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LEONARD SAVAGE'S MATHEMATICAL THEORY OF DECISION

Certain elements of the mathematical theory of decision were developed as early as at the end of the 19th century, or even earlier. The theory of probability and its applications to gambling were also highly developed. The concept of maximization of expected utility had already been introduced, however, it was not used for a long time. Still, the mature mathematical form of the theory of decision has been developed only recently. In the 1940s and 1950s a few works were published influencing greatly the development of mathematical research connected with the theory of decision, including *The Theory of Games and Economic Behaviour* by John von Neumann and Oskar Morgenstern (the theory of games) and *The Foundations of Statistics* by Leonard Savage (axiomatic foundations of the theory of subjective expected utility). These works formed the axiomatic bases for the contemporary theory of decision, and the results logically expanded in a number of theoretical findings from the beginning of the 20th century.

Probability

In the 1900s economists noticed that probability is a useful theoretical tool for modelling such phenomena as financial investments or decision-making in companies. However, the notion itself was ambiguously interpreted those days. The most famous interpretations are: classical, frequentist (objective), logical, and subjective[1].

[1] Three of these interpretations are still being used today.

ISBN 83-89031-54-X ISSN 0860-150X

Classical (Bernoulli, Laplace)

The probability of an event is the ratio of the number of “favorable” cases to the total number of cases, where the cases are equally likely.

Frequentist

The probability of an event means the frequency of its occurrence in a great (potentially infinite) number of repeated trials. This interpretation is a basis for common statistical methods of testing hypotheses.

Logical (Keynes)

Probability is connected with statements and can be deduced from truth-value of the premises of the statements for which it is being inferred. (However, it remains unclear how the logical value of premises is defined.)

Subjective[2] *(Savage, Finetti)*

Probability is a subjective degree of conviction, which could be attributed to any event, either repeated or not. It could be measured by psychometric methods, such as observing choices in gambling. This approach was criticised since two people using the same information may disagree as to the probability of an event. The concept of subjective probability has its shortcomings. From a mathematical point of view, the sum of probability of event A and its opposite should be one. However, this in not the case in the concept of subjective probability. Some researchers, e.g. Serik Suleimenov, claim that in order to avoid this problem it is essential to abandon the notion of subjective probability and apply special functions to objective probability.

Utility and preferences

At the beginning of the 20$^{\text{th}}$ century the idea of utility as a psychological term was notorious as it was doubtful when used to calculate the probability of events. Therefore, economists started emphasising the notion of preference as a primary psychological notion. An individual may not be able to attribute utility expressed in the number to an object but presumably can say which of the alternatives he prefers or decide that all

[2] The presentation expounds on the concept of subjective probability as it is the basis for Savage’s theory of decision.

the alternatives are equivalent to him. Thus, it seems that preference as a qualitative binary relation can be perfectly perceived as a primary operation, on which the axiomatic theory of decision can be built.

The theory of decision

Leonard Savage formed axiomatic bases for statistics, which combined the theory of inference with the theory of decision. According to his theory, it was possible to pose and try to answer the question:

Given specific data, what decision to make?

In his theory Savage employed the subjective interpretation of probability and followed Ramsey and von Neuman and Morgenstern in using preference as a primary psychological notion. The basic notions of his theory are acts and consequences, which are used to define the notion of decision. Thus, we say that a decision has been made when there was a choice between two or more acts. Deciding on a fact involves considering consequences that can be inferred for any possible state of the world.

The following exemplifies the notion of an act and a consequence:

Example[3]

Your wife has just broken five good eggs into a bowl when you come and volunteer to finish making the omelet. The sixth egg lies unbroken beside the bowl. For some reason it must either be used for the omelet or wasted altogether. You must decide what to do with this unbroken egg.

You must decide among three acts only. Namely:

the first act: to break an egg to join the other five eggs,

the second act: to break an egg into a saucer for inspection,

the third act: to throw an egg away without inspection.

We have the following acts and consequences:

[3] The example is taken from Savage's *The Foundations of Statistics*, p. 14.

