STUDIES IN LOGIC, GRAMMAR AND RHETORIC 5 (18) 2002

Michał Tempczyk
Institute of Philosophy and Sociology
Polish Academy of Science

THE DYNAMICS OF NONLINEAR SYSTEMS

Introduction

Nonlinear dynamics, commonly called the chaos theory, changes the scientific way of looking at the dynamics of natural and social systems. Since these changes are deep and great in number, it is impossible to discuss them all in one paper. In this introduction I will therefore try to show how the approach of physicists to dynamical systems has changed. It has led to the change of the role of physics amongst the other natural sciences. I shall start with the popular view of physics.

The success of physics as the most precise, fundamental natural science, held up as a model for other sciences (and not only the natural ones) was based on the fact that by using developed mathematical theories and making precise measurements, physics was able to describe, understand and explain the properties and behavior of many important kinds of bodies and dynamical systems, such as the Solar System, simple mechanical systems (e.g. the clock), atoms and so on. The results of physics considerably surpassed the achievements of scientists in other sciences, particularly in biology, geology, psychology, etc. Because of this, fundamental physical theories, such as classical mechanics and electrodynamics, were the model of scientific knowledge and other domains of science tried to reach a comparable level of generality and precision.

When studying physics I was proud to be learning such a perfect science, the ideal of scientific method, and I was not conscious that this picture of physics was not consistent with the actual practice. That inconsistency follows from the fact that even in general physical theories, which are mathematically well worked-out, there are not many phenomena which can be described and explained in a precise, theoretical way. Let us look from this viewpoint at mechanics for instance. Newton's equations give the universal

ISBN 83-89031-54-X ISSN 0860-150X

dependence of motion upon forces, but in the monographs on mechanics we find only a few simple situations in which the equations really describe what is going on and which can be solved exactly. Without precise solutions nobody can predict the future behavior of the process. So we have two main limitations of physical theories: the phenomena investigated may be too complicated to be fully modeled and even if we have a model, we may not be able to solve its equations. When one learns physics, we are hardly ever aware of these limitations because the lectures are devoted to those problems which have been satisfactory and generally solved. Thus during classes one learns how to solve examples that can be solved. In virtue of this, I was sure, after I finished studying physics, that it was the powerful general science. Through investigating philosophical and methodological problems of physics I was slowly led to the conclusion that physics was actually not so powerful and that it had important limitations. Those are not limitations connected with inaccuracy of observations, often discussed by physicists, but with its theoretical methods and equations. We will examine them in connection with the theory of the Solar System.

The first well known difficult problem is that the equations of motion for bodies interacting under a gravitational force can be solved only for two bodies. Such bodies move on ellipses around a common center of mass. It is easy to write the equations of motion for systems with many bodies: these equations have, however, no analytical solutions. The so-called reduced Hill's problem is the simplest system that has no precise solution. The system consists of two big interacting bodies moving around their center of mass with the third small body moving in their common gravitational field. The third body is so small that its action on the larger bodies can be neglected. It moves in the well-known regular field of its two neighbors. Its motion is not always regular because there are areas in which the forces generated by the larger bodies balance one another and a small change of position results in the enormous change of the motion and the trajectory of the smaller body. This instability of the motion was discovered by H. Poincaré in 1892, who called this kind of motion the homoclinic tangle (Stewart, ch. 4). The example shows how a small complication of the system, the addition of a small body to two larger bodies moving in a regular way, leads to an essential change of the motion. The motion becomes complicated and unpredictable. Having discovered this, Poincaré was not able to pursue the study of it as at that time the reduced Hill's problem was too difficult to be described and analyzed precisely. Now, thanks to the computer, it is well understood. One can imagine how complicated the motion of ten bodies interacting under gravity would be. The dynamics of the Solar System is simple because the mass of the Sun is so much bigger than masses of planets and the planets are far one from another. Because of this, one can separately study the motion of each planet with the Sun.

