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## POSSIBLE WORLDS IN THE LANGUAGE OF NON-FREGEAN LOGIC

### I. Introduction

The term “possible world” is used usually in the metalanguage of modal logic, and it is applied to the interpretation of modal connectives. Surprisingly, as it has been shown in Suszko [68a] certain versions of that notion can be defined in the language of non-Fregean logic exclusively, by means of sentential variables and logical constants. This is so, because some of the non-Fregean theories contain theories of modality, as shown in Suszko [71a]. Intuitively, possible worlds are maximal (with respect to an order of situations) and consistent situations, while the real world may be understood as a situation, which is a possible world and the fact.

Non-Fregean theories are theories based on the non-Fregean logic. Non-Fregean logic is the logical calculus created by Polish logician Roman Suszko in the sixties. The idea of that calculus was conceived under the influence of Wittgenstein’s *Tractatus*. According to Wittgenstein, declarative sentences of any language describe situations. Different explanations of what is a situation may be found in Wittgenstein [94], Wolniewicz [99], Barwise and Perry [83], Wójcicki [86] and others.

### II. W-languages

To speak in a formal way about the structure of the universe of situations, Suszko introduced languages which he called W-languages (in honor of L. Wittgenstein). In the alphabet of these languages, there are two kinds of variables: sentential variables:  $p, q, r, \dots$ , and nominal variables:  $x, y, z, \dots$ , identity connective and identity predicate, both symbolized by the sign “ $\equiv$ ”,

and quantifiers binding the two kinds variables. The intended interpretation of W-languages is such that nominal variables range over the universe of objects while the sentential variables run over the universe of situations. All other symbols in these languages, except the sentential and nominal variables, are interpreted as symbols of some functions both defined over the universe of situations and the universe of objects. The identity connective corresponds to the identity relation between situations, and the identity predicate corresponds to the identity relation between objects. It is obvious that the language of the ordinary predicate calculus with identity is a part of the W-language excluding sentential variables, but the most frequently used sentential languages are the part of the W-language without nominal variables and the identity predicate.

If a language does not contain sentential variables then according to Suszko it is not fit for a full formalization of a theory of situations, but at most for that of a theory of events considered as reified equivalents of situations (see Suszko [94]).

In contemporary science the majority of studied theories are those which are expressed in languages without sentential variables. The appearance of the Fregean sentential semantics, Leśniewski's prothotetics, Wittgenstein's *Tractatus* and the Non-Fregean Logic has been an important step in the development of logico-philosophical reflection, because these theories require a language with sentential variables. According to Suszko, *situations* are primitive with respect to *events*, for the latter are objects abstracted from the former. In Suszko [94] it is proved that:

- (i) certain theories of situations are mutually translatable into theories of events,
  - (ii) certain algebras of situations are isomorphic with algebras of events,
- Suszko posed the question:

*"... What, therefore, is the cause of the fact that our thought and the natural language to a certain degree discriminate sentence variables, and particularly, general and existential sentences about situations?... Why, therefore, should we prefer the theory of events to that of situations?"*

And in the same paper Suszko answers:

*"It is probably the symptom of some deep, historically determined attribute our thought and of natural language, the examination and explication of which will certainly be long and arduous".*

These features of our thinking induce an account of the world rather as the universe of objects possessing certain properties and connected by certain relations, not – as the totality of facts obtaining a logical space as in Wittgenstein's *Tractatus*. According to Suszko, one of the aspects of

this bias of our thinking is the tendency (originating from logical empiricism) of shifting philosophical problems from the object language to the metalanguage. E.g., such terms as: *the real world*, *a possible world*, *a fact*, *a situation* usually do not occur in the object language, but are used mostly in the metalanguage of modal logic applied to the interpretation of modal connectives.

### III. SCI with quantifiers

In this paper, for the sake of simplicity, I have restricted myself to languages without nominal variables, i.e. to languages whose alphabet contains only the sentential variables, the logical connectives, quantifiers, and some auxiliary symbols defined by them, that is, to the SCI languages with quantifiers.

In that language, a non-Fregean logic is generated by the Modus Ponens rule and by some logical axioms divided into three groups:

#### A1 – The axioms for the truth-functional connectives:

- (1)  $\forall p \forall q [p \rightarrow (q \rightarrow p)]$
- (2)  $\forall p \forall q \forall r [(p \rightarrow (q \rightarrow r)) \rightarrow ((p \rightarrow q) \rightarrow (p \rightarrow r))]$
- (3)  $\forall p \forall q [\neg p \rightarrow (p \rightarrow q)]$
- (4)  $\forall p [(\neg p \rightarrow p) \rightarrow p]$

A1 is the set of axioms characterizing the connectives:  $\neg, \wedge, \vee, \rightarrow, \leftrightarrow$  in the classical way.