Act	*State* [4]	
	Good	*Bad*
break into a bowl	*six-egg omelet*	*No omelet, and five good eggs destroyed*
break into a saucer	*six-egg omelet, and a saucer to wash*	*five-egg omelet, and a saucer to wash*
Throw away	*Five-egg omelet, and one good egg destroyed*	*five-egg omelet*

If two acts have the same consequences in all states of the world, there is no reason to view them as two separate acts. Thus, acts can be identified with sets of their consequences. Formally, an act is a function that attributes consequences to every state of the world.

The set of all acts available in a given situation will be marked as F (in the omelette example set F consits of three elements).

In a set of acts the relation of preference is introduced. Considering two acts, an individual may prefer act f to act g. In other words, if an individual has to decide between f and g, and no other acts are involved, he will choose f.

An individual cannot simultaneously prefer act f to g and act g to f. That is why Savage replaces the realtion *is preferred* with the relation *is not preferred*.

Formal notation

$s, s', \ldots$ – states of world,
S – set of states of world,
$\mathrm{A}, \mathrm{B}, \mathrm{C}, \ldots$ – events (subsets of set of states of world),
$c, c', \ldots$ – consequences,
C – set of consequences,

[4] States of eggs correspond to states of world in this example.

$f, g, h, \ldots$ – act[5] (functions from set S into set C). (If f denotes act, and s denotes state of world, thes $f(s)$ denotes the consequence assigned to state s.)
F – set of acts.
$\succcurlyeq$ – relation of preference on set F.

A constant act, is an act, whose consequences are independent from the state of world[6]. A formal definition of the constant act is as following:

Def. 1.
f $(\in F)$ is a constant act if and only if for any $s \in S$ holds $f(s) = c$, where $c \in C$.

F^{const} will be used to denote a set of constant acts.
By $[f, A; g, A']$ we denote an act h, such that, for $s \in S$:

$$h(s) = \begin{cases} f(s), & \text{if } s \in A \\ g(s), & \text{if } s \in A' \end{cases}$$

Savage introduces a relation of preference, when some event is given. The definition is as follows:

Def. 2
$f \succcurlyeq_A g$: if $[f, A; h, A'] \succcurlyeq [g, A; h, A']$ for a certain $h \in F$.

$f \succcurlyeq_A g$ is understood as: act f is prefered to act g, when event A is given.
Savage also defines the notion of null event.

Def. 3
An event A *is null* if $f \succcurlyeq_A g$ for any $f, g \in F$.

Postulates of Savage's theory of decisions[7]

P1. $\succcurlyeq$ *is complete and transitive.*

P2. *For any $f, g, h, h' \in F$, not null event $A \subseteq S$ holds:*
$[f, A; h, A'] \succcurlyeq [g, A; h, A']$ *if and only if* $[f, A; h', A'] \succcurlyeq [g, A; h', A']$.

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[5] An act is viewed as a particular attribution of consequences to the states of the world and corresponds with real or hypothetical alternatives that a decision-maker chooses from.

[6] Savage insists on the occurrence of all constant facts, which is problematic as it is possible that the act "Boeing 707 is flying over the Atlantic" is required while all oil sources are used up.

[7] The notation of axioms used here is not identical with Savage's original notation.

P3. *For not null event $A \subseteq S$, and for any $f, g \in F^{const}$:*

$$[f, A; h, A'] \succcurlyeq [g, A; h, A'] \text{ if and only if } f \succcurlyeq g.$$

P4. *For any $A, B \subseteq S$ and for $f, g, f', g' \in F^{const}$ such that $f \succ g$ and $f' \succ g'$:*

$$[f, A; g, A'] \succcurlyeq [f, B; g, B'] \text{ if and only if } [f', A; g', A'] \succcurlyeq [f', B; g', B'].$$

P5. *There exist $f, g \in f^{const}$ such that $f \succ g$.*

P6. *For any $f, g \in F$ such that $f \succ g$ and for any $h \in F^{const}$ there exists finite partition $\mathcal{P}$ of the set S such that, for any $H \in \mathcal{P}$:*

i) $[h, H; f, H'] \succ g$,

ii) $f \succ [h, H; g, H']$

P7. *For any $f, g, h \in F$, if $f(s) \succ g(s)$ for any state s of event A, then for any h*

$$[f, A; h, A'] \succcurlyeq [g, A; h, A'].$$

Savage's axioms resemble those of von Neuman and Morgenstern's theory of games. They both assume that the ordering relation is complete and transitive[8]. Both theories contain the *Sure Thing Principle*, which means that common elements in any pair of alternatives can be ignored or eliminated (axiom P2). Savage's system requires also some other special axioms, the most important of which are axioms P3 and P4. They help to achieve subjective probability from subjective utility.

