y = Z).$$

The expression "$\alpha(X) \longmapsto_{\varphi} \beta(Y)$" is read: the formula φ establishes the adequacy of the object X to the object Y with respect to the fact the object X possesses the property α, and the object Y possesses the property β.

The axiom of the adequacy can be verbally formulated in the following way: for any objects X and Y the formula φ establishes the adequacy of the object X to the object Y with respect to the fact that the object X possesses property α, while the object Y possesses property β iff 1) when objects X and Y satisfy the formula φ, 2) if any objects V and Z satisfy the formula φ, then the object V possesses the property α, while the object Z possesses the property β, 3) there exists exactly one object which – together with the object X – satisfies the formula φ.

When the accepted notation of the object X points, in an unambiguous manner, to that it has the property α, and the accepted notation of the object Y indicates, implicitly, that it possesses the property β, and the formula φ is assumed to have been well defined prior to that, then instead of writing:

"$\alpha(X) \longmapsto_{\varphi} \beta(Y)$", we shall write as follows: "$X \longmapsto_{Adq} Y$"

and read the expression as: *the object X is adequate to the object Y*.

For example: "$\boldsymbol{Re}$" and "$S^{\boldsymbol{Re}}$" are the notations of sets and points, respectively, to their being a relational structure and being a structure of tuples defined by the relational structure $\boldsymbol{Re}$. We shall thus write as follows:

$$\boldsymbol{Re} \longmapsto_{Adq} S^{\boldsymbol{Re}} \text{ instead of: } \alpha(\boldsymbol{Re}) \longmapsto_{\varphi} \beta(S^{\boldsymbol{Re}}),$$

where α is a property of being a relational structure and β is a property of being a structure of tuples determined by a relational structure.

Fact 5.

Let α, β, γ be any one-argument formulas. Then

a) *there exists a two-argument formula φ, such that $\alpha(X) \longmapsto_{\varphi} \alpha(X)$,*

b) *for any two-argument formulas* φ, φ_1 *there exist such a two-argument formula* φ_2, *that*

$$\alpha(X) \longmapsto_{\varphi} \beta(Y) \wedge \beta(Y) \longmapsto_{\varphi_1} \gamma(Z) \Rightarrow \alpha(X) \longmapsto_{\varphi_2} \gamma(Z).$$

Proof.

In proof a) it is enough to choose the formula of identity as the formula φ, while in proof b) to select, as the formula φ_2, a formula defined by the following expression:

$$\varphi_2(X_1, X_3) \Leftrightarrow \varphi(X_1, X_2) \wedge \varphi_1(X_4, X_3) \wedge X_2 = X_4. \qquad \blacksquare$$

Let us note that the object $\boldsymbol{Re}$ has the property of being a relational structure, while the object $S^{\boldsymbol{Re}}$ has the property of being a structure of tuples. Assuming that φ is a formula which states that the structure of tuples $S^{\boldsymbol{Re}}$ is determined, in an unambiguous way, by the relational structure $\boldsymbol{Re}$ (in accordance with Definition 8), we have:

Fact 6.

$$\boldsymbol{Re} \longmapsto_{\varphi} S^{\boldsymbol{Re}}.$$

Establishing the adequacy of different structures, we shall make use of the following definition of the adequacy of objects:

Definition 11.

For any formulas $\varphi_1, \varphi_2, \varphi_3, \varphi_4$, *the objects* X *and* Y *of the property* α *are adequate* (*symbolically:* $\alpha(X) \approx_{Adq} \alpha(Y)$) *iff there exist such objects* V *and* Z *of a certain property* β *that*

(i) $\alpha(X) \longmapsto_{\varphi_1} \beta(V) \wedge \alpha(Y) \longmapsto_{\varphi_2} \beta(Z)$,
and

(ii) $\alpha(X) \longmapsto_{\varphi_3} \beta(Z) \wedge \alpha(Y) \longmapsto_{\varphi_4} \beta(V)$.

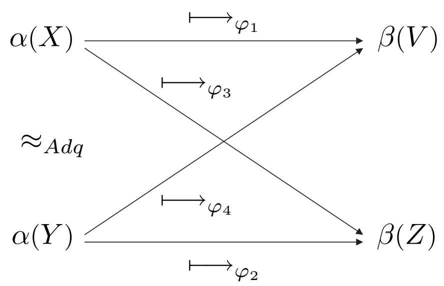
The expression "$\alpha(X) \approx_{Adq} \alpha(Y)$" is read: the objects X and Y are adequate with respect to the fact that the objects X and Y possess the property α.

