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TENSE LOGICS AND ARISTOTELIAN ARGUMENT ABOUT DETERMINISM

Logical determinism[1] is a point of view which proves the thesis that only logical principles are sufficient while discussing determinism. Logical determinists say that the principles of bivalency and excluded middle low are sufficient to construct an argument about determinism. The problem was clearly formulated by Aristotle in Chapter IX of *On Interpretation.* Aristotle assumes that sentences about the past and present are either true or false. He claims, however, that the assumption that sentences about the future are either true or false is sufficient while constructing an argument about determinism. As he puts it:

> When the subject, however, is individual, and that which is predicated of it relates to the future, the case is altered. For if all propositions whether positive or negative are either true or false, then any given predicate must either belong to the subject or not, so that if one man affirms that an event of a given character will take place and another denies it, it is plain that the statement of the one will correspond with reality and that of the other will not. For the predicate cannot both belong and not belong to the subject at one and the same time with regard to the future. Thus, if it is true to say that a thing is white, it must necessarily be white; if the reverse proposition is true, it will of necessity not be white. Again, if it is white, the proposition stating that it is white was true; if it is not white, the proposition to the opposite effect was true. And if it is not white, the man who states that it is making a false statement; and if the man who states that it is white is making a false statement, it follows that it is not white. It may therefore be argued that it is necessary that affirmations or denials must be either true or false. Now if this be so, nothing is or takes place fortuitously, either in the present or in the

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[1] Apart from logical determinism **physical determinism** is considered. Physical determinism is a point of view which states that every fact has immemorial causes in other earlier facts. Physical determinism is connected with the principle of causality.

future, and there are no real alternatives; everything takes place of necessity and is fixed[2].

If all sentences about the future are true or false, then the events described by these sentences are determined. If all future events are determined, there are not accidental events and everything is necessary. Therefore, the thesis that sentences about future events are either true or false implies that – apart from the past and present – the future is also logically determined.

Therefore, if we accept the principle of necessity and the principle of the excluded middle, the argument on determinism can be reconstructed as follows:

Let $T(p)$ means "*p is true*", and $\Box(p)$ means "*p is necessary*".

1) $T(p) \rightarrow \Box(p)$ – the principle of necessity,

2) $T(\neg p) \rightarrow \Box(\neg p)$ – the principle of necessity,

3) $T(p) \vee T(\neg p)$ – the principle of excluded middle,

4) $\Box(p) \vee \Box(\neg p)$ – 1, 2, 3 and $\dfrac{\alpha \rightarrow \gamma, \beta \rightarrow \delta, \alpha \vee \beta}{\gamma \vee \delta}$.

To get a very similar proof, we can construct an argument using the principle of bivalency.

Determination should be considered in a temporal context. Therefore, we will get a better approximation of the notion of determining if we introduce temporal expressions into the language. It is realized in the varied tense logics.

Standard tense operators are as follows:

$F\alpha$ – at least once in the future α,
$P\alpha$ – at least once in the past α,
$G\alpha$ – it is always going to be the α,
$H\alpha$ – it has always been the case α.

The expression which we should recognize as a notation of the thesis of determinism in the language of tense logic is:

$$F\alpha \vee F\neg\alpha.$$

With regard to understanding of F tense operator, we will read $F\alpha \vee F\neg\alpha$ as follows:

At least once in the future α or at least once in the future $\neg\alpha$.

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[2] Aristotle, *On Interpretation*, Chapter IX.

In our consideration we assume that sentences are semantic correlates of events, therefore, a possible interpretation of $F\alpha \lor F\neg\alpha$ is:

*For event **A** (which is a semantic correlate of the sentence α) is either determined with a moment of time in the future such when the event **A** holds at that moment, or is determined with a moment of time in the future when the event **A** does not hold at it.*

Because in the expression $F\alpha \lor F\neg\alpha$ α is any, therefore, for any event it is either determined that this event holds at some moment of time in the future, or it is determined that it does not hold at any moment of time in the future. In other words, all future events are determined and there are no future accidental events.

