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MATHEMATICAL METHODS ON COMMODITY EXCHANGES

1. Introduction

Using of various often quite sophisticated mathematical methods in different branches of economy is now so common that mathematics became an indispensable element of any economical curriculum. In the following we concentrate our attention on some mathematical models and methods that can be applied to the rational option pricing on commodity exchanges. Commodity markets are quite specific and full of different derivative instruments. In Poland there are two main "players" in the field, namely: Giełda Poznańska (GP) and Warszawska Giełda Towarowa (WGT). In general, on the commodity exchanges we can distinguish the following basic contracts (instruments):

- Options: giving rights to buy or sell a commodity,
- Forward: being an obligation for buying and taking/delivering an asset,
- Futures: transactions providing security against unfavorable changes of prices and not a real realization of a contract.

An *option* is a contract that gives the right to buy or sell a prescribed asset for a given price at a prescribed (expiration) time (European option) or at any time prior the expiration date (American option). Two main types of options are *call* and *put* options.

- Call option – the holder of the call option has the right but not the obligation to buy a given asset for a given price. The writer of the call option must sell the asset for a fixed exercise (or strike) price if the holder decides to buy it.
- Put option – the holder of the put option has the right but not the obligation to sell an asset for a given price. The writer of the put option must buy the asset under the prescribed conditions if the holder decides to sell it.

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The question of how much should be paid for the mentioned right or, in other words, what is a reasonable value of an option is the main issue discussed in this paper.

2. Statistics of commodity exchanges in Poland

As already mentioned, in Poland there are two institutions (GP and WGT) dealing with various derivative contracts and instruments for commodities. The first transaction of this sort was performed in 1995 at GP – 100 options for porkhalf were issued. The second emission took place in February 1996 – again 200 contracts for porkhalf. In both cases the writer was Agencja Rynku Rolnego (ARR). Almost all (198) options were actually sold. In 1997 options for milling wheat were emitted by a private business. Since 1998 transactions dealing with contracts on milling and feed wheat have been performed. In 1999 contracts on live hog were added. On WGT first contracts were futures contracts on currency exchange rates for USD/PLN and DEM/PLN as well as futures for milling and feed wheat. In March 1999 futures contracts for interest rates (one and three months) WIBOR have been introduced. In May 1999 the contract DEM/PLN was replaced with EUR/PLN. Also a new commodity futures contract for live hog has been introduced. All these futures contracts were emitted by ARR.

Table 1. Statistics of transactions on GP

Year	Number of clearing houses	Volume in thousands PLN	Number of transactions	
			cash	derivatives
1991	1			
1992	16	6104,1	246	
1993	22	65318,0	1328	
1994	20	169190,7	1090	
1995	33	269393,4	3118	111
1996	34	426298,0	3842	376
1997	27	362675,1	2914	204
1998	19	231341,6	4380	1119
1999	20	551874,9	12174	1607
2000	20	964932,4	15750	118
2001 (till 30.08)	21	277034,9	3558	–

Source: Prepared by the authors

The number of transactions was not large – after the dynamic growth in the period of 1998-1999 it drastically decreased in 2000.

On WGT trading of commodity options began on June 12, 1997. Options available on WGT are standardized:

- Fixed quantity of the basic asset (20 tons of wheat, 50 tons of corn, 5 tons porkhalf or beef carcasses).
- Fixed duration of 4, 7, 8, 13 or 26 weeks, starting from the emission date.
- Registered and settled via the WGT clearing house.
- There is a commission – the fee equal to the option premium.

In 1997 on WGT the total volume of derivatives traded was given by a six-digit figure: 113154 PLN. In 1998, 1400 options were emitted, with the total value of 362963 PLN. Many of them were not purchased at all. In 1999 the American call options for milling wheat were issued. The total volume was worth much less – 11994 PLN.