When the accepted notations of the object X and the object Y explicitly point to the fact that they have the property α, and – moreover – the formulae $\varphi_1, \varphi_2, \varphi_3, \varphi_4$ are assumed to have been well defined prior to that, then instead of writing:

"$\alpha(X) \approx_{Adq} \alpha(Y)$", we shall write the following: $X \approx_{Adq} Y$

and read this expression as: *the objects* X *and* Y *are adequate.*

The definition of the adequacy is illustrated by the following diagram, in which the arrows symbolize the adequacy of one object in relation to another one:



The introduced axiom of the adequacy and the definition of adequacy of objects allow formulating certain theorems and conclusions that can be regarded as certain criteria of adequacy of structures.

Theorem 5.

If two structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ are isomorphic, then the relational structure ***Re1*** *is adequate to the structure of tuples $S^{\boldsymbol{Re2}}$ and also the relational structure* ***Re2*** *is adequate to the structure of tuples $S^{\boldsymbol{Re1}}$. Symbolically:*

$$S^{\boldsymbol{Re1}} \approx_{isom} S^{\boldsymbol{Re2}} \Rightarrow \boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re2}} \wedge \boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re1}}.$$

Proof.

Let the structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ be isomorphic. Since the structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ are isomorphic, there exists a function f establishing the homomorphism of the structure $S^{\boldsymbol{Re1}}$ into the structure $S^{\boldsymbol{Re2}}$, and being a one to one function, while the function f^{-1} – the inverse of f, establishes the homomorphism of the structure $S^{\boldsymbol{Re2}}$ into the structure $S^{\boldsymbol{Re1}}$.

Hence that the relational structure ***Re1*** is adequate to the structure of tuples $S^{\boldsymbol{Re1}}$ (see Fact 6) and that the function f establishes the homomorphism of the structure of tuples $S^{\boldsymbol{Re1}}$ into the structure of tuples $S^{\boldsymbol{Re2}}$ it follows that there exists such a formula φ that states that ***Re1*** determines $S^{\boldsymbol{Re2}}$ in an unambiguous manner. Since the set ***Re1*** has the property of being a relational structure, while the set $S^{\boldsymbol{Re2}}$ has the property of being a structure of tuples, it follows from the axiom of adequacy that $\boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re2}}$.

We can show in an analogous way that $\boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re1}}$, as it is sufficient to observe that the function f^{-1} establishes the homomorphism of the structure $S^{\boldsymbol{Re2}}$ into the structure $S^{\boldsymbol{Re1}}$. ■

From Theorem 5 and Definition 10 there follows:

Corollary 1.

If two structures of tuples $S^{\boldsymbol{Re1}}$ *and* $S^{\boldsymbol{Re2}}$ *are identical, then the relational structure* **Re1** *is adequate to the structure of tuples* $S^{\boldsymbol{Re2}}$ *and also the relational structure* **Re2** *is adequate to the structure of tuples* $S^{\boldsymbol{Re1}}$*. Symbolically:*

$$S^{\boldsymbol{Re1}} = S^{\boldsymbol{Re2}} \Rightarrow \boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re2}} \wedge \boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re1}}.$$

Theorem 6.

The relational structure **Re1** *is adequate to the structure of tuples* $S^{\boldsymbol{Re2}}$ *and also the relational structure* **Re2** *is adequate to the structure of tuples* $S^{\boldsymbol{Re1}}$ *iff the relational structures* **Re1** *and* **Re2** *are adequate. Symbolically:*

$$\boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re2}} \wedge \boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re1}} \Leftrightarrow \boldsymbol{Re1} \approx_{Adq} \boldsymbol{Re2}.$$

Proof.