#### A2 – Axioms for quantifiers binding sentential variables, which are all the formulas represented by the following schemas:

- (5)  $\forall p \alpha \rightarrow \alpha[p/\beta]$ , where  $\alpha[p/\beta]$  is the result of correctly substituting in the formula  $\alpha$  the formula  $\beta$  for the variable  $p$ ;
- (6)  $\alpha \rightarrow \forall p \alpha$ , if  $p$  is not free in  $\alpha$ ;
- (7)  $\forall p (\alpha \rightarrow \beta) \rightarrow (\forall p \alpha \rightarrow \forall p \beta)$
- (8)  $\exists p \alpha \leftrightarrow \neg \forall p \neg \alpha$

#### A3 – axioms for the identity connective:

- (9)  $\forall p \forall q [(p \equiv q) \rightarrow (p \rightarrow q)]$ ;
- (10)  $\forall p \forall q [(p \equiv q) \rightarrow (\neg p \equiv \neg q)]$ ;
- (11)  $\forall p \forall q \forall r \forall s [(p \equiv q) \wedge (r \equiv s) \rightarrow (p \% r) \equiv (q \% s)]$ , where  $\%$  stands for  $\wedge, \vee, \rightarrow, \leftrightarrow$ ;

- (12)  $\alpha_1 \equiv \alpha_2$ , where  $\alpha_1, \alpha_2$  differ at most by bound variables;  
 (13)  $\forall p (\alpha \equiv \beta) \rightarrow (Qp \alpha \equiv Qp \beta)$ , where  $Q$  stands for  $\forall, \exists$ .

This calculus is a part of the non-Fregean logic and is called the non-Fregean sentential logic or ‘SCI with quantifiers’. Although every logical theorem is at the same time an ontological one, the non-Fregean logic doesn’t stipulate any structural and quantitative conditions on the universe of sentential variables, except that it must contain at least two-elements. This is expressed by the logical theorem:  $\exists p \exists q \neg(p \equiv q)$ . For example: ‘ $\forall p (p)$ ’ present a different situation than the sentence ‘ $\exists p (p)$ ’.

To express in the language of logic certain extra-logical presuppositions of the ontology pertaining to the structure of situations, let us adopt the following notations. If  $A$  is a set of formulas, then  $Gn(A)$  is the set of all generalizations of formulas in  $A$ , and  $Eqv(A)$  is the set of all equations  $\alpha \equiv \beta$ , such that  $(\alpha \leftrightarrow \beta) \in A$ .

#### IV. Ontology of situations

According to the principles of non-Fregean semantics as presented in Omyła [94] each sentence  $\alpha$  in any model  $M$  has a semantic correlate, which denote by  $|\alpha|_M$ . The models for the language of non-Fregean sentential logic are structures:

$$M = (\mathbf{U}, F),$$

where:  $\mathbf{U}$  is a generalization of the SCI-algebra on a given set  $U$ , and  $F$  is a suitable subset of  $U$ . In any model  $M$  the logical constants have the intended interpretation, therefore any model for the language of sentential logic will be treated as a formal representation of a certain universe of situations with a distinguished set of facts in it. In order to simplify the formulations a generalized algebra of any model  $\mathbf{M}$  for the non-Fregean sentential logic will be called the algebra of situations, and the set  $F$  will be called the set of all facts occurring in this model. The algebra situations is the same what Suszko understood by semi-models. Let  $\mathbf{M}$  be

$$\mathbf{M} = (\mathbf{U}, F),$$

any model for the language of non-Fregean sentential logic, and

$$\mathbf{K} = \{(\mathbf{U}, F_i) : i \in I\}$$

be the family of all models for the language of non-Fregean sentential logic  $L$  determined by the algebra of situations  $\mathbf{U}$ . The fact that the models from the

class  $\mathbf{K}$  have the same generalized algebra  $\mathbf{U}$  means that these models are determined by the same set of situations and in the same way are articulated in the language  $L$ . The models from the class differ at most from the set of facts realized in them.