If $F\alpha \lor F\neg\alpha$ is a thesis of some system of the tense logic then the language of such a system has limited possibilities to describe the world because if we use the language of this formal system, we can describe only the properties of such worlds which belong to the class of determined worlds. However, in this case, logic is ontological involved in the matter of determinism.

A following question arises:

What conditions should be fulfilled for the expression $F\alpha \lor F\neg\alpha$ to be a tautology of the system of tense logics?

We are going to consider two kinds of tense logic:
- tense logic based on classical logic and
- tense logic based on intuitionistic logic

First, let us consider tense logic based on classical logic. The most known deductive system of tense logic based on classical logic is K_t system. K_t is a minimal system with no conditions imposed upon R (earlier-later) relation, therefore, the structure of semantic time can be any. The expression $F\alpha \lor F\neg\alpha$ cannot be a tautology of K_t because for a semantic time we can accept the time which includes the last moment. If a moment t is the last moment of the time structure, then $F\alpha$ is not satisfied at the moment t whereas moment t does not satisfy $F\neg\alpha$. A necessary condition for the truth of the expressions $F\alpha$ and $F\neg\alpha$ at the moment t is the existence of the moment of time later than the moment t.

A following question arises:

Is $F\alpha \lor F\neg\alpha$ a tautology of classical tense logic of non-ending time?

The answer is YES. In the language of tense logic a class of non-ending time is characterized by the following expression: $G\alpha \Rightarrow F\alpha$. We will prove the following theorem:

THEOREM 1.
$$K_t \cup \{G\alpha \Rightarrow F\alpha\} \vdash (F\alpha \vee F\neg\alpha).$$

PROOF

1) $\alpha \vee \neg\alpha$ – excluded middle law,
2) $G(\alpha \vee \neg\alpha)$ – 1, R2[3],
3) $G(\alpha \vee \neg\alpha) \Rightarrow F(\alpha \vee \neg\alpha)$ – expression added to axioms of K_t,
4) $F(\alpha \vee \neg\alpha)$ – 2, 3, MP,
5) $F(\alpha \vee \neg\alpha) \Rightarrow (F\alpha \vee F\neg\alpha)$ – K_t tautology[4],
6) $(F\alpha \vee F\neg\alpha)$ – 4, 5, MP.

The thesis of determinism formulated as above is a tautology of classical tense logic of non-ending time.

Now let us consider the connections between the thesis of determinism formulated as $F\alpha \vee F\neg\alpha$ and the intuitionistic equivalent of K_t system. This system is called IT_m [5].

In the case of the assumption that the real world is not determined the question: *Is the thesis of determinism a tautology of the minimal intuitionistic tense logic?* is not important. It is the following question that is fundamental: *Is the thesis of determinism a tautology of the intuitionist tense logic where as a semantic time we accept time which posses the properties usually attributed to real time?* Therefore, it is the following question that appears to be essential: Is the thesis of determinism a tautology of the intuitionist tense logic of linear, dense and non-ending time?

We can prove that the thesis of determinism formulated as $F\alpha \vee F\neg\alpha$ is not a tautology of the intuitionistic tense logic of non-ending, dense and linear time.

THEOREM 2

$Fp \vee F\neg p$ is not a tautology of the intuitionistic tense logic of non-ending, dense and linear time.

PROOF

To prove the theorem we will show that the formula $Fp \vee F\neg p$ has a counter model.

Let model $\mathfrak{M}_{(T,\wp)}(= \{m_i : i \in I\})$ satisfies the conditions below when in any world $m_i(= \langle T_i, R_i, V_i \rangle)$ the relation R_i is as follows:

1) $\forall_{t \in T_i} \forall_{t_1 \in T_i} \forall_{t_2 \in T_i}$ [if $(t_1 R_i t$ and $t_2 R_i t)$, then $(t_1 = t_2$ or $t_1 R_i t_2$ or $t_2 R_i t_1)$] (left linearity),

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[3]   Axioms and rules of inference of K_t are presented in the appendix.
[4]   R. P. McArthur, *Tense Logics*, p. 22.
[5]   Axioms and rules of inference of IT_m are presented in the appendix.