In those areas of physics investigating more complicated systems, such as hydrodynamics and atomic physics, computational limitations of even simple models are well-known and it narrows the area of their effective applications. The theory is in principle general but the range of its efficient models is much smaller. Thus we see that the important and fascinating successes of physics, in fact, comprise a narrow domain of natural phenomena. Most observed phenomena are too complicated for the application of simple physical models to be able to yield precise results. One has to use simplified models, approximate methods and inaccurate descriptions. The complexity of natural processes was the obstacle in the development of science. Of course, scientists could not give up investigating complex phenomena because of their importance for us. Without understanding them people would not be able to act and to develop technology. The study of complex systems conducted over the centuries has produced many important results and about forty years ago they began to come together to create a new universal domain of research called the theory of chaos. The theory offers effective, precise methods of complex systems analysis. It is a mixture of mathematical, empirical and computational methods and results. In order to describe its achievements and possibilities I shall concentrate my attention on two problems: the instability and complexity of motion of simple systems and the order appearing in the behavior of complex systems.

The role of instability in the behavior of dynamic systems

I start with a statement that seemed obvious not long ago: a simple material system should act in a simple way. Led by this principle, scientists tried to study the simplest physical, chemical and biological structures, because their behavior should be equally simple and intelligible. However, it can be easily shown that a simple system, for instance, a mechanical one, does not need to act in a simple way. A good example is a big pendulum on the end of which a small pendulum is hung. Each of them separately works in a simple, predictable way but their combination is an irregular unpredictable system. The small pendulum disturbs the motion of the big pendulum, but itself also behaves in a complex way because its hanging point is moving. Their motion is given by two interconnected differential equations which can be solved only approximately, yielding irregular, complicated solutions. The simplicity of the structure of the system does not imply the simplicity of its dynamics.

If we wish to understand the features of the dynamics of complex systems, we should look at them in a new way. Classical physics, first of all, tried to solve the equations of motion in order to describe quantitatively the motion of the system studied: a planet, a pendulum, colliding balls, etc. The motion of a complex system cannot be known precisely so we pose questions concerning its kind and properties. The most important question concerns the stability of the motion. Stability means that small perturbations of the motion result in small, slowly increasing changes of the trajectory. The systems which are the most important for us, such as the motion of Earth around the Sun or the motion of a car, are stable but many important processes, such as atmospheric phenomena, are unstable. Another problem concerning the dynamics of complex systems is the kind of motion realized by the system. That motion can neither be described nor predicted, as it is too complicated. It can, however, be characterized approximately and quantitatively. For example, by studying the behavior of a system with friction one can easily predict that after some time it will stop if there is no energy inflow from outside.

The chaos theory uses specific concepts to examine such problems. The most important is the concept of phase space (Tempczyk, pp. 34-37). It is the space of parameters completely describing the motion of a given system. In classical mechanics, in studying the motion of a body we usually use the coordinates of its position, but this does not provide a full description since bodies can move on one trajectory with different velocities. Therefore, the phase space of material point is built from positions and velocities. For formal reasons physicists use momenta instead of velocities. Momentum is the product of mass and velocity of the body. The advantage of phase space is that it contains the entire history of the motion of the system – its trajectory. Because of the uniqueness of the solutions of the equations of motion, trajectories cannot cross. They are lines resembling the lines of the flow of water in a river. Looking at trajectory families, which are classes of the equations of motion solutions, one can answer questions concerning the kind of motion. In the case of stable motion, neighboring trajectories disperse slowly and are rather regular. If the motion is unstable, trajectories close at the beginning separate rapidly, frequently changing their direction in the phase space. If the motion of a typical system ends in the same way, for instance, by becoming slower and slower, then all trajectories tend to the same point or area, this being called an attractor.

An often used tool of the chaos theory is the iteration procedure. We take any starting point and study how it changes after 1 second, 2 seconds, 3 seconds and so on. By observing those points in the phase space we see how the system moves: whether its motion is regular, cyclic or chaotic. Observations of this kind yield a lot of information about the dynamics though we do not possess analytical solutions of the equations of motion. Such approach to the dynamics is called a qualitative theory of differential equations.

The next key concept of the theory of chaos is linearity. A system is linear if the differential or algebraic equations describing it are linear. Systems of linear equations are easy to solve, which is why the theory of those equations was well developed in the 19^{th} century. It was used by the empirical sciences and through it linear processes became well understood. A process is linear when its parts act in the same way, independently of their surroundings and other parts. An electric field is linear. Each charge generates a defined field and the global field is the vector sum of all partial fields. One might say that the electric charge 'does not know' its environment – the system which it belongs to – and its field is always the same. The classical Newtonian theory of gravitation is likewise linear. Linear systems are easily decomposed into parts. Scientists study those parts and then reconstruct the whole. This approach is ineffective with respect to nonlinear systems as their parts adapt to the environment and their behavior is unpredictable if they are examined in separation. Most natural technical and social processes are nonlinear and non-linearity presented a substantial obstacle for science even forty years ago (Tempczyk, pp. 24-26).