($\Rightarrow$) Let the relational structure $\boldsymbol{Re1}$ be adequate to the structure of tuples $S^{\boldsymbol{Re2}}$ and let the relational structure $\boldsymbol{Re2}$ be adequate to the structure of tuples $S^{\boldsymbol{Re1}}$. Let us note that this assumption can be in accordance with condition (i) or condition (ii) of Definition 11. Let us assume that the assumption: $\boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re2}} \wedge \boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re1}}$ is in agreement with condition (i) of Definition 11. It follows from Fact 6 that the relational structure $\boldsymbol{Re1}$ is adequate to the structure of tuples $S^{\boldsymbol{Re1}}$ and that the relational structure $\boldsymbol{Re2}$ is adequate to the structure of tuples $S^{\boldsymbol{Re2}}$, thus condition (ii) of Definition 11 is also satisfied, i.e. $\boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re1}} \wedge \boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re2}}$. Hence, the relational structures $\boldsymbol{Re1}$ and $\boldsymbol{Re2}$ are adequate.

($\Leftarrow$) Let the relational structure $\boldsymbol{Re1}$ be adequate to the relational structure $\boldsymbol{Re2}$. When in Definition 11 instead of X, V, Y, Z we accept, in turn, the objects $\boldsymbol{Re1}, S^{\boldsymbol{Re1}}, \boldsymbol{Re2}, S^{\boldsymbol{Re2}}$, then from condition (ii) of this definition, it follows directly that: $\boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re2}} \wedge \boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re1}}$. ■

Theorem 7.

If the base structures $BS^{\boldsymbol{B(Re1)}}$ *and* $BS^{\boldsymbol{B(Re2)}}$ *are isomorphic, then the relational structures* **Re1** *and* **Re2** *are adequate.*

Proof.

Let the formula P_1 determine the function establishing the isomorphism of the base structure $BS^{\boldsymbol{B(Re1)}}$ onto $BS^{\boldsymbol{B(Re2)}}$, and let the formula P_2 be determined by the equivalence: $P_2(t, t') \Leftrightarrow P_1(t', t)$. The formula P_2 defines the function which establishes the isomorphism $BS^{\boldsymbol{B(Re2)}}$ onto $BS^{\boldsymbol{B(Re1)}}$.

For the formulas determined by the equations:

$$BS^{\boldsymbol{B(Re1)}} = \{t' : \exists t \in BS^{\boldsymbol{B(Re2)}} P_2(t, t')\},$$
$$BS^{\boldsymbol{B(Re2)}} = \{t' : \exists t \in BS^{\boldsymbol{B(Re1)}} P_1(t, t')\},$$

we have:

$$BS^{\boldsymbol{B(Re1)}} \longmapsto_{Adq} BS^{\boldsymbol{B(Re2)}} \text{ and } BS^{\boldsymbol{B(Re2)}} \longmapsto_{Adq} BS^{\boldsymbol{B(Re1)}}.$$

Since for the definition formula which determines the base structure for the relational structure we have:

$$\boldsymbol{Re1} \longmapsto_{Adq} BS^{\boldsymbol{B(Re1)}} \text{ and } \boldsymbol{Re2} \longmapsto_{Adq} BS^{\boldsymbol{B(Re2)}},$$

thus

1. $\boldsymbol{Re1} \longmapsto_{Adq} BS^{\boldsymbol{B(Re1)}}$,
2. $BS^{\boldsymbol{B(Re1)}} \longmapsto_{Adq} BS^{\boldsymbol{B(Re2)}}$,
3. $BS^{\boldsymbol{B(Re2)}} \longmapsto_{Adq} S^{\boldsymbol{Re2}}$,
4. $\boldsymbol{Re2} \longmapsto_{Adq} BS^{\boldsymbol{B(Re2)}}$,
5. $BS^{\boldsymbol{B(Re2)}} \longmapsto_{Adq} BS^{\boldsymbol{B(Re1)}}$,
6. $BS^{\boldsymbol{B(Re1)}} \longmapsto_{Adq} S^{\boldsymbol{Re1}}$.

Hence and from Fact 5b) we obtain, for certain formulas establishing adequacy, the following relations referring to adequacy:

7. $\boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re2}}$,
8. $\boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re1}}$.

Thus, on the basis of Theorem 6 as well as the formulas 7 and 8 it follows that relational structures ***Re1*** and ***Re2*** are adequate. ■

There follows directly from Theorems 5 and 6:

Theorem 8.