2) $\forall_{t \in T_i} \forall_{t_1 \in T_i} \forall_{t_2 \in T_i}$ [if (tR_it_1 and tR_it_2), then ($t_1 = t_2$ or $t_1R_it_2$ or $t_2R_it_1$)] (right linearity)

3) $\forall_{t \in T_i} \forall_{t_1 \in T_i} \exists_{t_2 \in T_i}$ (tR_it_2 and $t_2R_it_1$) (density)

4) $\forall_{t \in T_i} \exists_{t_1 \in T_i}$ tR_it_1 (there is not a last moment of time).

Let $m_1(= \langle T_1, R_1, V_1 \rangle)$, $m_2(= \langle T_2, R_2, V_2 \rangle) \in \mathfrak{M}_{(T,\wp)}$ are such that $t_0 \in T_1$ and following conditions are satisfied:

5) $\forall_{t \in T_1}$ if t_0R_1t, then $p \notin V_1(t)$,

6) $\forall_{t \in T_2}$ if t_0R_2t, then $p \in V_2(t)$,

7) $m_1 \le m_2$.

In the world m_1 there is no moment t later than t_0 (t_0R_1t) where a sentence p is satisfied in t.

From the definition 5.f we arrive at

8) $\mathfrak{M}_{(T,\wp)} \nvDash Fp[t_0, m_1]$.

However, in the world m_1 there is no moment of time t later than t_0 when sentence $\neg p$ is satisfied in t. To satisfy sentence $\neg p$ at some moment of time t in the world m_1, it is necessary for sentence p not to be satisfied at the moment t in every world determined not less than world m_1 (definition 5.b[6]). The condition cannot be fulfilled because in the world m_2 (determined not less than the world m_1) at any moment t later than t_0 (t_0R_1t) sentence p is satisfied.

As a result, in the world m_1 there is no moment t, later than t_0 (t_0R_1t) such that, in t is satisfied the sentence $\neg p$. From the definition 5.f we have

9) $\mathfrak{M}_{(T,\wp)} \nvDash F\neg p[t_0, m_1]$.

8), 9) and definition 5.c present the following:

$$\mathfrak{M}_{(T,F)} \nvDash (Fp \vee F\neg p)[t_0, m_1].$$

From the above theorem we arrive at the following corollaries:

COROLLARY 1

$F\alpha \vee F\neg\alpha$ is not a tautology of non-ending, dense and linear time.

COROLLARY 2

$F\alpha \vee F\neg\alpha$ is not a tautology of minimal intuitionistic tense logic IT_m.

Because $F\alpha \vee F\neg\alpha$ is not even a tautology of intuitionistic tense logic, we assume that a semantic time is linear, dense and non-ending. Therefore, we can say that in the language of IT_m it is possible to describe that some

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[6] See appendix.

events are not determined, though in the real world there are no such events. However, intuitionistic tense logic is not determined by it.

$F\alpha \vee F\neg\alpha$ is a tautology of tense logic if a possible semantic world fulfills some conditions. A class of these semantics is determined by the following theorem:

THEOREM 3

For any p, $\vDash (Fp \vee F\neg p)$ if and only if

a) for any model $\mathfrak{M}$, for any world $m_i \in \mathfrak{M}$ a time in the world m_i is non-ending,

b) for any model $\mathfrak{M}$, for any $m_i, m_j \in \mathfrak{M}$ holds $m_i = m_j$.

PROOF

To prove this theorem, we have to prove two theses:

A) If $Fp \vee F\neg p$, then it fulfills conditions a) and b),

B) If conditions a) and b) are fulfilled, then $\vDash (Fp \vee F\neg p)$.

PROOF A)

We will construct our proof using the *ad absurdum* method.