Let us return to the behavior of simple systems. Their dynamics need not be simple. The Lorenz gas is an example. It is a model of electron motion in a crystal. Electrons move along straight lines and collide with atoms that are like balls arranged in a regular way. The collisions electrons make with atoms are unstable because an electron moving towards the center of the atom can turn right or left, depending on small deviations of its trajectory: Two electrons initially moving along close paths fly in different directions after colliding and their future is different. Because of this, the movements in the Lorenz gas are unstable. It is a linear system and its instability has a geometrical origin.

A well known example of simple system with complex dynamics is the system described by the logistic equation, first studied by R. May and next by M. Feigenbaum. Feigenbaum investigated the behavior of trajectories given by the simple square equation:

$$x_{n+1} = kx_n(1 - x_n) = f(x_n)$$

depending on the parameter k. It is the function that maps the interval [0,1] into itself, if $0 < k < 4$. Feigenbaum wanted to work out the motion beginning from any point x_0.

For $0 < k < 1$ the answer is simple as always $x_{n+1} < x_n$ and after many steps the value of x_n is close to 0. So 0 is the unique attractor of the system. We can imagine that x describes the population of grasshoppers in a meadow, where 1 corresponds with the maximum number of the insects in the meadow and k is the coefficient of their reproductiveness. With $k < 1$ grasshoppers reproduce too little to survive as each generation is smaller then the preceding one. They, therefore, perish.

For $1 < k < 3$ the situation is also simple. There exists the stable point $x_k = 1 - 1/k$ which is the attractor as all trajectories, except the one starting from $x = 0$, tend to it. For R. May, the biologist who used the logistic equation to describe population dynamics, the result was obvious. It proved that each population will tend to an state of equilibrium, depending on the reproductiveness of animals and environmental conditions. It was consistent with the scientific view on the nature of biological equilibrium.

The behavior of the systems changes radically when $k > 3$. The point $x_k = 1 - 1/k$ is still stable, but the value of $|dx/dt|$ becomes greater than 1 in its neighborhood causing x_k to change from an attractor to a repeller [repulsion point]. Instead of it appear two adjoint points x_1, x_2, such that $x_1 = f(x_2)$ and $x_2 = f(x_1)$ and these take on the role of the attractor. Each trajectory approaches one of them and oscillates with them in a two-element cycle. The attractor point changes into a two-element attracting cycle. The pair x_1, x_2 attract neighboring trajectories because the composition of functions $f(f(x_1))$ and $f(f(x_2))$ has the absolute value of its derivative smaller then 1. This situation changes again for $k = 1 + \sqrt{6}$. For this value each of the two branches bifurcate and there arises an attracting four-element cycle. Once again, all trajectories approach those points and jump with them in a definite order. It is easy to see that a further increase of k generates an 8-element cycle, a 16-element cycle and so on. At the limit value of $k = 3.5699456$ the cycle becomes infinite and one observes the characteristic picture of the Feigenbaum bifurcations (Tempczyk, p. 63).

Feigenbaum noticed, by studying the problem on his calculator, that successive bifurcation points become closer and closer and that the proportion of their distance remains constant. He calculated the constant $\delta = 4.6692016091$, which has been called the Feigenbaum constant in honour of him. Mathematicians were initially sure that the constant was related to the logistic function, but when Feigenbaum published his results, scientists from Los Alamos, N. Metropolis and M. and P. Steins, studied the dynamics of another function $x_{n+1} = rx_n \sin \pi x_n$, obtaining the same bifurcation scheme and, what was more important, the same constant. Many kinds of functions have since been examined and in each case the bifurcation structure has had the same constant δ. A new universal number appeared in mathematics.