Axioms P5, P6, P7 are mainly technical conditions (*non-triviality, continuity*, and *domination*). They are mostly non-controversial.

The advantage of Savage's theory is the fact that it does not assume *a priori* the existence of subjective probability but derives it from axioms connected with preferences. Moreover, the objects of preference in this theory are concrete and easily identifiable with the elements of real decision problems.

The drawback of Savage's theory is the fact that it allows a set S to be an infinity set. Most theories alternative to Savage's theory, which preserve the linear order and additive subjective probability and which use subjective probability on finity sets, use the notion of lottery. It is achieved by either

[8] The completeness means that any two objects are either exactly ordered or equivalent, while transitiveness guarantees that there are no cycles of exact preference. The S of completeness and transitiveness are included in the axiom P1.

the change of the set of consequences C for the set Pc (all the lotteries in C) so that acts attribute lotteries to the states of the world, or by forming mixed acts as lotteries whose results are acts in the meaning of Savages's theory[9].

In practice, in the case of any real application of the theory, the set of the states of the world and the set of consequences can be identified only for the finite number of elements. This was why Savage made a difference between *small worlds*, which are only models of real-life situations, and *big worlds*, which we live in. The rightness and usefulness of Savage's theory depends largely on the question whether it is possible to transfer the construction of *a small world* to *a big world*. Is *a small world* an adequate representation of *a big world* and does it fulfill the axioms?

Expected utility

The order of preference can be established on the basis of the expected value U (so called utility function) for the decisions. It means that:

$$f \succcurlyeq g \text{ if and only if } E(U(f)) \geq E(U(g)).$$

Two utility functions establishing the same order of preference are referred to as strategically equivalent. Otherwise, they are called strategically non-equvalent. The power of this result is rooted in the fact that it allows to attribute to preferences the function of real values defined on the results of these preferences. It could be used to help solve decision problems by identifying the utility function.

Example

Mr X is a student of marketing and management and his hobby is watching *Bay Watch*. The time has come to decide what path to take. Mr X has two alternatives: either a life-guard or a manager. Considering the options, Mr X takes two factors into account: Z which is health and B which is wealth (their lack will be marked NZ and NB, respectively). He thinks that being a life-guard is healthy but does not make you rich. On the other hand, a manager can be rich but his hard work and lifestyle are likely to lead

[9] Cf.e.g. Anscombe and Aumann (1963) and Fishburn (1967).

to his health deterioration. Let's assume that in our example there is a little chance of a life-guard becoming rich (0.1%). Thus, his career prospects are as follows: ZNB – 0.999, ZB – 0.001. On the other hand, however, we can assume that there is a little chance of a manager going bankrupt (as a result of his wrong decisions). Career prospects of a manager are thus as follows: NZ, B – 0.99 and NZ, NB – 0.01.

Supposing that the utility function U for Mr X is: $U(ZB) = 1$; $U(Z, NB) = 0.6$; $U(N, ZB) = 0.7$; $U(NZ, NB) = 0$, Mr X should choose to become a manager, since the expected utility of this decision is 0.693 and is higher than the expected utility of a life-guard's career, which stands at 0.6004.

Violation of axioms of Savage's theory

Allais'Paradox

Savage's model, and his Sure Thing Principle P2 in particular, was immediately attacked by a French economist Maurice Alais. He presented an example of paradoxical decision problem in which a decision-maker chooses a decision that violates the Sure Thing Principle.

Option	A $(p = 0.1)$	B $(p = 0.89)$	C $(p = 0.01)$
M	\$5,000,000	\$1,000,000	\$0
N	\$1,000,000	\$1,000,000	\$1,000,000
M'	\$5,000,000	\$0	\$0
N'	\$1,000,000	\$0	\$1,000,000

Most people choose M not N but also N' not M'. This violates the Sure Thing Principle as the same results (option B) should not influence the choice between the alternatives.