Let us assume that $\vDash (Fp \vee F\neg p)$. Imagine that a) condition is not fulfilled. Let us consider model $\mathfrak{M}_{(T,\wp)}$ and world $m_i (\in \mathfrak{M}_{(T,\wp)})$ with $m_i = \langle T_i, R_i, V_i \rangle$ and time in the world m_i possessing the last moment. If $t \ (\in T_i)$ is the last moment of time, then it is not true that $\exists_{t_1 \in T_1} t R_i t_1$. In the world m_i there is no moment of time later than t. Then

$$\mathfrak{M}_{(T,\wp)} \nvDash (Fp \vee F\neg p)[t, m_i].$$

It is in contradiction with the assumption that $\vDash (Fp \vee F\neg p)$.

Now let us assume that condition b) is not satisfied.

Let us consider $\mathfrak{M}_{(T,\wp)}$ model which we used in the proof of the theorem 2. For $m_1, m_2 (\in \mathfrak{M}_{(T,\wp)})$ $m_1 \neq m_2$. To prove the theorem 2 we showed that

$$\mathfrak{M}_{(T,\wp)} \nvDash (Fp \vee F\neg p)[t_1, m_1].$$

It contradicts the assumption that $\vDash (Fp \vee F\neg p)$.

PROOF B)

In this case, we will also construct our proof using the *ad absurdum* method.

Let $\mathfrak{M}_{(T,\wp)}$, $m_1 (\in \mathfrak{M}_{(T,\wp)})$ be any, but conditions a) and b) are given and satisfied.

Let $m_1 = \langle T_1, R_1, V_1 \rangle$. From the principle of bivalency for any moment $t \ (\in T_1)$ **1)** $\mathfrak{M}_{(T,\wp)} \vDash p[t, m_1]$ or **2)** $\mathfrak{M}_{(T,\wp)} \nvDash p[t, m_1]$. From b) condition for any $m_i, m_j \in \mathfrak{M}_{(T,\wp)}$ $m_i = m_j$. If $\mathfrak{M}_{(T,\wp)} \nvDash p[t, m_1]$, then for any m_j $\mathfrak{M}_{(T,\wp)} \nvDash p[t, m_j]$. So, from the definition 5.b we get **3)** $\mathfrak{M}_{(T,\wp)} \vDash \neg p[t, m_1]$. From 1) and 3) we get **4)** $\mathfrak{M}_{(T,\wp)} \vDash p[t, m_1]$ or $\mathfrak{M}_{(T,\wp)} \vDash \neg p[t, m_1]$. A mo-

ment t is any, then 4) is true at every moment of time of the world m_1. From a) condition it follows that for any moment t ($\in T_1$) there is a moment t' ($\in T_1$), when tR_1t'. So, in the moment t' **5)** $\mathfrak{M}_{(T,\wp)} \vDash p[t', m_1]$ or $\mathfrak{M}_{(T,\wp)} \vDash \neg p[t', m_1]$. Because of tRt', then from 5) and the definition 5.f we will get **6)** $\mathfrak{M}_{(T,\wp)} \vDash Fp[t, m_1]$ or $\mathfrak{M}_{(T,\wp)} \vDash F\neg p[t, m_1]$. From 6) and the definition 5.c we have **7)** $\mathfrak{M}_{(T,\wp)} \vDash (Fp \vee F\neg p)[t, m_1]$. Because of the model $\mathfrak{M}_{(T,\wp)}$, world m_1 ($\in \mathfrak{M}_{(T,\wp)}$) and moment t ($\in T_1$) were the satisfied conditions are a) and b), we will arrive at the conclusion that in the class of such models $\vDash (Fp \vee F\neg p)$.

Let us highlight that if the class of models satisfies condition b) of the above theorem, there will not be proper models for tense logic based on intuitionistic logic. Because for any i, j $m_i = m_j$, the multiplicity of worlds given in the possible world's semantics for intuitionistic tense logic is reduced to one world. The one-world semantics suits tense logic based on classical logic.