At that time Feigenbaum's discovery was semi-empirical as there was no mathematical theory describing and explaining the behavior of attractors and their bifurcation points for a given function $f(x)$. Such a theory was elaborated over the course of several years and it was proved that the Feigenbaum bifurcation scheme was universal for functions having one distinctive maximum in the interval [0,1]. Those functions with another shape, for instance, those with two maxima, have another way of arising and branching attractors. The currently developed bifurcation theory is presented in various monographs, for example, in the book by Schuster (ch. 3). It is an excellent example of mathematical theory created for the description of complex systems. There are hidden interesting universal properties in their complicated and hard-to-predict behavior. This kind of mathematics is useful in the study of biological, economic, and mechanical systems. In economic contexts, it reminds me of the once-popular theory of cyclic crises in capitalism. Marxist economists claimed that capitalists invested too much, causing overproduction and cyclic crises leading to reduction of the number of firms. Then the next boom appears, capitalists invest too much and the situation repeats. It resembles the bifurcation scheme for $k > 3$, where a two-element attracting cycle shapes the dynamics.

Complex systems with simple action

The chaos theory shows two aspects of complexity. One of them was described above. Now we shall discuss the second one – the arranging action of non-linearity. A nonlinear system is one in which particular elements adapt to the environment and the whole. The consequence of such a global adaptation is that there arises a global order which is different from the order of local interactions and exceeds their diversity. Such global dynamic structures have been studied by the empirical sciences: for example, patterns of flowing water, tornadoes, living organisms, and ecological systems. These studies were difficult and imprecise because of the lack of theoretical tools and enormous complexity of those systems. From time to time there, however, appeared curious and important results that will be the subject of the discussion below.

Benard cells provide an example of complex structures that organize themselves as a consequence of the process of heat flowing into them. Benard was a French physicist who in 1900 discovered and accurately studied the global order emerging in shallow water heated from below. It is a process familiar from everyday experience. Initially, when the temperature of the heated bottom is relatively low, warm water rises up as one volume, loses its heat at the surface and sinks as it becomes cooler and heavier. When the temperature of the bottom increases, the process becomes quicker and more intensive (Tempczyk, pp. 81-82). At a certain moment, when the temperature reaches a critical value, there is a rapid change of the movement of the water and the transfer of heat. Short, parallel cylinders come into existence which rotate in such a way that the friction of their neighbors becomes as small as possible. Water moving up and down in the cylinders carries heat quicker and with lesser friction than previously and the heat flow is more effective and less chaotic. At first, the Benard cells are stable: small fluctuations and disturbances do not destroy them. However, the increase of the heating temperature causes an increase of their rotation speed and at a certain point the cells become unstable. They start to oscillate and in the end the structure disintegrates. The water motion becomes chaotic again. Benard studied the process, photographed the cells and published the results. During dozen of years physicists tried to formulate the theory of how they arise but were unsuccessful. In the monograph by Chandrasekhar (1961) devoted to hydrodynamics, the discussion of Benard cells and their theories occupies a big part of the book. They are a good example showing how the global order arranges and facilitates the course of the process: in this case, the flow of heat.

In 1963 an American meteorologist, E. Lorenz, used the model of Benard cells to describe the dynamics of processes taking place in the atmosphere over ground heated by the sun's rays. The systems resemble those of a liquid heated from below, so Lorenz elaborated a similar model and wrote three equations describing its dynamics (Schuster, ch. 1):

$$dX/dt = -\delta X + \delta Y$$
$$dY/dt = rX - Y - XZ$$
$$dZ/dt = XY - bZ.$$

The three parameters of the model are: X the velocity of the air circulation; Y – the difference of the temperatures of the air going up and down; Z is proportional to the deviation of the temperature from the equilibrium state. Lorenz had a computer at his disposal and worked out a program solving his equations. It helped him to discover two essential features of those equations.

The first property, called by Lorenz 'the butterfly effect', is the instability of the solutions. The computer calculated the values of Y twice. The starting value of Y was slightly simplified the second time and it resulted in a completely different shape of the Y function. Lorenz came to the conclusion that atmospheric phenomena are unstable and the weather forecasting cannot be done effectively for periods of time longer than a few days because the errors are too great compared to the parameters calculated.