3. Forecasting of commodity prices

There are many methods that could be used to forecast prices for agricultural goods, being the underlying assets for any derivative contracts we discuss in this paper. Let us just mention the most common examples:

- Single-equation econometric models of different forms and types, including:
  - linear: $y_i = \beta_0 + \beta_1 x_{1i} + \ldots + \beta_k x_{ki} + \varepsilon_i$
  - nonlinear with respect to explanatory variables, e.g. $y_i = \beta_0 + \beta_1 x_i + \beta_2 x_i^2 + \varepsilon_i$
  - nonlinear with respect to both explanatory and parameters, but a transformation to linear models is possible, e.g. $y_i = \beta_0 \cdot x_{1i}^{\beta_1} \cdot x_{2i}^{\beta_2} + \varepsilon_i$
  - fully nonlinear, e.g. $y_i = \beta_1 + \beta_2 e^{\beta_3 \cdot x_{1i}} + \varepsilon_i$
- multi-equations econometric models:

$$Y_1 = \sum_{i=2}^{m} \beta_{1i} Y_i + \sum_{j=1}^{k} \lambda_{1j} Z_j + \varepsilon_1$$

$$Y_2 = \sum_{i=1, i \neq 2}^{m} \beta_{2i} Y_i + \sum_{j=1}^{k} \lambda_{2j} Z_j + \varepsilon_2$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$Y_m = \sum_{i=1}^{m-1} \beta_{mi} Y_i + \sum_{j=1}^{k} \lambda_{mj} Z_j + \varepsilon_m$$

- various types of neural networks
- nonlinear prediction models based on the deterministic chaos theory

In this paper we do not discuss further the problem of forecasting of the basic agricultural assets. We just assume that one or the other method can be used to model the price dynamics of the underlying commodity. Nevertheless, it is an important and nontrivial problem. In the following we introduce two simple versions of stochastic dynamics of underlying assets when presenting two fundamental and quite popular models frequently used for option pricing.

4. Basic models of option pricing

Here the Black-Scholes model and a binomial model of Cox-Ross--Rubinstein are briefly introduced. They will be elaborated in more details in the forthcoming sections. Later on they will also be used to analyze and illustrate the option valuation with data coming from Polish commodity markets.

The Black-Scholes model assumes (among other things) that prices of underlying assets are the subject to the continuous changes. The model is mainly used to European call and put options for the stock shares, stock indices, and the currency exchange rates but it can also be applied for American options pricing if underlying assets pay no dividends.

The Cox-Ross-Rubinstein model assumes (somewhat more realistically) that prices of the assets can change in a discrete way. In other words, a continuous stochastic process is replaced by a discrete random walk. The model is often used for pricing European options for stock shares, instruments with fixed dividend, currency exchange rates, indices, and for American options.

It should be mentioned that these two basic models are just the simplest examples of a much broader spectrum of methods and approaches used for option pricing. They are, however, very important examples, not only from the theoretical perspective but also from the practical point of view. Many other methods are just straightforward generalizations of them.

5. Black-Scholes model

Now classical, Black-Scholes model is based on the following assumptions:

- assets prices undergo a stochastic process of Ito's type (the Brownian motion),

– there are no overheads, taxes, and other transaction costs,
– assets pay no dividends during the time of option validity,
– there are no risk-free arbitrage opportunities possible,
– the so-called short selling is allowed,
– buying and selling of assets are possible continuously, i.e., assets can be traded in a continuous way
– all market participants can rent and invest money with the same risk--free interest rate,
– in the short term, a risk-free interest rate is constant.

Obviously, the option value V at time t is given by:

$$V(s,t) = \max(S(t) - W, 0)$$

where $S(t)$ denotes a market price of a given asset at time t and W is the exercise price.

A fundamental problem is to find a formula describing the option value (price) V as a function of an asset price $S(t)$ at an arbitrary given time moment t. Any solution to this problem requires a market model, which describes dynamics of prices of a given asset. Within the Black-Scholes approach we assume the following model of the asset (stochastic) dynamics:

$$dS(t) = \sigma S(t) dX(t) + \mu S(t) dt$$

where $dX(t)$ is a Wiener stochastic. Parameters μ and σ characterize the market.

Due to the presence of the stochastic element dX, the asset price S is a random variable at any given time moment. Still it is possible, and this is a great achievement of Black and Scholes, to construct a deterministic equation relating the option value to time and the asset price.

