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LATTICE PROPERTIES OF A PROTOLOGIC INFERENCE

A protologic construction of q-consequence in [1] has been designed as a formal counterpart of reasoning admitting rules of inference which lead from non-rejected premises to accepted conclusions. The very concept is a generalisation of Tarski consequence and, as such, it may be investigated similarly cf. [3]. In the paper we present results concerning the lattice $Q(L)$ of all q-consequence operations of a given sentential language L.

1. Q-consequence as a protologic inference

The concept of a q-consequence was introduced in [1] for the purpose of formalization of reasoning leading from non-rejected premisses to accepted conclusions. Accordingly, the first approach is based on extended logical matrices having two disjoint sets of distinguished values: rejected and accepted. The so-called q-matrix consequence relation, on its turn, imitated an inference not necessarily accepting the rule of unlimited repetition.

Let $L = (For, F_1, \ldots, F_m)$ be a sentential language. Formulas, i.e. elements of For, are then built from variables using the operations $F_1, \ldots, F_m$ representing the sentential connectives. In algebraic terms, L is freely generated by the set of its variables, $Var = \{p, q, r, \ldots\}$. A *q-matrix* is a triple

$$M^* = (A, D^*, D),$$

where A is an algebra similar to L and D^*, D are disjoint subsets of A interpreted as sets of *rejected* and *distinguished* values of M, respectively. For any such M^* one defines the relation $\vdash_{M*}$ between sets of formulae and formulae, a *matrix q-consequence of* M^* putting for any $X \subseteq For, \alpha \in For$

$$X \models_{M^*} \alpha \text{ iff for every } h \in Hom(L, A)(h\alpha \in D \text{ whenever } hX \cap D^* = \emptyset)$$

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The q-concepts reduce to usual concepts of matrix and consequence only if $D^* \cup D = A$, i.e. when the sets D^* and D are complementary. In that case the set of rejected elements coincides with the set of non-designated elements. So, the first set may be omitted and what then we have is a *matrix*

$$M = (A, D)$$

based on the same algebra of values and D serving as the set of distinguished elements.

The matrix consequence relation $\models_M \subseteq 2^{For} \times For$ for M,

$$X \models_M \alpha \text{ iff } h \in Hom(L, A)(h\alpha \in D \text{ whenever } hX \subseteq D),$$

may obviously be regarded as a special q-matrix consequence.

Given a matrix M for a language L, the system $\mathrm{E}(M)$ of sentential logic is defined as the *content* of M, i.e. the set of all formulas taking for every valuation h (a homomorphism) of L in M. Thus

$$\mathrm{E}(M) = \{\alpha \in For : \text{for every } h \in \mathrm{Hom}(L, A), h(\alpha) \in D\}.$$

Notice, that $\mathrm{E}(M) = \{\alpha : \emptyset \models_M \alpha\} = \{\alpha : \emptyset \models_{M^*} \alpha\}$. This obviously means that both, the matrix consequence, and q-matrix consequence may serve as inferential extensions of the single logical system.

With every $\models_{M^*}$ there may be uniquely associated an operation $Wn_{M^*} : 2^{For} \rightarrow 2^{For}$ such that

$$\alpha \in Wn_{M^*}(X) \text{ if and only if } X \models_{M*} \alpha.$$

called a *q-matrix consequence operation* of M^*. $Cn_M : 2^{For} \rightarrow 2^{For}$ defined by

$$\alpha \in Cn_M(X) \text{ if and only if } X \models_M \alpha,$$

a *matrix consequence operation* of M, is a special case of Wn_{M^*}.

As known, the concept of structural sentential logic is the ultimate generalisation of the notion of the matrix consequence operation. A structural logic for a given language L is identified with *Tarski's consequence* $\mathrm{C} : 2^{For} \rightarrow 2^{For}$,

(T0) $X \subseteq \mathrm{C}(X)$

(T1) $\mathrm{C}(X) \subseteq \mathrm{C}(Y)$ whenever $X \subseteq Y$

(T2) $\mathrm{C}(\mathrm{C}(X)) = \mathrm{C}(X)$,

satisfying the following condition of *structurality*

(S) $e\mathrm{C}(X) \subseteq \mathrm{C}(eX)$ for every *substitution* of L.

cf. [3].