We denote:  $TR(\mathbf{M})$  – the set of all formulas true in  $\mathbf{M}$  of the language of non-Fregean sentential logic,

$$\begin{aligned} Val(\mathbf{U}) &= \cap\{TR\{\mathbf{M}\} : \mathbf{M} \in \mathbf{K}\} \\ \alpha \in Val(\mathbf{U}) &\leftrightarrow \forall \mathbf{M} (\mathbf{M} \in \mathbf{K} \rightarrow \alpha \in TR(\mathbf{M})) \end{aligned}$$

**Definition 1.**

$T$  is an ontology of situations in the language of non-Fregean sentential logic  $L$  iff  $T$  is the theory in  $L$  and there exists an algebra of situations  $\mathbf{U}$  such that

$$T \subset Val(\mathbf{U}).$$

Symbolically:

$$T \text{ is } OS_L \leftrightarrow T \in TH(L) \text{ and } T \subset Val(\mathbf{U}).$$

where:  $OS_L$  – denotes the set of all ontology of situations in the language of non-Fregean sentential logic  $L$ ,  $TH(L)$  – the set of all theories in the language  $L$ .

The idea of ontology of situations defined in this way reflects those intuitions which state that ontology contains necessary statements, dependent only on the structure of the universe of situations and not on the occurring facts.

Three direct corollaries of the definition:

1.  $Cn(\emptyset)$  is the smallest ontology of situations, i.e.  $Cn(\emptyset)$  is an ontology of situations and moreover every ontology of situations includes  $Cn(\emptyset)$ .
2. If  $X$  is a set of equalities i.e.  $X = \{\alpha \equiv \beta : \alpha, \beta \in L\}$  such that  $Gn(X)$  is consistent, then  $Cn(Gen(X))$  is a certain ontology of situations (where  $Gen(X)$  is the set of all generalizations of all formulas from the set  $X$ ).
3. In the language of classical sentential logic there is only one ontology of situations, i.e. the set of all formulas true in two-element Boolean algebra, i.e. the set of all tautologies of the classical sentential logic.
4. The set  $TR(\mathbf{M}) - Val(\mathbf{M})$  is the set of sentences describing the facts holding contingently in the model  $\mathbf{M}$ , but which may not hold in other models defined on  $\mathbf{U}$ .

**Remarks:**

1. The term “ontology of situations” has been taken from the title of Wolniewicz [85], where it is used in a little different meaning.

2. What has been called “algebra of situations” and “ontology of situations” it was in Suszko [71b] called respectively “semi-model” and “truth in algebra”, symbolically  $Val(\mathbf{U})$ .

Let the set  $D$  be constituted by the definitions

- (d1)  $0 \equiv \forall p (p)$   
 (d2)  $1 \equiv \exists p (p)$   
 (d3)  $\forall p \forall q [(p \leq q) \equiv ((q \rightarrow p) \equiv 1)]$   
 (d4)  $SFp \equiv \{\forall q (q \rightarrow (q \leq p)) \wedge \forall r (\forall q (q \rightarrow (q \leq r)) \rightarrow (p \leq r))\}$   
 (d5)  $InfFp \equiv \forall q (q \rightarrow (p \leq q)) \wedge \forall r (\forall q (q \rightarrow (r \leq q)) \rightarrow (r \leq p))\}$   
 (d6)  $PWp \equiv \{\neg(p \equiv 0) \wedge (\forall q [(q \leq p) \vee (\neg q \leq p)])\}$   
 (d7)  $RWp \equiv (p \wedge \forall q (q \rightarrow (q \leq p)))$

The terms defined by (d1), (d2) have the following intuitive import: 0, 1 are sentential constants. They are the abbreviations of some sentences i.e. sentential formulas with no free variables. 0 is the designated impossible situation to the effect that all situations are facts. The sentence  $\exists p (p)$  is a theorem of the non-Fregean sentential logic, thus it represents an improper fact. Consequently, 1 is an improper fact to the effect that in any model the set of facts is not empty.