The key point is a construction of a secure portfolio consisting of a number of assets and a call option written for it. At any time t the value of such a portfolio is given by:

$$\Pi(t) = V(S,t) - nS(t)$$

This does not depend on the current price of the asset only if:

$$n = \frac{\partial V}{\partial S}$$

Such a situation corresponds to the so-called perfect hedging. Now we can make use of the assumption about the lack of arbitrage. Any potential gain of the portfolio value should be equal to the gain obtained from the risk-free investment:

$$r\Pi dt$$

From this we immediately obtain an equation describing dynamics of the option value:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2S^2\frac{\partial^2 V}{\partial S^2}+rS\frac{\partial V}{\partial S}-rV=0$$

From the mathematical point of view it is a partial differential equation of the parabolic type. There are two parameters: a risk-free interest rate r and volatility of the asset price σ. The solution to this equation (corresponding to the proper initial and boundary conditions) directly leads to the Black-Scholes formula for the option value V:

$$V(S,t)=SN(d)-We^{-r(T-t)}N(d-\sigma\sqrt{T-t})$$

Here N is given by the well-known integral:

$$N(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}e^{-\frac{1}{2}y^2}dy$$

Parameter d is defined as:

$$d=\frac{\ln\left(\frac{S}{W}\right)+\left(r+\frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}$$

Thus for a given time t the option value depends on two variables (time left to the expiration date $T-t$ and the current price of the asset S) and three parameters (risk-free interest rate r, volatility of the asset price σ and the exercise price W). One of the most important problems in the process of option pricing is estimation of σ.

6. Cox-Ross-Rubinstein binomial model

Some of the basic assumptions underlying the binomial model of Cox-Ross-Rubinstein for option pricing are the same as for the previously discussed Black-Scholes model. These are:

– there are no overheads, taxes, and other transaction costs,
– assets pay no dividends during the time of option validity,
– there are no risk-free arbitrage opportunities possible,

– all market participants can rent and invest money with the same risk--free interest rate,
– in the short term, a risk-free interest rate is constant.

There are, of course, also big differences. First of all, in the binomial model assets prices undergo a discrete random walk, what means that buying and selling of assets are possible only at some fixed time moments. Moreover, we assume that at any of these particular times, the asset price can either do up or down by a fixed amount and with a given constant probability.

Fortunately, the binomial model is very flexible and can easily accommodate a constant dividend paid by the underlying assets.

If there are no dividends, after the first step Δt we have just two possibilities: the asset price goes up to $S \cdot u$ or down to $S \cdot d$. After the second step at time $2 \cdot \Delta t$ there are already three possibilities: $S \cdot u^2$, S and $S \cdot d^2$. Generalizing this to "i" steps we see that after $i \cdot \Delta t$ there are $i + 1$ possible situations, which can be reached using 2^i ways. Any possible (reachable) price level can be computed using the following expression:

$$S \cdot u^j \cdot d^{i-j} \quad \text{for } j = 0, 1, 2, \ldots, i$$

Under the above assumptions the quantities u, d, p are given by:

$$u = e^{\sigma \cdot \sqrt{\Delta t}}$$

$$d = \frac{1}{u}$$

$$p = \frac{e^{r \cdot \Delta t} - d}{u - d}$$

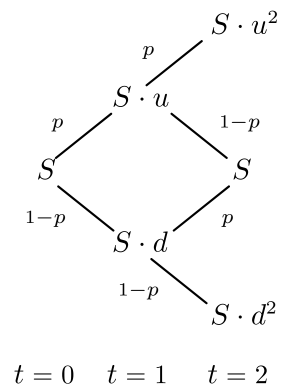
where:

σ – volatility of an asset price (measured by the standard deviation – usually computed as the historical volatility),
r – risk-free interest rate,
Δt – length of the time interval.

The parameters p, u and d have to take into account proper values of the average and variance for an asset price for the time period Δt (see Fig. 1).

It is worth noting that the Cox-Ross-Rubinstein model leads to a very effective computer code and usually provides a good quality numerical answers.

Figure 1. An idea leading to the binomial model



Source: Prepared by the authors

7. Case studies – empirical examples

Here we present a strategy "in action" of using these methods for the purpose of the derivatives pricing on Polish commodity markets. Our studies are based on empirical data obtained from both GP and WGT.

First, using empirical data from GP, we estimate the values of a futures contract and a call option. Then we present the analysis of European option pricing using data from WGT.