Aristorelian "sea fight"

Now let us consider connections between the thesis of determinism formulated as $F\alpha \vee F\neg\alpha$, and Aristotelian example "*There will be a sea battle tomorrow or there will not be a sea battle tomorrow*".

The Aristotelian example has no an equivalent in the language of tense logic with standard tense operators. We can suggest writing "*in the future there will be α*", but we cannot suggest "*tomorrow will be α*".

However, we can imagine a language where, except for F operator, there is F^T operator (this is an example of a metric tense operator) and the expression $F^T p$ is understood as "*there will be p tomorrow*".

If we assume that p means "*sea fight*", then in the language with a F^T operator the Aristotelian sentence:

"*There will be a sea battle tomorrow or there will not be a sea battle tomorrow.*"

can be written as:

$$F^T p \vee F^T \neg p.$$

Because of $F^T p \vee F^T \neg p$ is not an expression of the language of tense logic with standard tense operators, we cannot study syntactical connections between the following expressions: $Fp \vee F\neg p$ and $F^T p \vee F^T \neg p$. However, intuitive $Fp \vee F\neg p$ is a semantic result of $F^T p \vee F^T \neg p$.

$$F^T p \vee F^T \neg p \ \rightsquigarrow \ Fp \vee F\neg p$$

If the sentence "*There will be a p tomorrow or there will not be a p to-morrow.*" is true, then we can say that there is a moment of time in the future when p or *not p* is true at this moment. However, $F^T p \vee F^T \neg p$ is not a semantic result of $Fp \vee F\neg p$. If $Fp \vee F\neg p$ is true, then we can say that there is a moment of time in the future when p is true at this moment, or we can say that there is a moment of time in the future when $\neg p$ is true. However, it is not enough to say that p, or $\neg p$ will be true exactly tomorrow. Therefore, we can say that the truth of the sentence $F^T p \vee F^T \neg p$ is a necessary condition for the truth of the sentence $Fp \vee F\neg p$.

On the other hand, the falsity of the statement $Fp \vee F\neg p$ implies the falsity of the statement $F^T p \vee F^T \neg p$. Therefore, we arrive at the conclusion that the truth of the sentence $Fp \vee F\neg p$ is a necessary condition for the truth of the sentence $F^T p \vee F^T \neg p$.

The following conclusions can be drawn from the above considerations:

CONCLUSION 1

In tense logic of non-ending time based on classical logic necessary conditions are fulfilled to reconstruct the Aristotelian argument about determinism. If we add a specific tense operator F^T, to the language of such logic, it will be possible to arrive at a formal reconstruction of this argument.

CONCLUSION 2

In intuitionistic tense logic of non-ending time necessary conditions are not fulfilled to reconstruct the Aristotelian argument about determinism (even if we assume that the semantic time is linear and dense). If we add a specific tense operator F^T, to the language of the intuitionistic tense logic of non-ending time, we still cannot perform a formal reconstruction of the Aristotelian argument about determinism.

Therefore, intuitionistic tense logic seems to be a better formalism to describe the properties of non-determined worlds. Intuitionistic tense logic is constructed at the cost of the refutation of excluded middle law. However, it was already in the past that this law was questioned. Aristotle wrote:

> In the first place, though facts should prove the one proposition false, the opposite would still be untrue[7].

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[7] Aristotle, *On Interpretation*, Chapter IX.

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APPENDIX

THE SYSTEM K_t

SYNTACS

Alphabet:
- set of propositional letters Ψ,
- connectives: $\neg, \Rightarrow$,
- temporal operators: G, H[8],
- parentheses: $), ($.

Set of sentences of language K_t of is defined as follows:

DEFINITION 1

Set of sentences is the smallest set $\mathbf{Z}$ where:
- $\Psi \subseteq \mathbf{Z}$,
- if $\alpha, \beta \in \mathbf{Z}$, then $\neg\alpha, (\alpha \Rightarrow \beta), G\alpha, H\alpha \in \mathbf{Z}$.