More important was the second Lorenz's discovery – that of an attractor. Lorenz decided to study the long-term behavior of his system. He set the computer in motion and left it to work for a long period of time, having no idea what the results would be. After some time the solutions, the trajectories in the 3-dimensional space for given X, Y, Z parameters, started to arrange themselves in a 2-dimensional pattern of two leaves, now familiar from books on the chaos theory. The trajectory first wandered along on one of them, moving on circles, then rapidly jumped to the other one, again drawing circles, then jumped again back to the first leaf, and so on. The number of turns on one leaf was unpredictable. It was of a completely accidental nature, even though the system worked according to strict deterministic equations. The same attractor arose for different trajectories starting from different initial conditions and thus had a universal character. I am not going to describe the structure of the Lorenz attractor as it is well known (Schuster, ch. 5; Tempczyk, pp. 67-69), I would like to emphasize rather its great importance for science. In 1963 the idea of such an area attracting neighbor trajectories was incomprehensible and Lorenz's colleagues treated it simply as a by-product of the calculating procedure used. Lorenz published his results in a professional meteorological journal and stopped researching the problem. Ten years passed before mathematicians and naturalists began to understand the role of attractors. They then started to look for them in nature. Lorenz's work was rediscovered and its author became famous. Mathematicians found precise constructions leading to attractors, such as the Rossler and Henon attractors. Scientists started to discover attractors in data describing the dynamics of processes taking place in nature and society.

Presently, there is no mathematical theory of attractors. Mathematicians are not able to decide if given equations have attractors and for which values of the controlling parameter. One has to make one's calculations and observe whether the solutions reveal regularities corresponding to an attractor. One thing is certain. Trajectories can approach one another in the phase space to create an attractor only when there is inside the system the dissipation of energy supplied from outside. This explains why the energy-conserving Hamiltonian systems have no attractors. In the case of the Lorenz systems, energy is carried by the sun's rays and then transferred to higher levels of the atmosphere.

There are two methods for looking for the attractors of dynamic empirical systems. One of them consists in working with a mathematical model of the system. The equations of motion are solved regaedless whether they have an attractor or not. The history of the Lorenz equations was of this kind. Very often, however, scientists have no mathematical model of the phenomena under study but they do have a lot of empirical data which they try to order. Attractors are a type of order which is very difficult to observe. In 1981 F. Takens worked out a method of discovering attractors with delayed time series (Schuster, ch. 5.3). The method was successfully applied by a team of physicists led by R. Shaw to the study of a dripping faucet (Crunfield... 1986). They obtained interesting results which they published in Scientific American. They measured the temporal distance between succeeding drops and using Takens's method and they acquired a three-dimensional picture of the attractor. Next, the scientists elaborated the mathematical model of the process of the drop falling off and by solving its equations they found the same picture of the attractor. This example proved the effectiveness of the Takens's method. It is now widely used to search for regularities in biological, demographic, and physical systems.

Perspectives – the chaos theory in social sciences

In conclusion, I shall analyze the new possibilities the chaos theory gives to the social sciences. Its methods and results enable scientists to study in a new and effective way the behavior of complex systems which are too complicated to be analyzed by classical tools. The application of those methods has brought enormous progress in many well-developed domains, such as hydrodynamics, physics, chemistry and biology. Those are fields of science which study both simple and complex systems. However, the methods are most promising in those fields of research in which scientists are from the beginning dealing with complex phenomena and where they cannot use simplified models as applied in classical science. Such a situation is typical for sociology and economics with the result that in those sciences standard mathematical models based on differential equations are not efficient and their possibilities for gaining knowledge are fairly limited. Scientists have to use methods taking into account the high level of complexity of the processes under study. There are two different ways they can take.

The first one is the construction of nonlinear mathematical models of phenomena. The models help to understand and explain some of astonishing properties of self-organizing systems. They have been successfully applied by many researches investigating social processes. The application of such models is a straightforward affair; if there is a possibility of modeling mathematically complex processes, then such methods are used when necessary. More promising and of greater generality is the second way of analyzing complexity – the search for attractors in big databases lacking formal models. It is more general than mathematical modeling as one can draw important conclusions about complex processes while having no idea about the nature and course of those processes.

When we study complex processes whose representation requires many parameters, then any formal model of them is of necessity approximate. The use of such a model is efficient only when it enables one to grasp the essential properties of phenomena under study and to predict their future. Having the model, one can complicate it, making it closer and closer to the real process and obtaining better results. The agreement of theoretical predictions with observations proves that the scientist is heading in the right direction and that his theory adequately represents the reality. This route is, however, not open when there is no theory as all approximate models will give inaccurate results and researchers do not know how to describe theoretically the processes analyzed. They can then only use model-independent methods of data analysis. The most sophisticated method is the search for attractors in big sets of empirical data. The method helps to discover regularities hidden in the chaos of local relations and complex behavior. It is widely used by economists, sociologists and psychologists.

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