The investigation in [1] shows that similar generalisation of the q-matrix consequence aims at the theory of the *q-consequence operation* $\mathrm{W} : 2^{For} \to 2^{For}$ satisfying the following postulates:

(T1) $W(X) \subseteq W(Y)$ whenever $X \subseteq Y$

(T2) $W(X \cup W(X)) = W(X)$,

and, possibly,

(S′) $e\mathrm{W}(X) \subseteq \mathrm{W}(eX)$ for every *substitution* of L.

2. The lattice *Q(L)* – general properties

Let $L = (For, F_1, \ldots, F_m)$ be a given sentential language and let $Q(L)$ be a class of all q-consequence operations on L.

Consider W_1, $W_2 \in Q(L)$. If $W_1(X) \subseteq W_2(X)$ for any $X \subseteq For$, then we say that W_1, is weaker than W_2, or that W_2, is stronger than W_1 and we write $W_1 \leq W_2$. Since $\subseteq$ partially orders the powerset of the set of formulas, 2^{For}, we obtain

2.1. $\leq$ is a partial ordering in $Q(L)$.

Where $U \subseteq Q(L)$, let $Sup(U)$ and $Inf(U)$ denote the least upper bound and the greatest lower bound, respectively, i.e.

(1) $Sup(U)$ is the weakest q-consequence operation in $Q(L)$, such that for every $W \in U$, $W \leq Sup(U)$,

(2) $Inf(U)$ is the strongest q-consequence operation in $Q(L)$, such that for every $W \in U$, $inf(U) \leq W$.

In what follows, we adopt a standard concept of a rule inference R as a set of sequents (X, α), i.e. $R \subseteq 2^{For} \times For$; cf. [3]. Next, we also need the notion of a q-consequence operation $W_{n_{\mathbf{R}}} : 2^{For} \to 2^{For}$ based on the set $\mathbf{R}$ of rules of inference. Thus, for any $X \subseteq For$

$$W_{n_{\mathbf{R}}}(X) = \bigcap \{Y \subseteq For : Y \text{ is } \mathbf{R}\text{-closed relative to } X\}$$

Recall, that $Y \subseteq For$ is $\mathbf{R}$-closed relative to $X \subseteq For$ if and only if for each $(Z, \alpha) \in R \in \mathbf{R}$ if $Z \subseteq X \cup Y$, then $\alpha \in Y$; cf. [1].

Finally, let $Rule(W)$ denote the set of all rules of inference of a given q-consequence W:

$$Rule(W) = \{R : \text{ for every } (X, \alpha) \in R,\ \alpha \in W(X)\}.$$

Clearly, $Rule(W)$ is the biggest inferential basis for W.

2.2. For any $W_1, W_2 \in Q(L)$

$$W_1 \leq W_2 \text{ iff } Rule(W_1) \subseteq Rule(W_2).$$

Assuming that $U = \{W_i : i \in I\}$ we get the characterisation theorems concerning $Sup(U)$ and $Inf(U)$:

2.3. $Sup(U) = W_Q$, where $Q = \bigcup\{Q_i : W_i = Wn_{Q_i}\}$.

PROOF. Since $Rule(W_i) \subseteq Rule(Sup(U))$, for every $W_i \in U, W_i \leq Sup(U)$. On the other hand, for any W^* such that $W_i \leq W^*$ for all $W_i \in U$ we get $\bigcup\{Rule(W_i) : W_i \in U\} \subseteq Rule(W^*)$. So $W_Q = W_{\cup\{Rule(W_i):W_i \in U\}} \leq W^+$ and, therefore, $W_Q \leq W^+$.

2.4. $Inf(U) = \bigcap\{W_i : i \in I\}$, i.e. for every $X \subseteq For$ $Inf(U)(X) = \bigcap\{W_i(X) : i \in I\}$.

PROOF. Assume that $W^* : 2^{For} \to 2^{For}$ is an operation defined as:

$$W^*(X) = \bigcap\{W_i(X) : i \in I\}.$$

What then we have to prove as first is that W^* is a q-consequence, i.e. that it satisfies the conditions $(W1)$ and $(W2)$. W^* obviously satisfies $(W1)$. To check that it also satisfies $(W2)$ let us assume that for some $X \subseteq For$ and $\alpha \in For$, $\alpha \in W^*(W^*(X) \cup X)$. Then, also, for every $i \in I$, $\alpha \in W_i(W^*(X) \cup X)$ and due to $W^*(X) \subseteq W_i(X)$, $\alpha \in W_i(W_i(X) \cup X)$. Since each W_i is a q-consequence we obtain that $\alpha \in W_i(X)$ and, consequently, $\alpha \in W^*(X)$. So, $W^*(W^*(X) \cup X) \subseteq W^*(X)$. Taking into account that the reverse inclusion is implied by $(W1)$, the task is concluded.