*If the two structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ are isomorphic, then the relational structures **Re1** and **Re2** are adequate. Symbolically:*

$$S^{\boldsymbol{Re1}} \approx_{isom} S^{\boldsymbol{Re2}} \Rightarrow \boldsymbol{Re1} \approx_{Adq} \boldsymbol{Re2}.$$

Theorem 9.

*If the two relational structures **Re1** and **Re2** are isomorphic, then the relational structure **Re1** is adequate to the structure of tuples $S^{\boldsymbol{Re2}}$, as well as the relational structure **Re2** is adequate to the structure of tuples $S^{\boldsymbol{Re1}}$. Symbolically:*

$$\boldsymbol{Re1} \sim_{isom} \boldsymbol{Re2} \Rightarrow \boldsymbol{Re1} \longmapsto_{Adq} S^{\boldsymbol{Re2}} \wedge \boldsymbol{Re2} \longmapsto_{Adq} S^{\boldsymbol{Re1}}.$$

Proof.

Let the relational structures ***Re1*** and ***Re2*** be isomorphic. Hence and from Theorem 4 and Definition 10 it follows that the base structures $BS^{\boldsymbol{B(Re1)}}$ and $BS^{\boldsymbol{B(Re2)}}$ are isomorphic. Thus, on the strength of Theorem 7 it follows that the relational structures ***Re1*** and ***Re2*** are adequate. The thesis of the theorem is obtained on the basis of Theorem 6. ■

From Theorem 9 and Theorem 6 there follows:

Corollary 2.

*If the two relational structures **Re1** and **Re2** are isomorphic, then the relational structures **Re1** and **Re2** are adequate. Symbolically:*

$$\boldsymbol{Re1} \sim_{isom} \boldsymbol{Re2} \Rightarrow \boldsymbol{Re1} \approx_{Adq} \boldsymbol{Re2}.$$

Theorem 10.

*If the relational structure **Re1** is isomorphically embedded in the relational structure **Re2**, then in the relational structure **Re2** there exists such relational substructure **Re'2** that the relational structures **Re'2** and **Re1** are adequate. Symbolically:*

$$\boldsymbol{Re1} \sim_{emb} \boldsymbol{Re2} \Rightarrow \exists \boldsymbol{Re'2} \subset_p \boldsymbol{Re2}\ (\boldsymbol{Re'2} \approx_{Adq} \boldsymbol{Re1})^2.$$

Proof.

Let the relational structure ***Re1*** be embedded isomorphically in the relational structure ***Re2***. Hence, it follows that in the relational structure ***Re2*** there exists a relational substructure which is isomorphic with the structure ***Re1***. We denote this relational substructure by ***Re'2***. Since the relational structures ***Re1*** and ***Re'2*** are isomorphic, it follows from Corollary 2 that the relational structures ***Re'2*** and ***Re1*** are adequate. ■

Theorem 11.

*If the two structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ are isomorphic and also if there exists a well-formed formula which establishes, in an unambiguous manner, the isomorphism of the structures of tuples **Re1** and **Re2**, then the structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ are adequate. Symbolically:*

$$S^{\boldsymbol{Re1}} \approx_{isom} S^{\boldsymbol{Re2}} \Rightarrow S^{\boldsymbol{Re1}} \approx_{Adq} S^{\boldsymbol{Re2}}.$$

Proof.

Let the structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ be isomorphic. Each structure of tuples is identical to itself and from Fact 5a) it follows that:

$$S^{\boldsymbol{Re1}} \longmapsto_{Adq} S^{\boldsymbol{Re1}} \text{ and } S^{\boldsymbol{Re2}} \longmapsto_{Adq} S^{\boldsymbol{Re2}}.$$

Since the structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ are isomorphic and there exists a well-formed formula which establishes the isomorphism of structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$, hence:

[2] The notation: ***Re'2*** $\subset_p$ ***Re2*** was applied as an indication that ***Re'2*** is a relational substructure of the relational structure ***Re2***.

$$S^{\boldsymbol{Re1}} \longmapsto_{Adq} S^{\boldsymbol{Re2}} \text{ and } S^{\boldsymbol{Re2}} \longmapsto_{Adq} S^{\boldsymbol{Re1}}.$$

We showed that Conditions (i) and (ii) of Definition 11 are satisfied. Then the structures of tuples $S^{\boldsymbol{Re1}}$ and $S^{\boldsymbol{Re2}}$ are adequate. ■

Now we shall formulate certain theorems on the adequacy of relational structures which have the same universe.