In K_t we have the following definitions:

DEFINITION 2

$$
\begin{aligned}
(\alpha \vee \beta) &\equiv (\neg\alpha \Rightarrow \beta), \\
(\alpha \wedge \beta) &\equiv \neg(\alpha \Rightarrow \neg\beta), \\
(\alpha \Leftrightarrow \beta) &\equiv \neg[(\alpha \Rightarrow \beta) \Rightarrow \neg(\beta \Rightarrow \alpha)], \\
F\alpha &\equiv \neg G\neg\alpha, \\
P\alpha &\equiv \neg H\neg\alpha.
\end{aligned}
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[8] Some formulations of systems of tense logic accept operators F i P as the basic ones.

SEMANTICS

In semantic considerations in K_t we assume that time has a point structure.

Let T be any non-empty set (elements of this set are called "moment of time"). A time $\mathfrak{T}$ is an ordered couple $\langle T, R \rangle$, where R is a binary relation (earlier-later) on T. Let V be a function mapping points $t \in T$ to subsets $V(t)$ of the set of propositional letters ($V : \Psi \to 2^T$). Model "$\mathfrak{M}$" is an ordered triple $\langle T, R, V \rangle$.

The so-called "truth definition" explains what it means for α to be true model $\mathfrak{M}$ in a moment of time ($\mathfrak{M}, t \vDash \alpha$):

DEFINITION 3

a) $\mathfrak{M}, t \vDash \alpha$ $\quad \equiv \quad t \in V(\alpha)$, if $\alpha \in \Psi$,

b) $\mathfrak{M}, t \vDash \neg\alpha$ $\quad \equiv \quad$ not $\mathfrak{M}, t \vDash \alpha$,

c) $\mathfrak{M}, t \vDash (\alpha \Rightarrow \beta) \equiv$ if $\mathfrak{M}, t \vDash \alpha$, then $\mathfrak{M}, t \vDash \beta$,

d) $\mathfrak{M}, t \vDash G\alpha$ $\quad \equiv \quad$ for any t_1 such that tRt_1 holds $\mathfrak{M}, t_1 \vDash \alpha$,

e) $\mathfrak{M}, t \vDash H\alpha$ $\quad \equiv \quad$ for any t_1 such that t_1Rt holds $\mathfrak{M}, t_1 \vDash \alpha$.

AXIOMS

1. $\alpha \Rightarrow (\beta \Rightarrow \alpha)$,

2. $[\alpha \Rightarrow (\beta \Rightarrow \gamma)] \Rightarrow [(\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)]$,

3. $(\neg\alpha \Rightarrow \neg\beta) \Rightarrow (\beta \Rightarrow \alpha)$,

4. $G(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow G\beta)$,

5. $H(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow H\beta)$,

6. $\alpha \Rightarrow GP\alpha$,

7. $\alpha \Rightarrow HF\alpha$.

RULES OF INFERENCES

R1: Modus Ponens.

Rules of temporal generalization

R2: $\dfrac{\alpha}{G\alpha}$ $\qquad\qquad$ R3: $\dfrac{\alpha}{H\alpha}$

Axioms 1,2 and 3 are substitutions of tautologies of classical propositional calculus. Axioms 4, 5, 6, 7 are specific axioms of K_t system.

IT_m – MINIMAL INTUITIONISTIC TENSE LOGIC

SYNTAX

Alphabet:

- set of propositional letters: Ψ,
- intuitionistic unary connective: $\neg$,
- intuitionistic binary connectives: $\wedge, \vee, \Rightarrow, \Leftrightarrow$,
- temporal operators: G, H, F, P,
- parentheses: $), (.$

Set $\boldsymbol{Z}$ of sentences of language of IT_m system is defined according to the standards.