Example 1.

First, consider a futures contract, valid for three months, issued for 20 tons of rye, emitted in June 2000, with the following parameters (to perform the actual computations we used the software *The Black-Scholes And Beyond Interactive Toolkit*):

S – 426,50 PLN/ton (a price of the underlying asset, an average price in June 2000),

W – 426,50 PLN/ton (an exercise price),

r – 17% (a risk-free interest rate),

T – 90 days (time to the expiration date),

σ – 25,7% (historical volatility estimated using the data from the last 12 time periods)

To estimate the option premium both Black-Scholes and Cox-Ross--Rubinstein models were used. The results are summarized in Table 2.

According to the Black-Scholes model, the option value was computed to be 31,14 PLN. A very similar number was obtained using the binomial

Table 2. Results of call option pricing for rye

Method	*Black-Scholes model*	Binomial model ($n = 200$ steps)	
Type of option	European option	European option	American option
Option value (PLN/ton)	31,14	31,11	31,11
Delta	0,65	0,65	0,65
Gamma	0,01	0,01	0,01
Theta	−0,23	−0,23	−0,23
Vega	0,78	0,78	0,78
Rho	60,94	60,93	60,93

Source: Prepared by the authors

model – 31,11 zł. In September 2000 (the expiration time), the average price for rye on the cash market was equal to 412,50 PLN/ton. Of course, it was unreasonable to exercise the call option at that time because it would generate the loss of 280 PLN per option.

The five "Greek" parameters listed in the table 2, provide in a compact and easily accessible form some additional information and measure the degree of dependence of the option price on the asset price, the time left to the expiration date, volatility, etc.

Example 2.

In June 5, 1997, ARR issued on GP a call option for 10 tons of porkhalfs, the expiration date being August 5, 1997. The initial option price proposed by ARR was 300 PLN. The actually negotiated average price was higher – 1210 PLN.

Input data:

S – 4,90 PLN/kg (a price of the underlying asset),
W – 5,20 PLN/kg (an exercise price),
r – 19% (a risk-free interest rate),
T – 60 days (time to the expiration date),
σ – 22% (historical volatility),

The Black-Scholes model produced the following value for the option premium:

V – 0,12 PLN/kg (1200 PLN for each option),
σ_{imp} – 22,59% (implied volatility for $V = 1210$ PLN),

Historical volatility was computed using the data from GP, giving $\sigma = 22\%$. As the result of using the Black-Scholes formula we get the option premium of 0,12 PLN/kg, what gives together 1200 PLN per option issued for 10 tons. This is more than three times more that the initial (proposed) price and by just 10 PLN less than the average negotiated price. Then we estimated the implied volatility obtaining (for average premium 1210 PLN) 22,59%. This number is quite close to the historical volatility.

Example 3.

In June 23, 1998, on WGT, European call options for milling wheat valid for 8 weeks were emitted.

Input data:

S – 490 PLN/ton (a price of the underlying asset),

W – 500 PLN/ton (an exercise price),

r – 17% (a risk-free interest rate),

T – 56 days (time to the expiration date),

σ – 18,5% (historical volatility),

Using the Black-Scholes model we obtained the following results:

V – 16,7 PLN/kg (an actual price 12,5 PLN/kg, loss of 4,2 PLN/kg),

σ_{imp} – 13,9%.

In this case the actually negotiated exercise price was much less that the option value suggested by the model.

8. Brief Summary

The development of Polish futures and options markets, where various derivative products could be traded, is rather slow. There are several reasons for this unsatisfactory situation. First of all, the ARR plays an absolutely dominant role in the intervention market. This should change after the access of Poland to the European Community, allowing the derivatives trading to grow up dynamically. We believe that the mathematical methods of option pricing, presented in this paper, will still be useful in determining the option value. Our investigations show that despite the fact that some of the assumptions underlying both models are apparently not satisfying, both methods can be fruitfully used in practical estimations at the Polish commodity markets. Of course, it is possible to generalize the models, to relax some of the constraints, and to use a bit more realistic assumptions. Still both Black-Scholes and Cox-Ross-Rubinstein models will remain popular as the simplest examples of very useful and fruitful theoretical simplifications, providing a good starting point for further investigations.

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