Theorem 12.

*If the base $\boldsymbol{B}(\boldsymbol{Re1})$ is a subbase of the base $\boldsymbol{B}(\boldsymbol{Re2})$ and $BS^{\boldsymbol{B}(\boldsymbol{Re2})} \subseteq S^{\boldsymbol{Re1}}$, then relational structures **Re1** and **Re2** are adequate.*

Proof.

Let the base $\boldsymbol{B}(\boldsymbol{Re1})$ be a subbase of the base $\boldsymbol{B}(\boldsymbol{Re2})$ and $BS^{\boldsymbol{B}(\boldsymbol{Re2})} \subseteq S^{\boldsymbol{Re1}}$.

From the fact that the base $\boldsymbol{B}(\boldsymbol{Re1})$ is a subbase of the base $\boldsymbol{B}(\boldsymbol{Re2})$ and from Fact 1 it follows that $BS^{\boldsymbol{B}(\boldsymbol{Re1})} \subseteq BS^{\boldsymbol{B}(\boldsymbol{Re2})}$. Let us analyze the following two cases:

a) $BS^{\boldsymbol{B}(\boldsymbol{Re1})} = BS^{\boldsymbol{B}(\boldsymbol{Re2})}$,

b) $BS^{\boldsymbol{B}(\boldsymbol{Re1})} \subset BS^{\boldsymbol{B}(\boldsymbol{Re2})}$.

The proof in the case of a) is obvious. Let us analyze case b). From the assumption that the base $\boldsymbol{B}(\boldsymbol{Re1})$ is a subbase of the base $\boldsymbol{B}(\boldsymbol{Re2})$ and on the strength of Fact 4 we have: $S^{\boldsymbol{Re1}} \subseteq S^{\boldsymbol{Re2}}$.

From the assumption of the theorem: $BS^{\boldsymbol{B}(\boldsymbol{Re2})} \subseteq S^{\boldsymbol{Re1}}$, as well as from Definiction 8 it follows that the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re1})}$ generates tuples belonging to the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re2})}$ according to the Conditions C1-C4 of this definition. Thus, $BS^{\boldsymbol{B}(\boldsymbol{Re1})}$ determines $S^{\boldsymbol{Re2}}$. Thus, the structure of tuples $S^{\boldsymbol{Re2}}$, which is obtained from the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re2})}$ according to the conditions C1-C4 of Definition 8, is included in the structure of tuples $S^{\boldsymbol{Re1}}$.

Since $S^{\boldsymbol{Re1}} \subseteq S^{\boldsymbol{Re2}}$ and $S^{\boldsymbol{Re2}} \subseteq S^{\boldsymbol{Re1}}$, then $S^{\boldsymbol{Re1}} = S^{\boldsymbol{Re2}}$. Hence and from Corollary 1 and Theorem 6 it follows that the relational structures ***Re1*** and ***Re2*** are adequate. ■

Theorem 13.

*If $BS^{\boldsymbol{B}(\boldsymbol{Re2})} \subseteq S^{\boldsymbol{Re1}}$ and $BS^{\boldsymbol{B}(\boldsymbol{Re1})} \subseteq S^{\boldsymbol{Re2}}$, then relational structures **Re1** and **Re2** are adequate.*

Proof.

From the assumption that $BS^{\boldsymbol{B}(\boldsymbol{Re2})} \subseteq S^{\boldsymbol{Re2}}$ and from Definition 8 it follows that the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re1})}$ generates tuples belonging to the base structure $BS^{\boldsymbol{B}(\boldsymbol{Re2})}$ according to the Conditions C1-C4 of this definition. Thus, $BS^{\boldsymbol{B}(\boldsymbol{Re1})}$ determines $S^{\boldsymbol{Re2}}$. Hence, the structure of tuples $S^{\boldsymbol{Re2}}$, which is determined by the base structure $BS^{\boldsymbol{B(Re2)}}$ according to the Conditions C1-C4 of Definition 8, is included in the structure of tuples $S^{\boldsymbol{Re1}}$.