Assume now that W^- is a q-consequence weaker than all W_i'. Thus $W^-(X) \subseteq W_i(X)$ for every $X \subseteq For$ and every $i \in I$. Therefore, $W^-(X) \subseteq W^*(X) = \bigcap\{W_i(X) : i \in I\}$. This justifies that $W^* = Inf(U)$. $Inf(U)(X) = W^*(X) = \bigcap\{W_i(X) : i \in I\}$.

2.5. Corollary. $Q(L)$ is a complete lattice under $\leq$.

Now, let $R \subseteq Rule(L)$ be any set of inference rules in L. Next, let $Q_R(L)$ be the subclass of all q-consequence operations with the very property of having (at least) all rules in R:

$$Q_R(L) = \{W : W \in Q(L) \text{ and } R \subseteq Rule(L)\}$$

2.6. $Q_R(L)$ is a complete sublattice of $Q(L)$.

PROOF. The property follows easily from 2.2, 2.3 and 2.4.

3. Important sublattices of *Q(L)*

Since adding the unrestricted rule of repetition,

$$rep = \{(\{\alpha\},\ \alpha) : \alpha \in For\}$$

to the set of rules of any q-consequence W changes it into the consequence operation, cf. [1]. p. 51, from 2.6 we immediately get

3.1. The class $C(L)$ of all consequence operations on L is a complete sublattice of $Q(L)$.

3.2. The class $Q_S(L)$ of all structural q-consequence operations on L is a complete sublattice of $Q(L)$.

Further to 3.2 and also 3.1 we easily get another characterisation Sup and Inf in $Q(L)$. Namely,

3.3. For every $U \subseteq Q(L)$:

(1) If at least one of $W \in U$ is a consequence operation, then $Sup(U)$ is a consequence operation as well.

(2) If at least one of $W \in U$ is not a consequence operation, then $Inf(U)$ is not a consequence operation.

Using the last corollary we may localize the greatest and the smallest elements of $Q(L)$. Let us consider the following two operations on L:

$$W_S \text{ such that for every } X \subseteq For, W_S(X) = For$$

$$W_\emptyset \text{ such that for every } X \subseteq For, W_\emptyset(X) = \emptyset$$

The former, W_S, is the inconsistent consequence and the biggest element of $Q(L)$. The latter, $W_\emptyset$, is the smallest element of $Q(L)$. To complete the view, let us recall that the identity operator:

$$W_T \text{ such that for every } X \subseteq For, W_T(X) = X$$

the smallest element of $Q_C(L)$

Given a language L, let $Q(\emptyset, T) = \{W \in Q(L) : W_\emptyset < W < W_T\}$, where $<$ means that $\leq$ and $\neq$.

3.4. $Q(\emptyset, T) \geq 2^\omega$.

Proof. $card(For) = \omega$. For every $Z \in 2^{For}$ let us put:

$$W_Z(X) = \begin{cases} Z & \text{if } X = For \\ \emptyset & \text{if } X \neq For. \end{cases}$$

It is obvious, that every such W_Z is a q-consequence operation in between $W_\emptyset$ and W_T. The aproximation comes, directly from the theory of characteristic functions.

4. Two negative results

Every consequence operation C has the property that all C-indistinquistable formes are C-equivalent. We say that $\alpha, \beta \in For$ are *C-indistinguishable*, $\alpha =_C \beta$, if

$$\alpha =_C \beta \text{ iff } \forall_X C(X, \varphi(\alpha/p)) = C(X, \varphi(\beta/p))$$

and *C-equivalent*, $\alpha \approx_C \beta$, if

$$\alpha \approx_C \text{ iff } \forall_X (\varphi(\alpha/p) \in C(X) \text{ iff } \varphi(\beta/p)) \in C(X)).$$

The inclusion

(*) $$=_C \subseteq \approx_C.$$

exemplifying the property does not hold generally, i.e. it fails for some $W \in Q(L)$.