Using these notations we are in a position to determine the following theories of situation known in the literature:

$$\begin{aligned} WBQ &=_{\text{def}} Cn(Eqv(Cn_0(\emptyset)) \cup D); \\ WTQ &=_{\text{def}} Cn(Eqv(Cn(\emptyset) \cup D)), \\ WHQ &=_{\text{def}} Cn(WBQ \cup H) \end{aligned}$$

where:

- $Cn_0$  is a subconsequence of  $Cn$  generated by the sets of axioms (A1), (A2) and the rules (MP),
- $H$  is the set consisting of the two formulae:

$$\begin{aligned} \forall p \forall q [(p \equiv q) \equiv ((p \equiv q) \equiv 1)] \\ \forall p \forall q [\neg(p \equiv q) \equiv ((p \equiv q) \equiv 0)] \end{aligned}$$

The theory  $WBQ$  contains all generalizations of the valid Boolean equations written out by means of sentential variables, connectives and quantifiers only. Particularly, theorems  $WBQ$  are all generalization of formulas of the following form:

$$\begin{aligned} \alpha(p) \leq \forall p \alpha(p), & \quad \exists p \alpha(p) \leq \alpha(p), \\ 1 \leq p, & \quad p \leq 0 \end{aligned}$$

The formula ‘ $(p \leq q)$ ’ under any theory containing the set  $WBQ$ , is read: “the situation  $p$  is contained in the situation  $q$ ” or “the situation  $p$  obtains in the situation  $q$ ”, or “the situation  $p$  occurs in the situation  $q$ ”. This is justified by the circumstance that in any  $WBQ$ -model the counterpart of the connective “ $\leq$ ” is an ordering relation on the universe of situations, called *ordering of situations*. The formula: “ $SFp$ ” is to say that *the situation  $p$*  is – under the ordering of situations – the least upper bound of the set of all facts, i.e. it is the sum of all facts. Similarly,  $InfFp$  says that  $p$  is its greatest lower bound. Formula ‘ $RWp$ ’ is to say that  $p$  is a fact containing all other facts.

Any theory  $T$  expressed in the language of the non-Fregean sentential logic  $L$  such that  $WBQ \subset T$  is called a  $WBQ$ -theory, and its model a  $WBQ$ -model. (With  $W$  – for Wittgenstein,  $B$  – for Boolean algebra, and  $Q$  – for quantifiers).

To be able to read the formula ‘ $PWp$ ’ as ‘the situation  $p$  is a possible world’, we assume that 0 is the only impossible situation. This holds for these models  $M$  of considered language  $L$  in which  $WHQ \subset TR(M)$ , where  $TR(M)$  is the set of all formulas true in the model  $M$ .

Furthermore, any theory  $T$  such that  $WHQ \subset T$  is called a  $WHQ$ -theory, and its model a  $WHQ$ -model.

We get the following:

**Metatheorem 1.**

In any  $WBQ$ -theory these formulae are theorems:

- (i)  $\exists p SFp$
- (ii)  $\exists p InfFp$

By the completeness theorem for the non-Fregean logic metatheorem 1 says that in every  $WBQ$ -model there exists the sum of all facts, and also their infimum which is the Boolean unit 1.

**Metatheorem 2.**

For any  $WHQ$ -model  $M = (\mathbf{U}, F)$  we have:

- (i) with regard to the ordering of situations, the least upper bound of a set of possible worlds is  $|\forall p (PWp \rightarrow p)|$ ;
- (ii) the greatest lower bound of such a set is  $|\exists p (p \wedge PWp)|$ .

Simple proofs of the metatheorems may be found in Omyla [86]. From metatheorem 2, and in view of the connection between truth in algebra of situations and truth in the model defined on that algebra, we get the following:

- (i) No situation is a possible world in the algebra of situation  $\mathbf{U}$  iff  
$$\langle \exists p (p \wedge PWp) \equiv 0 \rangle \in Val(\mathbf{U})$$
- (ii) In the algebra of situations  $\mathbf{U}$  there are situations which are possible worlds iff  
$$\langle \neg[\exists p (p \wedge PWp) \equiv 0] \rangle \in Val(\mathbf{U}).$$
- (iii) In  $M$  there is a situation which is the real world iff ' $\exists p (p \wedge PWp)$ ' is true in  $M$ .
- (iv) In  $\mathbf{U}$  each possible situation is contained in some possible world iff  
$$\langle \exists p (p \wedge PWp) \equiv 1 \rangle \in Val(\mathbf{U}).$$

Finally, let me add that in this paper I have considered certain particular kinds of situations, such that as facts and possible worlds, and I have analysed them from the point of view of ordering of situations determined by the definition (d3). The definitions (d1)–(d7) are little modifications of definitions considered by Suszko in the papers Suszko [68a], [68b]. This paper is a reconstruction and systematization Suszko's consideration about possible worlds contained in those papers.

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