In an analogous way, we show that $S^{\boldsymbol{Re1}} \subseteq S^{\boldsymbol{Re2}}$. Because $S^{\boldsymbol{Re1}} \subseteq S^{\boldsymbol{Re2}}$ and $S^{\boldsymbol{Re2}} \subseteq S^{\boldsymbol{Re1}}$, then $S^{\boldsymbol{Re1}} = S^{\boldsymbol{Re2}}$. Hence, from Corollary 1 and Theorem 6 it follows that the relational structures ***Re1*** and ***Re2*** are adequate. ∎

4. Ontological structures of an object and the ontological problem of the adequacy

Finally, we shall discuss the problem area connected with applications of the theory of the adequacy, taking as an example selected applications of this theory in ontology. First, let us observe that there have continuously been undertaken certain attempts at a precise determination of the notion of the substantial structure of object (structure of the matter, physical structure of bodies, chemical structure of substances, etc.), that is the notion of the ontological structure of the object. Making reference to certain theoretical propositions by J. Perzanowski (2004) and the concepts of adequacy introduced in the present paper, we can propose the following description of this notion:

Through the ***ontological structure of an object*** *we shall understand a structure described by the general theory of analysis and synthesis relativised to description of this object.*

The general theory of analysis and synthesis was formulated by Perzanowski (2004). In the sense of this theory, elements (components) of an object are the objects into which the given object can be decomposed or from which this object can be composed. The predicate that corresponds in this sense to being a component of the object is denoted by "$\prec$". The expression "$x \prec y$" is read: *the object x is a component of the object y or the object x is simpler, than the object y, its component.*

Let the object o be a distinguished object, whose structure we describe. Let us make the description relative only to components of this object. We accept, thus, that all the considered objects are components of the object o, and hence $\forall x\ (x \prec o)$.

Let us consider the following notions:

The object x is a superelement of the object o (symbolically: $SE(x)$) iff

(SE) $\quad \forall y\ (x \prec y)$.

The object x is a simple component of the object o (symbolically: $S(x)$) iff

(S) $\quad \forall y\ (\neg y \prec x)$.

The object x is an atom of the object o (symbolically: $A(x)$) iff

(A) $\quad \forall y\ (y \prec x \Rightarrow x = y)$.

The object x is an element of the object o (symbolically: $E(x)$) iff

(E) $\quad \forall y\ (y \prec x \Rightarrow x = y \vee SE(x))$.

The ontological structure of the object o is well defined, when for any condition $\alpha \in \{(SE), (S), (A), (E)\}$ there exists such a component x of the object o that $\alpha(x)$ is satisfied.

In order to precisely verify the knowledge about whether the ontological structure of the object o is well described, in other words, to answer the question whether this knowledge is adequate, the best thing to do is to interpret this knowledge on the basis of formal ontology, e.g. the set theory.

Thus, let U^o be the set of all components of the object o possible to be distinguished in the process of conceptualization or through experience by means of analysis or synthesis. A descriptive model of the ontological structure of the object o will then be described set-theoretically by the relational structure $\boldsymbol{Re}^o = \langle U^o, <, \subseteq \rangle$, where "$<$" denotes a *relation of being a component in the process of analysis*, and "$\subseteq$" denotes a *relation of being a component in the process of synthesis*. Predicate "$\prec$" will be interpreted as the first or the second relation ($x \prec y \Leftrightarrow x \subseteq y \vee x < y$). Because the appearance of objects in the process of synthesis and their decomposition into components in the process of analysis denotes that the components are connected to form tuples (chains), and relationships among the elements (links) of tuples (chains) are determined through analysis and synthesis. The set of all such tuples S^o is a certain structure of tuples. This structure, set by experience, can be or may not be determined by means of a theoretical model $\boldsymbol{Re}^o$. In other words: theoretical knowledge can be adequate to experimental knowledge or not. However, we can verify it precisely on the ground of formal ontology. Therefore there appears the following question: **is structure $\boldsymbol{Re}^o$ adequate to structure** S^o? We identify this question with the **ontological problem of the adequacy**. Offering, on the grounds of the theory of adequacy, some criteria of adequacy that are necessary to solve this problem, goes considerably beyond the intended framework of this paper.

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