For the purpose of discerning between two kinds of operations in $Q(L)$ in [2] the notions of extensional and intensional q-consequence were introduced. Assuming the same definitions for $=_W$, $\approx_W$, we say that a q-consequence is *extensional* provided that

(*) $$=_W \subseteq \approx_W.$$

Otherwise, W is called intensional.

A natural question obtains whether or not the two classes of q-consequences are sublattices of $Q(L)$. Below, we give two examples showing that the answer in both cases is negative.

Example 1. Showing that Inf of two extensional q-consequence operations may be intensional. Given $p, q, r \in Var(L)$ we put

$$W_1(\emptyset) = \emptyset$$
$$W_1(\{p\}) = W_1(\{q\}) = W_1(\{p, q\}) = \{p, q\}$$
$$W_1(X) = For, \ \text{ otherwise.}$$

It is easy to verify that W_1 is a consequence operation and, as such, it is an extensional q-consequence operation.

Next, we put

$W_2(\emptyset) = \emptyset$

$W_2(\{r\}) = \{r\}$

$W_2(\{q\}) = \{q\}$

$W_2(\{p\}) = W_2(\{p, q\}) = W_2(\{p, r\}) = W_2(\{q, r\}) =$
$\quad = W_2(\{p, q, r\}) = \{q, r\}$

$W_2(X) = For$, otherwise.

Now, note that

$$p =_{W_2} p,\ q =_{W_2} q,\ r =_{W_2} r$$

and

$$\alpha =_{W_2} \beta$$

for $\alpha, \beta \in For - \{p, q, r\}$. Moreover, each variable p, q, r, is not W_2 equal to other formula different from itself, i.e.

$p \neq_{W_2} \alpha$ for every $\alpha \neq p$

$q \neq_{W_2} \alpha$ for every $\alpha \neq q$

$r \neq_{W_2} \alpha$ for every $\alpha \neq r$.

On the other hand, we have

$q \approx_{W_2} q, r \approx_{W_2} r$

and

$\alpha \approx_{W_2} \beta$

for $\alpha, \beta \in For - \{q, r\}$. And, further to that,

$q \not\approx_{W_2} \alpha$ for every $\alpha \neq q$

$r \not\approx_{W_2} \alpha$ for every $\alpha \neq r$

One may easily verify that W_2 is an extensional consequence operation, i.e. that W_2 is a q-consequence and that $=_{W_2} \subseteq \approx_{W_2}$.

In the end, let $W = Inf(W_1, W_2)$.

Then, due to ..., we get that

$W(\emptyset) = \emptyset$

$W(\{p\}) = \{q\}$

$W(\{q\}) = \{q\}$

$W(\{r\}) = r$

$W(\{p, q\}) = \{p\}$

$W(\{p, r\}) = W(\{q, r\}) = W(\{p, q, r\}) = \{q, r\}$

$W(X) = For$ otherwise.

One may check that $p =_W q$. However, at the same time $p \notin W(\{p\})$ and $q \in W(\{q\})$. Consequently, $p \not\approx_W p$. This means that W is not extensional.

Example 2. (Showing that inf of two intensional q-consequence operations may be intensional). Given $p, q \in Var(L)$, we put

$$W_1(\emptyset) = \emptyset$$
$$W_1(\{p\}) = W_1(\{q\}) = W_1(\{p,q\}) = \{p\}$$
$$W_1(X) = For, \text{ otherwise.}$$

The q-consequence thus defined is intensional since $p =_{W_1} q$, but $p \in W_1(\{p\})$, $q \notin W_1(\{p\})$ and, therefore $p \not\approx_W q$.

Next, we put

$$W_2(\emptyset) = \emptyset$$
$$W_2(\{p\}) = W_2(\{q\}) = W_2(\{p,q\}) = \{q\}$$
$$W_2(X) = For \text{ otherwise.}$$

For this q-consequence we have:

$$p =_{W_2} q, p \notin W_2(\{q\}) \text{ and } q \in W_2(\{p\})$$

and thus, W_2 is also intensional.

Now consider $W = Inf\{W_1, W_2\}$. W is then characterised as below:

$$W(\emptyset) = W(\{p\}) = W(\{q\}) = W(\{p,q\}) = \emptyset$$
$$W(X) = For, \text{ otherwise.}$$

Accordingly, every two formulas α and β of the language are q-equivalent, $\alpha \approx_W \beta$. So, W is an extensional q-consequence operation.

References

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