Ellsberg's Paradox

Although Ssavage's theory has two subjective functions (probability and utility) and seems difficult to test, in 1961 Ellsberg presented an example of a decision problem which contradicts the theory of subjectively expected utility. Ellsberg's paradox can be illustrated by the following:

Supposing we have an urn with 90 bowls: 30 red ones and 60 blue or green ones in an unknown proportion, we will consider the following game:

Option	Payoffs for drawing a ball of each color		
	Red	*Blue*	*Green*
F	\$100	\$0	\$0
G	\$0	\$100	\$0
F'	\$100	\$0	\$100
G'	\$0	\$100	\$100

Most of the pollees choose F not G and G' not F'. People who prefer F to G should also prefer F' to G', as the only difference is the result for the green bowl, which does not differenciate between F and G or F' to G'. If an individual prefers F to G, the theory claims that $U(\mathit{red}) > U(\mathit{blue})$, whereas if an individual prefers G' to F', $U(\mathit{red}) < U(\mathit{blue})$. Considering the preferences of the pollees, we arrive at a contradiction. Many people confirmed this paradoxical choice even if they knew Ellsberg's paradox. One of the possible explanations of this phenomenon is aversion to uncertainty. The other one is that decision-makers do not believe that urns in both cases are the same. This paradox falsifies the hypothesis that convictions can be represented by subjective probability.

Asymmetrical Domination

According to the axioms of Savage's theory, if acts f and g are given, the additional act h should not influence the preference between acts f and g. However, this proves untrue.

Let's consider the sales of beer. Brand X costs 1.80 zl per bottle and its quality is rated at 50. Brand Y costs 2.60 zl with the quality rated at 70. Some prefer X, some Y. Let's add brand Z costing 2.00 zl with the quality rated at 50. It is obvious that Z is worse than X. We could say that Z is dominated by X as it is cheaper and of the same quality. Thus, Z should not change the preferences between X and Y. However, it does. People tend to choose X more often when given the choice of X, Y and Z than when given the choice of X and Y only. Why? There seems to be another reason for such a choice. It is not clear whether X or Y is better but it is clear that X is better than Z. Thus, it can be said that X is at least better than some other brand, while Y is not.

The Reversal of Preference

Let us consider two holiday destinations:

A: average weather, average beach, average hotel, average water temperature, average nightlife.

B: plenty of sunshine, great beaches and reefs, luxurious hotel, freezing water, extremely strong winds, no nightlife.

33% of the pollees chose A, while 67% preferred B. Those who had booked two destinations, paid advance payments and had to choose between A and B made the following decisions: 52% abandoned A and 48% abandoned B. According to the theory of expected utility the preference to choose or abandon should be the same. There is the following explanation of this phenomenon: as much can be said in favour of B as against it.

The reversal of preference is explained in the following way: the choice is relative and depends on the way the question is asked. When pollees are asked what they want or what they do not want their attention is drawn to either positive or negative aspects. This mostly accounts for differences in preferences.

Conclusions

Savage's theory, as its author claims, could be viewed as an immature and superficial empirical theory of foreseeing human behaviour while making decisions. It can be applied only to limited fields and everybody can use it to predict some aspects of human behaviour. Concurrently, human behaviour often contradicts theories, sometimes to an outstanding degree. Usually, such results are attributed to chance or subconscious motives.

If we compare Savage's theory of decision and von Nauman and Morgenstern's theory of games, von Neuman and Morgenstern's theory of expected utility seems to be more applicable to games in which matrix of payments and players' choices from possible random strategies are known because of the construction of the game. Savage's theory of subjective utility seems more applicable for modelling games that are created by nature. In these games the agent has to shape his subjective convictions regarding both: payments (consequences) and strategic intentions of his opponents. Formally, Savage's model refers to a decision problem in which a single agent is engaged in a battle against impersonal forces of nature.

Both theories seem to be incomplete when applied to economic decision problems. This is because:

- expected (or subjectively expected) results of utility can contradict the market rating,
- neither of the theories takes temporal component,
- neither of the theories takes prior changes in consumption into account.

Savage's thought was the basis for other axiomatizations of theories of subjective probability and theories of utility, namely the works of Anscombe and Aumann (1963), Pratt, Raiff and Schlaifer (1964) and Fishburn (1967) among the others.

One of the more promising solutions seems to be the idea to represent individual convictions by non-additive probability.

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