NOTATION

I – nonempty set of indexes of worlds,

T_i $(i \in I)$ – nonempty set of moments of time in world indexed by i,

R_t $(\subseteq T_i \times T_i)$ (Binary relation on T_i),

$\mathcal{T}_i$ $(= \langle T_i, R_i \rangle)$ (A time in a world indexed by i),

$\mathcal{T} = \bigcup\limits_{i \in I} \mathcal{T}_i$ (Set of all moments of time),

R $(= \bigcup\limits_{i \in I} R_i)$ (Binary relation on the set of all moments of time),

V_i $(\subseteq T_i \times 2^{\Psi})$, where $i \in I$ (V_i is a function mapping to elements t $(\in T_i)$ subsets $V_i(t)$ of the set of propositional letters),

$\wp = \{V_i : i \in I\}$,

m_i $(= \langle T_i, R_i, V_i \rangle)$, where $i \in I$ (m_i is a world indexed by i),

$\mathfrak{M}_{(T,\wp)} = \{\langle T_i, R_i, V_i \rangle : V_i \in \wp, i \in I\}$, then $\mathfrak{M}_{(T,\wp)} = \{m_i : i \in I\}$

($\mathfrak{M}_{(T,\wp)}$ is a model based on time $\mathcal{T}$ and class of functions $\wp$).

Between elements of a model $\mathfrak{M}_{(T,\wp)}$ we introduce a relation $\leq$ $(\subseteq \mathfrak{M}_{(T,\wp)} \times \mathfrak{M}_{(T,\wp)})$ defined as follows:

DEFINITION 4

For any $i, j \in I$:

$$m_i \leq m_j \equiv (T_i \subseteq T_j \text{ and } R_i \subseteq R_j \text{ and } \forall_{t \in T_i} V_i(t) \subseteq V_j(t)).$$

$m_i \leq m_j$ means that the world m_j is no less determined than the world m_i.

REMARK

m_i^* (where $i \in I$) means any m_j $(\in \mathfrak{M}_{(T,\wp)})$ such that $m_i \leq m_j$.

DEFINITION 5

For a model $\mathfrak{M}_{(T,\wp)}$, world m_i ($= \langle T_i, R_i, V_i \rangle$), element t ($\in T_i$), a tense-logical formula α $\mathfrak{M}_{(T,\wp)} \vDash \alpha[t, m_i]$ is defined by the following conditions:

a) $\mathfrak{M}_{(T,\wp)} \vDash \alpha[t, m_i] \qquad \equiv \alpha \in V_i(t)$, if $\alpha \in \Psi$,

b) $\mathfrak{M}_{(T,\wp)} \vDash \neg\alpha[t, m_i] \qquad \equiv \forall_{m_i^* \in \mathfrak{M}_{(T,\wp)}} \mathfrak{M}_{(T,\wp)} \nvDash \alpha[t, m_i^*]$,

c) $\mathfrak{M}_{(T,\wp)} \vDash (\alpha \vee \beta)[t, m_i] \equiv \mathfrak{M}_{(T,\wp)} \vDash \alpha[t, m_i]$ or $\mathfrak{M}_{(T,\wp)} \vDash \beta[t, m_i]$,

d) $\mathfrak{M}_{(T,\wp)} \vDash (\alpha \wedge \beta)[t, m_i] \equiv \mathfrak{M}_{(T,\wp)} \vDash \alpha[t, m_i]$ and $\mathfrak{M}_{(T,\wp)} \vDash \beta[t, m_i]$,

e) $\mathfrak{M}_{(T,\wp)} \vDash (\alpha \Rightarrow \beta)[t, m_i] \equiv \forall_{m_i^* \in \mathfrak{M}_{(T,\wp)}} (\mathfrak{M}_{(T,\wp)} \nvDash \alpha[t, m_i^*]$ or $\mathfrak{M}_{(T,\wp)} \vDash \beta[t, m_i^*])$,

f) $\mathfrak{M}_{(T,\wp)} \vDash F\alpha[t, m_i] \equiv \exists_{t_1 \in T_i} (tR_i t_1$ and $\mathfrak{M}_{(T,\wp)} \vDash \alpha[t_1, m_i])$,

g) $\mathfrak{M}_{(T,\wp)} \vDash G\alpha[t, m_i] \equiv \forall_{m_i^* \in \mathfrak{M}_{(T,\wp)}} \forall_{t_1 \in T_i^*}$ (if $tR_i^* t_1$, then $\mathfrak{M}_{(T,\wp)} \vDash \alpha[t_1, m_i^*])$,

h) $\mathfrak{M}_{(T,\wp)} \vDash P\alpha[t, m_i] \equiv \exists_{t_1 \in T_i} (t_1 R_i t$ and $\mathfrak{M}_{(T,\wp)} \vDash \alpha[t_1, m_i])$,

i) $\mathfrak{M}_{(T,\wp)} \vDash H\alpha[t, m_i] \equiv \forall_{m_i^* \in \mathfrak{M}_{(T,\wp)}} \forall_{t_1 \in T_i^*}$ (if $t_1 R_i^* t$, then $\mathfrak{M}_{(T,\wp)} \vDash \alpha[t_1, m_i^*])$,

AXIOMS

For any $\alpha, \beta, \gamma \in \mathbf{Z}$:

A1) $\alpha \Rightarrow (\beta \Rightarrow \alpha)$,

A2) $(\alpha \Rightarrow \beta) \Rightarrow \{[\alpha \Rightarrow (\beta \Rightarrow \gamma)] \Rightarrow (\alpha \Rightarrow \gamma)\}$,

A3) $[(\alpha \Rightarrow \gamma) \wedge (\beta \Rightarrow \gamma)] \Rightarrow [(\alpha \vee \beta) \Rightarrow \gamma]$,

A4) $(\alpha \wedge \beta) \Rightarrow \alpha$,

A5) $(\alpha \wedge \beta) \Rightarrow \beta$,

A6) $\alpha \Rightarrow [\beta \Rightarrow (\alpha \wedge \beta)]$,

A7) $\alpha \Rightarrow (\alpha \vee \beta)$,

A8) $\beta \Rightarrow (\alpha \vee \beta)$,

A9) $(\alpha \wedge \neg\alpha) \Rightarrow \beta$,

A10) $(\alpha \Rightarrow \neg\alpha) \Rightarrow \neg\alpha$,

H1) $H(\alpha \Rightarrow \beta) \Rightarrow (H\alpha \Rightarrow H\beta)$,   G1) $G(\alpha \Rightarrow \beta) \Rightarrow (G\alpha \Rightarrow G\beta)$,

H2) $H(\alpha \Rightarrow \beta) \Rightarrow (P\alpha \Rightarrow P\beta)$,   G2) $G(\alpha \Rightarrow \beta) \Rightarrow (F\alpha \Rightarrow F\beta)$,

H3) $\alpha \Rightarrow HF\alpha$,   G3) $\alpha \Rightarrow GP\alpha$,

H4) $PG\alpha \Rightarrow \alpha$,   G4) $FH\alpha \Rightarrow \alpha$,

H5) $P(\alpha \vee \beta) \Rightarrow (P\alpha \vee P\beta)$,   G5) $F(\alpha \vee \beta) \Rightarrow (F\alpha \vee F\beta)$,

H6) $(P\alpha \Rightarrow H\beta) \Rightarrow H(\alpha \Rightarrow \beta)$,   G6) $(F\alpha \Rightarrow G\beta) \Rightarrow G(\alpha \Rightarrow \beta)$,

H7) $P\alpha \Rightarrow \neg H\neg\alpha$,   G7) $F\alpha \Rightarrow \neg G\neg\alpha$,

RULES

Modus Ponens

$$MP: \frac{\alpha \Rightarrow \beta, \alpha}{\beta}$$

Temporal generalization rules:

$$RH: \frac{\vdash_{IT_m} \alpha}{\vdash_{IT_m} H\alpha}$$

$$RG: \frac{\vdash_{IT_m} \alpha}{\vdash_{IT_m} G\alpha}$$